Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

Semidefinite Programs with Interval Uncertainty: Reduced Vertex Results [1]

Giuseppe Calafiore [*,2] **Fabrizio Dabbene** [**]

[*] *DAUIN Politecnico di Torino, Italy*
[**] *IEIIT-CNR Politecnico di Torino, Italy*

Abstract: In this paper, we derive a reduced vertex result for robust solution of uncertain semidefinite optimization problems subject to interval uncertainty. If the number of decision variables is m and the size of the coefficient matrices in the linear matrix inequality constraints is $n \times n$, a direct vertex approach would require satisfaction of $2^{n(m+1)(n+1)/2}$ vertex constraints: a huge number, even for small values of n and m. The conditions obtained here are instead based on the introduction of m slack variables and a subset of vertex coefficient matrices of cardinality 2^{n-1}, thus reducing the problem to a practically manageable size, at least for small n. A similar size reduction is also obtained for a class of problems with affinely dependent interval uncertainty.

1. INTRODUCTION

Semidefinite convex optimization problems (SDPs) deal with the minimization of a linear objective subject to a set of linear matrix inequality (LMI) constraints on the design variable $x \in \mathbb{R}^m$, that is

$$\min c^\top x \text{ subject to } F(x) = F_0 + \sum_{k=1}^{m} x_k F_k \preceq 0,$$

where $F_k \in \mathbb{S}^n$, see e.g. Boyd et al. (1994). Efficient polynomial-time solution techniques exist for this class of problems, such as those based on primal-dual interior-point methods, see Nesterov and Nemirovski (1994); Vandenberghe and Boyd (1996).

In the last years, the consideration that most real-world problems unavoidably entail a certain degree of uncertainty stimulated the research on robust solutions to *uncertain* SDP problems, see e.g. Ben-Tal and Nemirovski (1998); El Ghaoui et al. (1998). In this approach, the problem data $(F_0, \ldots, F_k)$ are assumed to be affected by bounded uncertainty, and a solution is said to be robust if it is guaranteed to satisfy the constraints for all admissible uncertainty values.

Unfortunately, tractable necessary and sufficient conditions for the solution of robust SDP problems are available only for very special problem classes, while the general situation is known to be NP-hard, see for instance Ben-Tal and Nemirovski (1998); Blondel and Tsitsiklis (2000); Nemirovski (1993). Various relaxation approaches have been hence proposed to conservatively solve these problems. In particular, in El Ghaoui et al. (1998) the authors provide upper bounds on the optimal solution (i.e. the objective is minimized subject to sufficient conditions for robust satisfaction of the uncertain LMIs) for the case when the uncertainty enters the data in a linear fractional form, while in Ben-Tal and Nemirovski (2002) the case of polytopic uncertainty is considered and a numerically tractable relaxation of the problem is provided, together with an a-priori bound on the degree of conservativeness of the approximation.

In this paper, we consider the case when the LMI coefficient matrices F_k, $k = 0, 1, \ldots, m$, are *symmetric interval matrices*, that is, symmetric matrices whose entries are bounded independently in given intervals. It is a well-known fact, easily proven by convexity arguments, that in this case the uncertain LMI condition is robustly satisfied whenever it is satisfied for all the $2^{n(m+1)(n+1)/2}$ *vertex matrices*, that is the matrices obtained by setting each matrix entry to its upper or lower limit, see Section 2.1. However, recent results in the literature (see Alamo et al. (2007, also in *Systems & Control Letters*, in press, 2008); Rohn (1994a) and the references therein) suggest that the number of vertices can actually be reduced in certain special cases. Motivated by these ideas, we develop in Section 2 an equivalent formulation of a robust interval SDP which is based on the introduction of m slack variables and requires satisfaction of only 2^{n-1} specially selected vertex matrices. Although this result is still exponential in the matrix dimension n, the number of required vertex constraints can be manageable by currently available solvers, for reasonable values of n. Exponential growth of the number of vertices is not surprising and cannot be avoided in general (unless P=NP), since it was already proven in Rohn (1994b) that even the simpler problem of checking robust positive semidefiniteness of a symmetric interval matrix is NP-hard. As a by-product of our reduced vertex set result, we also obtain that for a very specific class of uncertain SDPs, namely linear programs (LP) affected by interval uncertainty, the robust optimization problem can be recast exactly as a standard linear program with slack variables and hence solved in polynomial time. Finally, in Section 3 we present a result that holds when the uncertainty affecting the data is not completely independent, but it is instead represented by a linear transformation of a $p \times q$ interval matrix. In this case, a weaker result holds, which prescribes to impose satisfaction of the LMI constraints at 2^{p+q-1} selected vertices.

[1] This work is supported by MIUR under the FIRB project "Learning, randomization and guaranteed predictive inference for complex uncertain systems," and by CNR-RSTL funds.
[2] Corresponding author (`giuseppe.calafiore@polito.it`).

Related literature. The results of this paper are related to a classical work of Rohn (1994a) on interval symmetric matrices and to a recent paper of Alamo et al. (2007, also in *Systems & Control Letters*, in press, 2008), which provides a new and interesting vertex set result for a class of problems arising in a robust control setting. In particular, our interest towards the subject and the developments in this paper have been directly inspired by the results in Alamo et al. (2007, also in *Systems & Control Letters*, in press, 2008).

More precisely, in Rohn (1994a) a reduced vertex set result is provided for checking negative-definiteness of a symmetric interval matrix. Specifically, Theorem 2 of Rohn (1994a) states that a symmetric interval matrix of dimension $n \times n$ is robustly negative semidefinite if and only if 2^{n-1} special vertex matrices are negative semidefinite. Notice that the result of Rohn (1994a) is an *analysis* result, related to the problem of *checking* if a property (negative semidefiniteness) holds robustly. Instead, the problem considered in this paper is a *design* one, that is, the goal is to find a design vector x such that an interval linear matrix inequality in x is robustly satisfied. It is thus interesting to notice that the same vertex growth factor applies to both robustness analysis and design problems. The result of Rohn (1994a) is recovered in our framework as a special case for $m = 0$ (no design variables).

The developments in Section 3 deal instead with a class of uncertain SDP problems affected by affinely transformed interval uncertainty. Section 3 is closely related to the work in Alamo et al. (2007, also in *Systems & Control Letters*, in press, 2008). In particular, in Remark 2 we show that, under the proposed setting, the main vertex cardinality result in Alamo et al. (2007, also in *Systems & Control Letters*, in press, 2008) can be re-derived and improved by an halving factor.

Notation. For a vector x, the i-th element is denoted by x_i. The element in i-th row and j-th column of a matrix X is denoted either by $[X]_{ij}$, or by X_{ij}. For $X \in \mathbb{R}^{n,m}$, the notation $X \leq 0$ (resp. $X < 0$) denotes element-wise non-strict (resp. strict) inequality. The notation $|X|$ denotes a matrix whose (i,j)-th element is $|X_{ij}|$. $\mathbb{S}^n$ denotes the subspace of symmetric $n \times n$ real matrices. For $X \in \mathbb{S}^n$, the notation $X \preceq 0$ (resp. $X \prec 0$) specifies that X is negative semi-definite (resp. negative definite). If $X_1, \ldots, X_k$ are matrices, the notation $\mathrm{diag}(X_1, \ldots, X_k)$ denotes a block-diagonal matrix having $X_1, \ldots, X_k$ as diagonal blocks. If $x \in \mathbb{R}^n$, the notation $\mathrm{diag}(x)$ denotes a diagonal matrix with the elements of x on the diagonal. The operator $\odot$ denotes the Hadamard (entry-wise) matrix product. The set of diagonal matrices of signs is defined as

$$\mathcal{S}^n \doteq \{\mathrm{diag}(s_1, \ldots, s_n) : s_i = \pm 1,\ i = 1, \ldots, n\}.$$

The cardinality of this set is $\mathrm{card}\,(\mathcal{S}^n) = 2^n$.

2. INTERVAL SDPS

Consider an uncertain linear matrix inequality restriction on variables $x_1, \ldots, x_m$

$$F(x) = F_0 + \sum_{k=1}^{m} x_k F_k \preceq 0 \qquad (1)$$

where $F_k \in \mathbb{S}^n$, $k = 0, 1, \ldots, m$ are symmetric interval coefficient matrices. Namely, we assume that

$$F_k = F_k(\Delta_k) = \bar{F}_k + \Delta_k, \quad k = 0, 1, \ldots, m$$

where $\bar{F}_k \in \mathbb{S}^n$ are given, and Δ_k are only known to belong to the interval sets

$$\mathcal{D}_k \doteq \{\Delta \in \mathbb{S}^n : |\Delta| \leq B_k\},$$

where $B_k \geq 0$ is a symmetric matrix containing the bounds on the entries of Δ.

A robust interval LMI is then defined as the following semi-infinite convex constraint

$$F_0(\Delta_0) + \sum_{k=1}^{m} x_k F_k(\Delta_k) \preceq 0, \quad \forall \Delta_k \in \mathcal{D}_k. \qquad (2)$$

In this paper, we treat the two robustness problems defined next.

Problem 1. (Robust feasibility of interval LMI).
Given $x \in \mathbb{R}^m$, check if (2) holds.

Problem 2. (Robust solution of interval SDP).
Given $c \in \mathbb{R}^m$, find $x \in \mathbb{R}^m$ such that $c^\top x$ is minimized subject to the constraints in (2).

These interval problems found many applications in different fields, ranging from numerical analysis and robust linear algebra to engineering design. As a simple motivating example, we next illustrate a problem arising in the context of topology optimization and vibration control of discrete (or discretized) structures.

Example 1. (Truss topology optimization). A classical problem in structural design is to determine the cross-sectional areas x_i of a truss structure so to minimize the total weight while guaranteeing a lower bound on the structure fundamental modal frequency. Formally, for a desired frequency $\Omega \geq 0$, one has to solve an optimization problem of the form (see, e.g., Ben-Tal and Nemirovski (1997); Vandenberghe and Boyd (1999))

$$\min_x V(x) \quad \text{subject to:} \qquad (3)$$
$$x_{\mathrm{lb}} \leq x \leq x_{\mathrm{up}}$$
$$M(x)\Omega^2 - K(x) \preceq 0, \qquad (4)$$

where the weight of the structure, $V(x)$, is a linear function of the design variable x, the matrices $K(x)$ and $M(x)$ represent respectively the stiffness and the mass matrix of the structure, and constraint (4) specifies that the fundamental modal frequency should be higher than Ω. Vectors x_{lb}, x_{ub} contain lower and upper bounds on the cross-sectional areas, respectively. The mass and stiffness matrices are affine functions of x, that is

$$K(x) = K_0 + \sum_{k=1}^{m} x_k K_k, \quad M(x) = M_0 + \sum_{k=1}^{m} x_k M_k$$

where $M_k, K_k,\ k = 0, \ldots, m$ are symmetric matrices. Since these matrices depend on geometric and material characteristics of the structure, it is natural to assume an interval uncertainty over them, in which case

$$K_k = \bar{K}_k + \Delta_k^K, \quad k = 0, \ldots, m$$
$$M_k = \bar{M}_k + \Delta_k^M, \quad k = 0, \ldots, m.$$

In this situation, the designer may be interested in determining a design that works best in the worst-case scenario, that is in considering a robust version of problem (3). It can be easily seen that the robust version of constraint (4) is readily rewritten as an interval LMI of the form (2) by letting

$$\bar{F}_k = \bar{M}_k\Omega^2 - \bar{K}_k, \quad \Delta_k = \Delta_k^M\Omega^2 - \Delta_k^K; \quad k = 0, \ldots, m.$$

2.1 Naive vertex solution approach

We remark that there exist a straightforward (but computationally inefficient) way to solve Problems 1 and 2. Indeed, it can be easily seen that every matrix in the set $\mathcal{D}_k$ can be written as a convex combination of vertex matrices belonging to the set

$$\mathcal{D}_k^v \doteq \{\Delta \in \mathbb{S}^n : |\Delta_{ij}| = [B_k]_{ij},\ 1 \le i \le j \le n\},$$

which has cardinality $2^{n(n+1)/2}$. Then, the following lemma can be proved by elementary convexity arguments.

Lemma 1. The semi-infinite LMI constraint (2) is equivalent to the following finite set of vertex LMI contraints

$$F_0(\Delta_0) + \sum_{k=1}^m x_k F_k(\Delta_k) \preceq 0, \quad \forall \Delta_k \in \mathcal{D}_k^v,\ k = 0, 1, \ldots, m. \tag{5}$$

Problem 2 is thus equivalent to the SDP

$$\min c^\top x \quad \text{subject to (5)}.$$

In the latter minimization problem, the infinite number of constraints of Problem 2 has been replaced by a finite number of vertex constraints. Notice however that this number can be very large already for very small n and m. For instance, for $n = m = 3$ the number of constraints is $16, 777, 216$, and it becomes 1.1259×10^{15} for $n = m = 4$. Hence, Lemma 1 has a mainly theoretical interest, and it can rarely be applied in practice. In the next section, in the spirit of Alamo et al. (2007, also in *Systems & Control Letters*, in press, 2008), we derive a results that shows how the number of vertices can be drastically reduced and made independent of m, thus leading to a more manageable solution approach.

2.2 A result with reduced vertex set

We first establish the following preliminary lemma, which is instrumental for proving the main result.

Lemma 2. Given $x \in \mathbb{R}^m$, the robust condition (2) is satisfied if and only if

$$v^\top \bar{F}(x)v + |v|^\top B(|x|)|v| \le 0, \quad \forall v \in \mathbb{R}^n,$$

where we defined

$$\bar{F}(x) \doteq \bar{F}_0 + \sum_{k=1}^m x_k \bar{F}_k \tag{6}$$

$$B(\xi) \doteq B_0 + \sum_{k=1}^m \xi_k B_k. \tag{7}$$

Proof. The proof of this result follows the same lines of reasoning of Alamo et al. (2007, also in *Systems & Control Letters*, in press, 2008) and Rohn (1994a). We see that robust interval LMI (2) is satisfied if and only if

$$v^\top F_0(\Delta_0)v + \sum_{k=1}^m x_k v^\top F_k(\Delta_k)v \le 0$$

holds for all $\Delta_k \in \mathcal{D}_k$ and all $v \in \mathbb{R}^n$, that is, if and only if

$$\max_{\Delta_0 \in \mathcal{D}_0} v^\top F_0(\Delta_0)v + \sum_{k=1}^m \max_{\Delta_k \in \mathcal{D}_k} x_k v^\top F_k(\Delta_k)v \le 0,\ \forall v \in \mathbb{R}^n.$$

Notice that

$$\begin{aligned} &\max_{\Delta_0 \in \mathcal{D}_0} v^\top F_0(\Delta_0)v + \sum_{k=1}^m \max_{\Delta_k \in \mathcal{D}_k} x_k v^\top F_k(\Delta_k)v \\ &= v^\top \bar{F}_0 v + \sum_{k=1}^m x_k v^\top \bar{F}_k v + \max_{\Delta_0 \in \mathcal{D}_0} v^\top \Delta_0 v + \sum_{k=1}^m \max_{\Delta_k \in \mathcal{D}_k} x_k v^\top \Delta_k v \\ &= v^\top \bar{F}(x)v + \max_{\Delta_0 \in \mathcal{D}_0} v^\top \Delta_0 v + \sum_{k=1}^m \max_{\Delta_k \in \mathcal{D}_k} x_k v^\top \Delta_k v. \end{aligned} \tag{8}$$

Considering the second term in the previous summation, we have that

$$\max_{\Delta_0 \in \mathcal{D}_0} v^\top \Delta_0 v = \max_{\Delta_0 \in \mathcal{D}_0} \left(\sum_{i=1}^n v_i^2 [\Delta_0]_{ii} + 2 \sum_{1 \le i \le j \le n} v_i v_j [\Delta_0]_{ij} \right).$$

The maximum in this expression is attained by choosing $[\Delta_0]_{ii} = [B_0]_{ii}$ and $[\Delta_0]_{ij} = \mathrm{sign}(v_i v_j)[B_0]_{ij}$, which yields

$$\max_{\Delta_0 \in \mathcal{D}_0} v^\top \Delta_0 v = \sum_{i=1}^n v_i^2 [B_0]_{ii} + 2 \sum_{1 \le i \le j \le n} |v_i v_j| [B_0]_{ij} = |v|^\top B_0 |v|.$$

Similarly, considering the third term in (8), we have

$$\max_{\Delta_k \in \mathcal{D}_k} x_k v^\top \Delta_k v = \max_{\Delta_k \in \mathcal{D}_k} \left(\sum_{i=1}^n x_k v_i^2 [\Delta_k]_{ii} + 2 \sum_{1 \le i \le j \le n} x_k v_i v_j [\Delta_k]_{ij} \right).$$

For given x_k, the maximum in this expression is attained by choosing $[\Delta_k]_{ii} = \mathrm{sign}(x_k)[B_k]_{ii}$ and $[\Delta_k]_{ij} = \mathrm{sign}(x_k v_i v_j)[B_k]_{ij}$, which yields

$$\max_{\Delta_k \in \mathcal{D}_k} x_k\, v^\top \Delta_k v = \sum_{i=1}^n |x_k| v_i^2 [B_k]_{ii} + 2 \sum_{1 \le i \le j \le n} |x_k| |v_i v_j| [B_k]_{ij} = |x_k| |v|^\top B_k |v|,$$

thus concluding the proof. □

We are now in position to state the following corollary, which provides a reduced vertex set solution for Problem 1.

Corollary 1. (Robust feasibility with reduced vertex set). Given $x \in \mathbb{R}^m$, the semi-infinite condition (2) is satisfied if and only if

$$\bar{F}(x) + SB(|x|)S \preceq 0, \quad S = \mathrm{diag}(1, \tilde{S}), \quad \forall \tilde{S} \in \mathcal{S}^{n-1}, \tag{9}$$

where $\bar{F}(x)$ and $B(|x|)$ are defined in (6), (7), and $\mathcal{S}^{n-1}$ is the set of $(n-1) \times (n-1)$ diagonal matrices of signs. Condition (9) consists of a finite number 2^{n-1} of vertex conditions.

Proof. Suppose first that (2) holds, that is

$$\bar{F}(x) + \Delta_0 + \sum_{k=1}^m x_k \Delta_k \preceq 0,\ \forall \Delta_k \in \mathcal{D}_k,\ k = 0, 1, \ldots, m.$$

Then, in particular for any $\tilde{S} \in \mathcal{S}^{n-1}$, let $S = \mathrm{diag}(1, \tilde{S})$, and choose

$$\bar{\Delta}_0 = SB_0S \in \mathcal{D}_0$$
$$\bar{\Delta}_k = \mathrm{sign}(x_k)SB_kS \in \mathcal{D}_k, \quad k = 1, \dots, m.$$

Then, it must hold that $\bar{F}(x) + SB_0S + \sum_{k=1}^{m} |x_k| SB_kS \preceq 0$, which proves the first implication.

Conversely, suppose that (9) holds, and notice that $SB(|x|)S = (-S)B(|x|)(-S)$. This implies that the condition in (9) actually holds for all $S \in \mathcal{S}^n$. Therefore, for all $v \in \mathbb{R}^n$ it holds that

$$v^\top \bar{F}(x)v + v^\top SB(|x|)Sv \le 0, \quad \forall S \in \mathcal{S}^n.$$

Hence, choosing $S = \mathrm{diag}(\mathrm{sign}(v_1), \dots, \mathrm{sign}(v_n))$, it must hold that

$$v^\top \bar{F}(x)v + |v|^\top B(|x|)|v| \le 0, \quad \forall v \in \mathbb{R}^n.$$

By Lemma 2, this latter condition implies satisfaction of (2), thus concluding the proof. □

The result given in Corollary 1 is useful for robustness analysis (Problem 1). Notice that in the special case of $m = 0$ (no design variables) the vertex matrices in (9) coincide with those required in Theorem 2 of Rohn (1994a) for checking negative-semidefiniteness of symmetric interval matrices.

We next state the main result of the paper, which provides a reduced vertex set condition for the solution of the robust design problem (Problem 2).

Theorem 1. (Robust optimization w. reduced vertex set). The robust interval SDP problem (Problem 2) is equivalent to the following SDP in the variables $x, \xi \in \mathbb{R}^m$

$$\min c^\top x \ \text{ subject to}$$
$$\bar{F}(x) + SB(\xi)S \preceq 0, \quad S = \mathrm{diag}(1, \tilde{S}),\ \forall \tilde{S} \in \mathcal{S}^{n-1}; \tag{10}$$
$$x_k \le \xi_k, \quad k = 1, \dots, m \tag{11}$$
$$-x_k \le \xi_k, \quad k = 1, \dots, m. \tag{12}$$

Proof. We show that the robust LMI conditions (2) and (10)–(12) are equivalent. Suppose first that (2) holds for some $x \in \mathbb{R}^m$. Then, (9) holds by Corollary 1, hence (10)–(12) hold by taking $\xi_k = |x_k|$.

Conversely, suppose that (10)–(12) hold, and notice that (11), (12) imply $|x_k| \le \xi_k$. Since $SB(\xi)S = (-S)B(\xi)(-S)$, condition (10) implies $\bar{F}(x) + SB(\xi)S \preceq 0$ for all $S \in \mathcal{S}^n$. Hence, for all $S \in \mathcal{S}^n$

$$v^\top \bar{F}(x)v + v^\top SB_0Sv + \sum_{k=1}^{m} \xi_k v^\top SB_kSv \le 0, \quad \forall v \in \mathbb{R}^n.$$

In particular, for $S = \mathrm{diag}(\mathrm{sign}(v_1), \dots, \mathrm{sign}(v_n))$ we have

$$v^\top \bar{F}(x)v + |v|^\top B_0|v| + \sum_{k=1}^{m} \xi_k |v|^\top B_k|v| \le 0, \quad \forall v \in \mathbb{R}^n.$$

Now, since $B_k \ge 0$, then $|v|^\top B_k|v| \ge 0$. Therefore $|x_k| \le \xi_k$ implies that $\xi_k |v|^\top B_k|v| \ge |x_k||v|^\top B_k|v|$, thus for all $v \in \mathbb{R}^n$

$$v^\top \bar{F}(x)v + |v|^\top B_0|v| + \sum_{k=1}^{m} |x_k||v|^\top B_k|v| \le$$
$$v^\top \bar{F}(x)v + |v|^\top B_0|v| + \sum_{k=1}^{m} \xi_k |v|^\top B_k|v| \le 0.$$

This implies satisfaction of (2), by Lemma 2, which concludes the proof. □

Remark 1. (Vertex complexity). Theorem 1 shows that the infinite set of constraints in Problem 2 can be substituted by an equivalent finite set of of 2^{n-1} vertex LMIs. The number of vertices is thus independent of m and it is drastically reduced with respect to the case considered in Section 2.1. ◊

Looking more closely to condition (10) in Theorem 1, we see that when $B(\xi)$ and S commute (i.e. when $B(\xi)$ is diagonal) then, since $SS = I$, equation (10) reduces to the single LMI constraint $\bar{F}(x) + B(\xi) \preceq 0$. This situation happens for instance in the special case of interval linear programs, as briefly illustrated in the next example.

Example 2. (A special case: interval linear programs). Consider a standard linear programming problem (LP)

$$\min c^\top x \ \text{ subject to } Ax - b \le 0$$

with $A \in \mathbb{R}^{n,m}$ and $b \in \mathbb{R}^n$. It is straightforward to verify that the linear constraints in this problem can be cast in an equivalent LMI format (1) by taking

$$F_0 = \mathrm{diag}(-b); \quad F_k = \mathrm{diag}(a_k),\ k = 1, \dots, m,$$

where a_k denotes the k-th column of A. Now, if the entries of A and b are assumed to lie in independent intervals, we may write

$$A = A(\Delta) = \bar{A} + \Delta \odot R, \quad b = b(\delta) = \bar{b} + \delta \odot d,$$

where $\bar{A} \in \mathbb{R}^{n,m}$, $\bar{b} \in \mathbb{R}^n$ are the nominal matrices, $R \in \mathbb{R}^{n,m}$, $d \in \mathbb{R}^n$ are given positive matrices containing the interval limits, and $\Delta \in \mathbb{R}^{n,m}$, $\delta \in \mathbb{R}^n$ are the uncertainties, which are subject to $|\Delta| \le 1$, $|\delta| \le 1$. The robust interval LP problem

$$\min\ c^\top x \ \text{ subject to}$$
$$A(\Delta)x - b(\delta) \le 0, \quad \forall \Delta, \delta : |\Delta| \le 1,\ |\delta| \le 1$$

can therefore be equivalently rewritten in the format of a robust interval SDP, with diagonal nominal coefficient matrices $\bar{F}_0 = \mathrm{diag}(-\bar{b})$, $\bar{F}_k = \mathrm{diag}(\bar{a}_k)$ and diagonal bound matrices $B_0 = \mathrm{diag}(d)$, $B_k = \mathrm{diag}(r_k)$, $k = 1, \dots, m$, where $\bar{a}_k$, r_k are the k-th column of $\bar{A}$ and of R, respectively.

Let us now apply Theorem 1 to this interval SDP. Since $S \in \mathcal{S}^n$ and B_i are diagonal and $SS = I$, the S terms disappear in equation (10). It follows that problem (13) is equivalent to

$$\min c^\top x \,\text{subject to}$$
$$\bar{F}_0 + \sum_{k=1}^{m} x_k \bar{F}_k + B(\xi) \preceq 0$$
$$x_k \le \xi_k, \quad k = 1, \dots, m$$
$$-x_k \le \xi_k, \quad k = 1, \dots, m,$$

where $\xi^\top = [\xi_1 \cdots \xi_m]$ is a vector of slack variables. Converting this diagonally structured SDP back to standard LP format, we finally obtain that (13) is equivalent to the following standard linear program in variables x and ξ:

$$\begin{aligned} \min c^\top x \, & \text{subject to} \\ & \bar{A}x - b + R\xi + d \leq 0 \\ & x_k \leq \xi_k, \quad k = 1, \ldots, m \\ & -x_k \leq \xi_k, \quad k = 1, \ldots, m. \end{aligned}$$

This simple result can be of independent interest and improves upon a previous solution approach proposed in Chinneck and Ramadan (2000), which required exponential enumeration. Robust linear programs with more general uncertainty structures are studied in Ben-Tal and Nemirovski (1999). $\star$

3. SPDS WITH LINEARLY TRANSFORMED INTERVAL UNCERTAINTY

The main result treated in Section 2.2 refers to the situation when all entries of the LMI coefficient matrices are affected by independent interval uncertainty. In some specific applications, such as robust control, interval LMIs arise where the uncertainties are *not* independent. However, reduced vertex set results can be obtained also in these cases, as recently shown in Alamo et al. (2007, also in *Systems & Control Letters*, in press, 2008).

Here, we examine one of such non-independent uncertainty situations, where the LMI constraints are additively perturbed by a linear function of an interval matrix Δ. Namely, we consider the uncertain LMI constraint

$$F(x) = \bar{F}(x) + L\Delta R(x) + R^\top(x)\Delta^\top L^\top \preceq 0, \qquad (13)$$

where $\bar{F}(x)$ is an $n \times n$ symmetric affine matrix function of $x \in \mathbb{R}^m$, $R(x)$ is a $q \times n$ affine matrix function of x, $L \in \mathbb{R}^{n,p}$ is a given matrix, and Δ is a $p \times q$ interval matrix, i.e. $\Delta \in \mathcal{D}^{p,q}$, where

$$\mathcal{D}^{p,q} \doteq \{\Delta \in \mathbb{R}^{p,q} : |\Delta| \leq B\}$$

being $B \in \mathbb{R}^{p,q}$ a given nonnegative matrix representing the entry-wise bounds on the absolute values of the elements of Δ.

The uncertain LMI representation (13) is a special case of the classical linear fractional transformation (LFT; see, e.g., Section 2.2 of El Ghaoui et al. (1998)), which frequently arises in robust control applications, see for instance Zhou et al. (1996). The following theorem holds, see Calafiore and Dabbene (2008, in press) for a proof.

Theorem 2. A vector $x \in \mathbb{R}^m$ satisfies the robust LMI condition

$$\bar{F}(x) + L\Delta R(x) + R^\top(x)\Delta^\top L^\top \preceq 0, \quad \forall \Delta \in \mathcal{D}^{p,q} \quad (14)$$

if and only if it satisfies

$$\bar{F}(x) + LS_L B S_R R(x) + R^\top(x) S_R B^\top S_L L^\top \preceq 0, \quad (15)$$
$$S_L = \mathrm{diag}(1, \tilde{S}_L), \quad \forall \tilde{S}_L \in \mathcal{S}^{p-1}, \ \forall S_R \in \mathcal{S}^q.$$

Condition (15) consists of a finite number 2^{p+q-1} of vertex LMI constraints.

Remark 2. Theorem 2 and its proof are closely related to the main result in Alamo et al. (2007, also in *Systems & Control Letters*, in press, 2008). In particular, in Alamo et al. (2007, also in *Systems & Control Letters*, in press, 2008) the authors consider a problem arising in a robust control setting which, restated in the notation of this paper, takes the following form:

$$\bar{F}(x) + \Delta_a + \Delta_a^\top + \Delta_b Q(x) + Q^\top(x)\Delta_b \preceq 0,$$
$$\forall \Delta_a \in \mathcal{D}^{n,n}, \Delta_b \in \mathcal{D}^{n,m}. \quad (16)$$

It is proved in Alamo et al. (2007, also in *Systems & Control Letters*, in press, 2008) that this condition is equivalent to 2^{m+n} conditions on specific vertex matrices. We next show that (16) is a special case of LMI (14), and that Theorem 2 can be modified and specialized to this case, thus providing a vertex cardinality result that improves by an halving factor the 2^{m+n} vertex set cardinality result of Alamo et al. (2007, also in *Systems & Control Letters*, in press, 2008). To this end, notice that (16) can be written in the form of (14), by taking

$$\Delta = [\Delta_a \ \Delta_b] \in \mathbb{R}^{n,n+m}, \quad L = I_n, \quad R(x) = \begin{bmatrix} I_n \\ Q(x) \end{bmatrix}.$$

Then, it can be easily verified that all steps in the proof of Theorem 2 would go through with $S_L = \mathrm{diag}(1, \tilde{S}_L)$, $\tilde{S}_L \in \mathcal{S}^{n-1}$, and with S_R taking the specific block structure $S_R = \mathrm{diag}(S_L, S_Q)$, with $S_Q \in \mathcal{S}^m$. The resulting condition of type (15) would thus involve only 2^{m+n-1} vertex constraints:

$$\bar{F}(x) + S_L B_a S_L + S_L B_b S_Q Q(x) \preceq 0,$$
$$S_L = \mathrm{diag}(1, \tilde{S}_L), \quad \forall \tilde{S}_L \in \mathcal{S}^{n-1}, \ S_Q \in \mathcal{S}^m,$$

where $[B_a \ \ B_b]$ is the matrix of bounds for the interval matrix $\Delta = [\Delta_a \ \Delta_b]$. $\Diamond$

Finally, we provide a result which is the analog of Theorem 3 in Alamo et al. (2007, also in *Systems & Control Letters*, in press, 2008). This result gives an efficiently computable sufficient condition for satisfaction of (14) and it is reported in Corollary 2, see Calafiore and Dabbene (2008, in press) for a proof.

Corollary 2. If $x \in \mathbb{R}^m$, $\Theta = \mathrm{diag}(\theta_1, \ldots, \theta_q) \succ 0$, $T = \mathrm{diag}(t_1, \ldots, t_p)$ satisfy the LMIs

$$\begin{bmatrix} \bar{F}(x) + LTL^\top & R^\top(x) \\ R(x) & -\Theta \end{bmatrix} \prec 0 \qquad (17)$$

$$B\Theta B^\top \prec T \qquad (18)$$

then x satisfies (14).

4. NUMERICAL EXAMPLE

We revisit a problem originally considered in Fan and Nekooie (1992), dealing with the minimization of the largest eigenvalue of an affine combination of symmetric matrices. Namely, in Fan and Nekooie (1992) the following problem is considered:

$$\min_{x \in \mathbb{R}^5} \lambda_{\max} \left(\bar{A}_0 + \sum_{i=1}^{5} x_i \bar{A}_i \right) \qquad (19)$$

where $\lambda_{\max}$ denotes the largest eigenvalue of a symmetric matrix, and $\bar{A}_0, \ldots, \bar{A}_5$ are symmetric matrices, whose numerical value is given in Fan and Nekooie (1992).

Problem (19) can be recast in SDP form as

$$\min_{x\in\mathbb{R}^5,\lambda\in\mathbb{R}} \lambda \text{ subject to: } \bar{A}_0 + \sum_{i=1}^{5} x_i \bar{A}_i - \lambda I \preceq 0,$$

which has the optimal solution $\lambda_{\min} = 0.70888$.

Assume now that the matrices $\bar{A}_1, \ldots, \bar{A}_5$ represent *nominal* values, while the actual entries are only known to lie in independent intervals centered around the nominal values, with width equal to ρ percent of the nominal, that is

$$A_k = \bar{A}_k + \Delta_k, \quad |\Delta_k| \le \rho|\bar{A}_k|; \quad k = 1, \ldots, 5.$$

In this situation, the problem becomes that of minimizing the worst-case largest eigenvalue of the interval matrix family, that is:

$$\min_{x\in\mathbb{R}^5,\lambda\in\mathbb{R}} \lambda \text{ subject to:}$$

$$\bar{A}_0 + \sum_{i=1}^{5} x_i \bar{A}_i + \sum_{i=1}^{5} x_i \Delta_i - \lambda I \preceq 0,$$
$$\forall\, |\Delta_k| \le \rho|\bar{A}_k|,\ k = 1, \ldots, 5.$$

This problem is an interval SDP of the form (2).

Determining a robust solution for this interval SDP using a naive vertex approach would require considering $2^{50} \approx 10^{15}$ vertices. Application of Theorem 1 require instead only 16 vertices. Hence, solving this problem for increasing values of ρ ranging in the interval $[0, 1]$ (which corresponds to uncertainty level from 0% to 100%), we obtained the plot shown in Figure 1. This numerical example was coded in Matlab using the YALMIP Lofberg (2004) interface and the SeDuMi SDP solver. The numerical solution of the problem for each fixed value of ρ required about 0.13 seconds on an AMD Dual Opteron workstation.

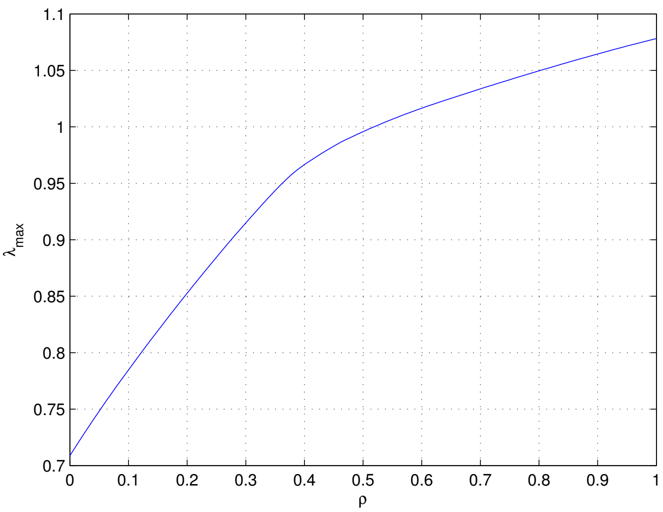

Fig. 1. Worst-case largest eigenvalue as a function of ρ.

5. CONCLUSIONS

Robust solutions to uncertain SDP problems with interval coefficient matrices require considering an exponential number of vertex constraints. We have shown in this paper that when all entries of the LMI coefficient matrices lie in independent intervals, the number of vertices to be considered in the optimization is 2^{n-1}, being n the size of the LMI. Interval linear programs are a special case of the considered class of problems, and can be solved efficiently without resorting to vertexization. When the LMI constraint is expressed as a linear transformation of a $p \times q$ matrix of uncertain coefficients, the required number of vertices becomes 2^{p+q-1}.

REFERENCES

T. Alamo, R. Tempo, D.R. Ramirez, and E.F. Camacho. A new vertex result for robustness problems with interval matrix uncertainty. In *Proceedings of the European Control Conference*, 2007, also in *Systems & Control Letters*, in press, 2008.

A. Ben-Tal and A. Nemirovski. On tractable approximations of uncertain linear matrix inequalities affected by interval uncertainty. *SIAM Journal on Optimization*, 12 (3):811–833, 2002.

A. Ben-Tal and A. Nemirovski. Robust truss topology design via semidefinite programming. *SIAM Journal on Optimization*, 7(4):991–1016, 1997.

A. Ben-Tal and A. Nemirovski. Robust convex optimization. *Math. Op. Res.*, 23:769–805, 1998.

A. Ben-Tal and A. Nemirovski. Robust solutions of uncertain linear programs. *Operations Research Letters*, 25(1):1–13, 1999.

V.D. Blondel and J.N. Tsitsiklis. A survey of computational complexity results in systems and control. *Automatica*, 36:1249–1274, 2000.

S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory.* SIAM, Philadelphia, 1994.

G. Calafiore and F. Dabbene. A reduced vertex set result for interval semidefinite optimization probelms. *Journal of Optimization Theory and Applications*, 2008, in press.

J.W. Chinneck and K. Ramadan. Linear programming with interval coefficients. *Journal of the Operational Research Society*, 51:209–220, 2000.

L. El Ghaoui, F. Oustry, and H. Lebret. Robust solutions to uncertain semidefinite programs. *SIAM Journal on Optimization*, 9:33–52, 1998.

M.K. Fan and B. Nekooie. On minimizing the largest eigenvalue of a symmetric matrix. In *Proceedings of the IEEE Conference on Decision and Control*, 1992.

J. Lofberg. Yalmip: A toolbox for modeling and optimization in MATLAB. In *Proceedings of the CACSD Conference*, 2004.

A. Nemirovski. Several NP-hard problems arising in robust stability analysis. *Mathematics of Control, Signals, and Systems*, 6:99–195, 1993.

Y. Nesterov and A.S. Nemirovski. *Interior Point Polynomial Algorithms in Convex Programming.* SIAM Journal on Applied Mathematics, Philadelphia, 1994.

J. Rohn. Positive definiteness and stability of interval matrices. *SIAM Journal on Matrix Analysis and Applications*, 15:175–184, 1994a.

J. Rohn. Checking positive definiteness or stability of symmetric interval matrices is NP-hard. *Comment. Math. Univ. Carolinæ*, 35(4):785–797, 1994b.

L. Vandenberghe and S. Boyd. Semidefinite programming. *SIAM Review*, 38:49–95, 1996.

L. Vandenberghe and S. Boyd. Applications of semidefinite programming. *App. Num. Math.*, 29(3):283–299, 1999.

K. Zhou, J.C. Doyle, and K. Glover. *Robust and Optimal Control.* Prentice-Hall, Upper Saddle River, 1996.