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A Notion of Zero Dynamics for Linear, Time-delay System

G. Conte A. M. Perdon

DIIGA, Università Politecnica delle Marche, via Brecce Bianche, 60131 Ancona, Italy (e-mail: gconte@univpm.it - perdon@univpm.it)

AbstractThe aim of this paper is to introduce a notion of zero dynamics for linear, time-delay systems. To this aim, we use the correspondence between time-delay systems and system with coefficients in a ring, so to exploit algebraic and geometric methods. By combining the algebraic notion of zero module and the geometric structure of the lattice of invariant submodules of the state module, we point out a natural way to define zero dynamics in particular situations. Then, we extend our approach, in order to encompass more general situations, and we provide a definition of zero dynamics which applies to a number of interesting cases, including that of time-delay systems with commensurable delays. Relations between this notion and fixed poles in closed loop control schemes, as well as with a concept of phase minimality, are discussed.

1. INTRODUCTION

The notion of zero of a linear, dynamical system has been investigated and studied by several authors from many different points of view (see Schrader and Sain [1989] for a comprehensive discussion of the literature). Among others, the approach based on the notion of Zero Module, introduced in Wyman and Sain [1981] and recalled below provides conceptual and practical tools that, besides being useful in the analysis and synthesis of classical linear systems, can be effectively generalized to a larger class of dynamical systems. In particular, an algebraic notion of zero in terms of zero module has been given in Conte and Perdon [1984] for linear, dynamical system with coefficients in a ring, instead of a field.

By exploiting the possibility to associate to any linear, time-delay system a system with coefficients in a suitable ring, the algebraic notion of zero introduced in the ring framework can be employed for defining a notion of zeros and of zero dynamics for time-delay systems. This idea has been partially developed in Conte and Perdon [2007], where some properties of zeros for time-delay systems with commensurable delays and their role in inversion and matching problems for the SISO case have been studied.

Here, extending previous results of Conte and Perdon [1984] and Conte and Perdon [2007], we consider the notion of zeros for time-delay systems from a geometric point of view. More precisely, we analyze the relation existing between the zero module and the quotient module of two controlled invariant submodules, endowed with the structure induced by a closed loop dynamics. This relation motivates the introduction of a notion of zero dynamics for time-delay systems in the general case of non commensurable delays. In addition, it is shown that the zero dynamics so defined determines the dynamics that remains fixed by closing (dynamic) feedback loops which make the system maximally unobservable.

The paper is organized as follows. In Section 2, we recall basic notions concerning system with coefficients in a ring and the related notion of Zero Module. In Section 3, we describe briefly the relation between time-delay systems and systems with coefficients in a ring and we introduce, on that basis, the algebraic notion of zero, in terms of zero module, for time-delay-systems. In Section 4, we recall some basic notions of the geometric approach and we give a meaningful characterization of the zero module in terms of controlled invariant submodules. This characterization suggests a possible notion of zero dynamics, which applies to the case of time-delay systems and which is formally defined in Section 5. Finally, in Section 6 we discuss the role of the zero dynamics so introduced in closed feedback loops, also in relation to stability and minimality of phase.

2. PRELIMINARY RESULTS

Let R denote a commutative ring. By a system with coefficients in R, or a system over R, we mean a linear dynamical system $\Sigma = (A, B, C, \mathcal{X})$ whose evolution is described by a set of difference equations of the form

$$\begin{cases} x(t+1) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases} \quad (1)$$

where $t \in I\!N$ is an independents variable, $x(\cdot)$ belongs to the free module $\mathcal{X} = R^n$, $u(\cdot)$ belongs to the free module $\mathcal{U} = R^m$, $y(\cdot)$ belongs to the free module $\mathcal{Y} = R^p$ and A, B, C are matrices of suitable dimensions with entries in R. By analogy with the classical case of linear, dynamical, discrete-time systems with coefficients in the field of real number $I\!R$, we view the variables x, u and y as, respectively, the state, input and output of Σ.

Besides being interesting abstract algebraic objects, systems with coefficient in a ring have been proved to be useful for modeling and studying particular classes of dynamical systems, such as discrete-time systems with integer coefficients, families of parameter dependent systems and time-delay systems. General results concerning the theory of systems with coefficients in a ring and a number of related control problems can be found in Sontag [1976], Sontag [1981], Brewer et al. [1986], Kamen [1991], Conte and Perdon [2000a] and the references therein. In particular, a geometric theory, similar to the existing

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one for linear systems over a field (see Basile and Marro [1992], Wohnam [1985]), has been developed and related concepts, like the notion of controlled invariance and that of conditioned invariance, maintain their relevance in the solution of several design problems in the ring framework (see Conte and Perdon [2000a]).

In the following, we will generally assume that the considered rings are Noetherian rings, that is rings in which non decreasing chains of ideals are stationary, having no zero divisors. Examples of rings of that kind are the rings of polynomials in one or several variables with real coefficients, that is $\mathbb{R}[\Delta_1, ..., \Delta_k]$, $k \geq 1$, which play a basic role in dealing with time-delay systems. For $k = 1$, $\mathbb{R}[\Delta]$ is also a principal ideal domain (p.i.d), that is a ring in which any ideal has a single generator (see e.g. Lang [1984]).

Introducing the ring $R[z]$ of polynomials in the indeterminate z with coefficients in R and its localization $R(z) = S^{-1}R[z]$ at the multiplicatively closed set S of all monic polynomials (that is the ring of all rational functions in the indeterminate z with monic denominator), we can associate to the system Σ its transfer function matrix $G(z) = C(zI-A)^{-1}B$, whose elements are in $R(z)$, and the induced $R(z)$-morphism $G : \mathcal{U} \otimes R(z) \to \mathcal{Y} \otimes R(z)$. Each element $u(z)$ of $\mathcal{U} \otimes R(z)$ can be written as $u(z) = \sum_{t=t_0}^{\infty} u_t z^{-t}$, with $u_t \in \mathcal{U}$, and it can be naturally interpreted as a time sequence, from some time t_0 to ∞, of inputs. Respectively, each element $y(z)$ of $\mathcal{Y} \otimes R(z)$ can be written as $y(z) = \sum_{t=t_0}^{\infty} y_t z^{-t}$, with $y_t \in \mathcal{Y}$, and it can be naturally interpreted as a time sequence, from some time t_0 to ∞, of outputs. Therefore, G can be interpreted as a transfer function between the space of input sequences and the space of output sequences.

From the point of view we adopt here, following Wyman and Sain [1981], the zeros of Σ are determined by the transfer function G in an abstract algebraic way. To this aim, let us recall that the $R[z]$-modules $\mathcal{U} \otimes R[z]$ and $\mathcal{Y} \otimes R[z]$, usually denoted by $\Omega\mathcal{U}$ and by $\Omega\mathcal{Y}$, are naturally embedded into $\mathcal{U} \otimes R(z)$ and into $\mathcal{Y} \otimes R(z)$, respectively. Then, as in Conte and Perdon [1984], we can extend to the framework of systems with coefficients in a ring the definition of Zero Module introduced in Wyman and Sain [1981].

Definition 1. (see Conte and Perdon [1984] Definition 2.1 and compare with Wyman and Sain [1981]) Given the system $\Sigma = (A, B, C, X)$ with coefficients in the ring R and transfer function G, the Zero Module of Σ is the $R[z]$-module $\mathcal{Z}$ defined by

$$\mathcal{Z} = \frac{G^{-1}(z)(\Omega Y) + \Omega U}{KerG(z) + \Omega U}. \tag{2}$$

The reader is referred to Conte and Perdon [1984] and Wyman and Sain [1981] for a discussion of the above definition and for a description of $\mathcal{Z}$ in terms of generator of zeroing signals. Here, we recall that the zero module is related to the numerator matrix in polynomial matrix factorizations of $G(z)$.

Proposition 1. (Conte and Perdon [1984] Proposition 2.5) Let $G(z) = D^{-1}N$ be a factorization where $D = D(z)$ and $N = N(z)$ are coprime polynomial matrices of suitable dimensions, with $D(z)$ invertible over $R(z)$. Then, the canonical projection $p_N : \Omega\mathcal{Y} \to \Omega\mathcal{Y}/N\Omega\mathcal{U}$ induces an injective $R[z]$-homomorphism $\alpha : \mathcal{Z} \to Tor(\Omega\mathcal{Y}/N\Omega\mathcal{U})$. When R is a field, α, as shown in Wyman and Sain [1981], is actually an isomorphism. In the ring case, additional conditions are required in order to assure this (see Conte and Perdon [1984], Section 2).

An important property of the zero module $\mathcal{Z}$ of Σ is that it is a finitely generated R-module (see Conte and Perdon [1984] Proposition 2.4).

The relations between zero modules and inverse systems in the ring framework has been investigated in Conte and Perdon [1984] and part of the results found there have been extended to the case of SISO time-delay systems in Conte and Perdon [2007]. The reader is referred to those papers for comments and examples. Here, we simply recall that the zero module of a left (right) invertible system Σ is, in a natural way, a submodule (a quotient module) of the state module of any left (right) inverse.

3. TIME-DELAY SYSTEMS AND SYSTEMS OVER RINGS

Let us consider a linear, time-invariant, time-delay system Σ_d with non commensurable delays $h_1, \ldots, h_k$, $h_i \in \mathbb{R}^+$, for $i = 1, \ldots, k$, described by equations of the the form

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{k}\sum_{j=0}^{a} A_{ij}x(t-jh_i) + \sum_{i=1}^{k}\sum_{j=0}^{b} B_{ij}u(t-jh_i) \\ y(t) = \sum_{i=1}^{k}\sum_{j=0}^{c} C_i x(t-jh_i) \end{cases} \tag{3}$$

where A_{ij}, B_{ij}, and C_{ij} are matrices of suitable dimensions with entries in the field of real number $\mathbb{R}$.

In the last years, a great research effort has been devoted to the development of analysis and synthesis techniques for this kind of systems, mainly extending tools and methods from the framework of classical linear systems (see e.g. the Proceedings of the IFAC Workshops on Linear Time Delay Systems from 1998 to 2006). Many of the difficulties in dealing with systems of the form (3) is due to the fact that their state space has infinite dimension. In order to circumvent this, it is useful to associate to a time-delay system a suitable system with coefficients in a ring, as described in the following.

For any delay h_j, let us introduce the delay operator δ_j defined, for any time function $f(t)$, by $\delta_j f(t) = f(t-h_j)$. Accordingly, we can re-write the system (3) as

$$\Sigma_d = \begin{cases} \dot{x}(t) = \sum_{i=1}^{a}\sum_{j=0}^{k} A_{ij}\delta_j^i x(t) + \sum_{i=1}^{b}\sum_{j=0}^{k} B_{ij}\delta_j^i u(t) \\ y(t) = \sum_{i=1}^{c}\sum_{j=0}^{k} C_{ij}\delta_j^i x(t) \end{cases}$$

Now, by formally replacing the delay operators δ_j by the algebraic unknowns Δ_j, it is possible to associate to Σ_d the discrete-time system Σ over the ring $R = \mathbb{R}[\Delta_1, ..., \Delta_k]$ defined by equations of the form (1) where the matrices A, B, C are given by $A = \sum_{i=1}^{a}\sum_{j=0}^{k} A_{ij}\Delta_j^i$, $B = \sum_{i=1}^{b}\sum_{j=0}^{k} B_{ij}\Delta_j^i$, $C = \sum_{i=1}^{c}\sum_{j=0}^{k} C_{ij}\Delta_j^i$. Actually, the time-delay system Σ_d and the associated system Σ over $R = \mathbb{R}[\Delta_1, ..., \Delta_k]$ are quite different objects from a dynamical point of view, but they share the structural properties that depend on the defining matrices.

Therefore, control problems concerning the input/output behavior of Σ_d can be naturally formulated in terms of the input/output behavior of Σ, transferring them from the time-delay framework to the ring framework. Since systems with coefficients in a ring have finite dimensional state modules, algebraic methods, similar to those of linear algebra, as well as geometric methods apply. Solutions to specific problems found in the ring framework often can be interpreted in the time-delay framework for solving the original problem (see Conte and Perdon [2000a], Conte and Perdon [2005] and the references therein).

Here, we use the correspondence between time-delay systems and systems over rings to derive a notion of zero modules for the first ones.

Definition 2. Given a time-delay system Σ_d of the form (3), the Zero Module of Σ_d is the zero module $\mathcal{Z}$ of the associated system Σ over the ring $R = I\!R[\Delta_1, ..., \Delta_k]$.

4. GEOMETRIC TOOLS

Extension of the geometric theory developed in Basile and Marro [1992] and in Wohnam [1985] to systems over rings has been considered by many authors (see Conte and Perdon [2000a] for an account of the geometric approach to systems with coefficients in a ring). The basic notion of the geometric approach we will need in the following are briefly recalled below.

Definition 3. (Hautus [1982]) Given a system Σ, defined over a ring R by equations of the form (1), a submodule $\mathcal{V}$ of its state module $\mathcal{X}$ is said to be

i) (A, B)-invariant, or controlled invariant, if and only if $A\mathcal{V} \subseteq \mathcal{V} + ImB$;
ii) (A, B)-invariant of feedback type if and only if there exists an R-linear map $F : \mathcal{X} \to \mathcal{U}$ such that $(A + BF)\mathcal{V} \subseteq \mathcal{V}$.

Any feedback F as in ii) above is called a friend of $\mathcal{V}$.

While (A, B)-invariance is a purely geometric property, controlled invariance is a notion related to system dynamics which is equivalent to invariance with respect to a closed loop dynamics. For systems with coefficients in a ring, an (A, B)-invariant submodule $\mathcal{V}$ is not necessarily of feedback type and therefore it cannot always be made invariant with respect to a closed loop dynamics, as it happens in the special case of systems with coefficients in the field of real numbers $I\!R$. Equivalence between the (generally weaker) geometric notion of (A, B)-invariance and the (generally stronger) dynamic notion of feedback type invariance holds if $\mathcal{V}$ is a direct summand of $\mathcal{X}$, that is $\mathcal{X} = \mathcal{V} \oplus \mathcal{W}$ for some submodule $\mathcal{W}$ (see Conte and Perdon [1998]).

Given a submodule $\mathcal{K} \subseteq \mathcal{X}$, there exists a maximum (A, B)-invariant submodule of $\mathcal{X}$ contained in $\mathcal{K}$, denoted by $\mathcal{V}^*(\mathcal{K})$, but there may not be a maximum (A, B)-invariant submodule of feedback type contained in $\mathcal{K}$.

The computation of $\mathcal{V}^*(\mathcal{K})$ is not difficult for systems with coefficients in the field of real numbers $I\!R$, since $\mathcal{V}^*(\mathcal{K})$ coincides with the limit of the sequence $\{\mathcal{V}_k\}$ defined by

$$\begin{aligned} \mathcal{V}_0 &= \mathcal{K} \\ \mathcal{V}_{k+1} &= \mathcal{K} \cap A^{-1}(\mathcal{V}_k + ImB) \end{aligned} \tag{4}$$

and the limit itself is reached in a number of steps lesser than or equal to the dimension of the state space. For systems with coefficients in a ring, the sequence (4), which is non-increasing, may not converge in a finite number of steps and, in such case, an algorithm for computing $\mathcal{V}^*(\mathcal{K})$ is in general not available. In case R is a principal ideal domain, however, using a different characterization, the problem of computing $\mathcal{V}^*(\mathcal{K})$ has been satisfactorily solved (see Assan et al. [1999b]).

Together with the notion of controlled invariance, it is useful to consider the following one.

Definition 4. Given a system Σ, defined over a ring R by equations of the form (1), a submodule $\mathcal{S}$ of its state module $\mathcal{X}$ is said to be

i) (A, C)-invariant, or conditioned invariant, if and only if $A(\mathcal{S} \cap KerC) \subseteq \mathcal{S}$;
ii) injection invariant if and only if there exists an R-linear map $G : \mathcal{Y} \to \mathcal{X}$ such that $(A + CG)\mathcal{S} \subseteq \mathcal{S}$.

Any output injection G as in ii) above is called a friend of $\mathcal{S}$.

In the ring framework, (A, C)-invariance is a weaker property than injection invariance. Given a submodule $\mathcal{K} \subseteq \mathcal{X}$, there exists a minimum (A, C)-invariant submodule of $\mathcal{X}$ containing $\mathcal{K}$, usually denoted by $\mathcal{S}^*(\mathcal{K})$, but there may not be a minimum injection invariant submodule containing $\mathcal{K}$. As in the field case, it is not difficult to show that, denoting simply by $\mathcal{V}^*$ the (A, B)-invariant submodule $\mathcal{V}^*(KerC)$ and by $\mathcal{S}^*$ the (A, C)-invariant submodule $\mathcal{S}^*(ImB)$, the submodule $\mathcal{R}^*$ defined by

$$\mathcal{R}^* = \mathcal{V}^* \cap \mathcal{S}^* \tag{5}$$

is the smallest (A, B)-invariant submodule of $\mathcal{V}^*$ containing $\mathcal{V}^* \cap ImB$. Moreover, if $\mathcal{V}^*$ is of feedback type with a friend F, also $\mathcal{R}^*$ is of feedback type and it has the same friends (see Basile and Marro [1992], Wohnam [1985]).

In dealing with geometric objects, it is useful to consider the following notion, first introduced in Conte and Perdon [1982].

Definition 5. Let $\mathcal{M} \subseteq \mathcal{N}$ be R-modules. The closure of $\mathcal{M}$ in $\mathcal{N}$, denoted by $CL_N(\mathcal{M})$ or simply $CL(\mathcal{M})$ if no confusion arises, is the R-module defined by

$$\begin{aligned} CL_N(\mathcal{M}) = \{x \in \mathcal{N},\ & \text{such that } ax \in \mathcal{M} \\ & \text{for some } a \neq 0,\ a \in R\}. \end{aligned} \tag{6}$$

If $\mathcal{M} = CL_N(\mathcal{M})$, $\mathcal{M}$ is said to be closed in $\mathcal{N}$.

The main result relating the zero module with the geometric objects we have introduced can now be stated.

Proposition 2. (compare with Conte and Perdon [1984], Proposition 4.2) Given a system Σ, defined over a ring R by equations of the form (1), with zero module $\mathcal{Z}$, assume that Σ is reachable and observable and that $G(\mathcal{U}(z)$ is closed in $\mathcal{Y}(z)$. Then, $\mathcal{V}^*/\mathcal{R}^*$ is R-isomorphic to $\mathcal{Z}$. If, in addition, $\mathcal{V}^*$ is of feedback type and F is one of its friends, $\mathcal{V}^*/\mathcal{R}^*$ endowed with the $R[z]$-module structure induced by the R-morphism $(A + BF)|_{\mathcal{V}^*/\mathcal{R}^*} : \mathcal{V}^*/\mathcal{R}^* \to \mathcal{V}^*/\mathcal{R}^*$ is $R[z]$-isomorphic to $\mathcal{Z}$.

In the hypothesis of the above Proposition, if one could assume that $\mathcal{V}^*/\mathcal{R}^*$ is a free R-module, say $\mathcal{V}^*/\mathcal{R}^* = R^r$, taking a matrix Z that represents $(A + BF)|_{\mathcal{V}^*/\mathcal{R}^*}$ with respect to the canonical basis of R^r, it would be quite natural to define the zero dynamics of Σ as the dynamics given by

$$\xi(t+1) = Z\xi(t) \tag{7}$$

with $\xi \in R^r$ or, equivalently, as the pair (R^r, Z). Although this approach may be inspiring, it is too limiting, due to the many restrictive hypothesis at its basis. However, motivated by the above discussion and following the same line of reasoning, we will provide a definition which applies to more general cases in the next Section.

5. ZERO DYNAMICS

Since the notion of zero we are considering is related to the input/output behavior of the system, it is clear that, for systems represented in state space form by equation (1), such notion makes sense only if the representation is minimal. Very roughly, this means that the dimension of the state module cannot be reduced without altering the transfer function. Minimality in this sense is guaranteed by the requirement of reachability and observability in Proposition 2, but reachability is a very strong property. Actually, in the framework of systems over rings, we can have state space representations of the form (1) that are minimal, in the sense explained above, but not reachable and, of course, cannot be transformed into a reachable representation by a change of basis in the state module. Minimal representation are characterized by the fact that both the observability matrix $[C^T \; C^T A^T .... C^T (A^T)^{n-1}]^T$ and the reachability magtrix $[B \; AB .... A^{n-1}B]$ are full rank. Reachability requires in addition that $[B \; AB .... A^{n-1}B]$ has a right inverse over R.

We will assume, in the rest of the paper, that the representation of the system Σ given by equations (1) is minimal. Then, in order to handle general situations, in which the strong hypothesis made in Proposition 2 do not necessarily hold, let us consider, instead of $\mathcal{S}^*$, its closure $CL(\mathcal{S}^*)$ in $\mathcal{X}$ and, instead of $\mathcal{R}^* = \mathcal{V}^* \cap \mathcal{S}^*$, the module $CL(\mathcal{S}^*) \cap \mathcal{V}^*$.

Proposition 3. LetΣ be a system defined over the ring R by equations of the form (1) and let $\mathcal{V}^*$, $\mathcal{S}^*$, $\mathcal{R}^*$ be as above. Then,

i) the closure $CL(\mathcal{S}^*)$ of $\mathcal{S}^*$ in $\mathcal{X}$ is a conditioned invariant submodule;
ii) the module $CL(\mathcal{S}^*) \cap \mathcal{V}^*$ coincide with the closure $CL_{(\mathcal{V}^*)}(\mathcal{R}^*)$ in $\mathcal{V}^*$ of $\mathcal{R}^*$;
iii) $CL_{(\mathcal{V}^*)}(\mathcal{R}^*)$ is the minimum closed submodule of $\mathcal{V}^*$ that contains $\mathcal{V}^* \cap ImB$;
iv) $CL_{(\mathcal{V}^*)}(\mathcal{R}^*)$ is a controlled invariant submodule of $\mathcal{X}$.

In the following, we will denote $CL_{(\mathcal{V}^*)}(\mathcal{R}^*)$ simply by $CL(\mathcal{R}^*)$.

Remark 1. Remark that $CL(\mathcal{R}^*)$ is not necessarily closed in $\mathcal{X}$. Since direct summands are closed, in case $\mathcal{R}^*$ is a direct summand of $\mathcal{V}^*$ (as it happens if $\mathcal{V}^*/\mathcal{R}^*$ is a free module), there is no difference in considering $\mathcal{R}^*$ or $CL(\mathcal{R}^*)$. On the other hand, if R is p.i.d. (as the ring $I\!R[\Delta]$), $CL(\mathcal{R}^*)$, being closed, is a direct summand of $\mathcal{V}^*$.

With the above ingredients, we can now generalize the situation described in Proposition 2 to the cases in which $\mathcal{V}^*$ is not of feedback type. In such an occurrence, letting $dim\mathcal{V}^* = s$, construct the extended system $\Sigma_e = (A_e, B_e, C_e, \mathcal{X}_e)$, with state module $\mathcal{X}_e = \mathcal{X} \oplus R^s$, for which

$$A_e = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}; B_e = \begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix}; C_e = [\, C \;\; 0 \,] \tag{8}$$

where I and 0 denote, respectively, the identity matrix and null matrices of suitable dimensions. Denoting by V a matrix whose columns span $\mathcal{V}^*$ in $\mathcal{X}$, the submodule $\mathcal{V}_e$ spanned in $\mathcal{X} \oplus R^s$ by the columns of the matrix $[V^T I]^T$ is easily seen to be (A_e, B_e)-invariant and direct summand of $\mathcal{X} \oplus R^s$. Hence, $\mathcal{V}_e$ is of feedback type and, since the canonical projection $\pi : \mathcal{X} \oplus R^s \to \mathcal{X}$ is such that $\pi(\mathcal{V}_e) = \mathcal{V}^*$, it can be viewed as an extension of $\mathcal{V}^*$. In addition, we have that $\mathcal{V}_e$ is contained in the kernel $KerC_e$ of the output map of Σ_e.

Proposition 4. In the above situation and with the above notations, let $\mathcal{R}_e$ be defined as $\mathcal{R}_e = \pi^{-1}(CL(\mathcal{R}^*))$. Then:

i) $\mathcal{R}_e$ is an (A_e, B_e)-invariant submodule of $\mathcal{X}_e$;
ii) $\mathcal{R}_e$ is the minimum closed submodule of $\mathcal{V}_e$ that contains $\mathcal{V}_e \cap ImB_e$;
iii) the canonical projection π induces an R-isomorphism between the quotient module $\mathcal{V}_e/\mathcal{R}_e$ and the quotient module $\mathcal{V}^*/(CL(\mathcal{R}^*))$.

It follows from the above Proposition, that the system extension produces an R-module $\mathcal{V}_e/\mathcal{R}_e$ and an R-morphism $(A_e + B_e F_e)|_{\mathcal{V}_e/\mathcal{R}_e}$, where F_e is a friend of $\mathcal{V}_e$, that form a pair akin to the pair $\mathcal{V}^*/\mathcal{R}^*$ and $(A + BF)|_{\mathcal{V}^*/\mathcal{R}^*}$ considered, under the restrictive hypothesis of Proposition 2, at the end of Session 4. So, we can state a Definition of zero dynamics for Σ which is meaningful also when not all the hypothesis of Proposition 2 necessarily hold.

Definition 6. With the above notation, assume that $\mathcal{V}_e/\mathcal{R}_e$ is a free R-module, say $\mathcal{V}_e/\mathcal{R}_e = R^r$, and that Z is a matrix representing $(A_e + B_e F_e)|_{\mathcal{V}_e/\mathcal{R}_e}$ with respect to the canonical basis of R^r. Then, the Zero Dynamics of Σ is the dynamics given by

$$\xi(t+1) = Z\xi(t) \tag{9}$$

with $\xi \in R^r$ or, equivalently, that determined by the pair (R^r, Z).

Remark that, in case $\mathcal{V}_e/\mathcal{R}_e$ is not a free R-module, we do not define a zero dynamics. However, zero dynamics exist in interesting cases, like, in particular, when $\mathcal{S}^* \cap \mathcal{V}^* = 0$ (as it happens if Σ is left invertible) and $\mathcal{V}^*$ is a free R-module. Moreover, in case R is a ring of polynomials in one or more indeterminates with coefficients in $I\!R$, i.e. $R = I\!R[\Delta_1, ..., \Delta_k]$, as it happens for systems over ring associated to time-delay ones, we have the following Proposition.

Proposition 5. If $R = I\!R[\Delta_1, ..., \Delta_k]$, then $\mathcal{V}_e/\mathcal{R}_e$ is a torsion free R-module. If $\mathcal{V}_e/\mathcal{R}_e$ is a free R-module, then $\mathcal{R}_e$ is a free direct summand of $\mathcal{V}_e$. In particular, if $k = 1$, $\mathcal{V}_e/\mathcal{R}_e$ is a free R-module of the form R^r for some r.

Proof. Being a direct summand of a free module over a ring of polynomials with real coefficients, $\mathcal{V}_e$ is a free R-module by Lam [1978], hence $\mathcal{V}_e = R^s$ for some s. Being the inverse image of a closed submodule, $\mathcal{R}_e$ is closed in $\mathcal{V}_e$. In the general case, this implies that $\mathcal{V}_e/\mathcal{R}_e$ is a torsion free R-module, which is the first statement. In particular, if $k = 1$, the ring of coefficients is a principal ideal domain and the last statement follows. If $\mathcal{V}_e/\mathcal{R}_e$ is free, that is $\mathcal{V}_e/\mathcal{R}_e = R^r$, $\mathcal{V}_e$ is isomorphic to $\mathcal{R}_e \oplus R^r$ and the second statement follows, as above, from Lam [1978].

Definition 7. Let Σ_d be a time-delay systems of the form (3). Then, the Zero Dynamics of Σ_d is the zero dynamics of the associated system Σ, if that is defined.

Remark 2. Remark that, by Proposition 5, the zero dynamics is always defined for time-delay system with commensurable delays, since the ring of coefficients of the associated systems is, in that case, $I\!R[\Delta_1]$.

Remark 3. Note that, in case the zero dynamics of Σ is given by the pair (R^r, Z), by substituting in Z the delay operators δ_j to the indeterminates Δ_j, we can consider, instead of (9), a time-delay dynamics of the form $\dot{\xi}(t) = Z\xi(t)$.

6. FIXED DYNAMICS AND PHASE MINIMALITY

A basic property of the zero dynamics introduced above is that it characterizes the fixed dynamics with respect to feedbacks which make the system maximally unobservable. To see this, recall that $\mathcal{V}^*$ represents the largest submodule of the state module $\mathcal{X}$ that can be made unobservable by means of a feedback, either a static one, in case $\mathcal{V}^*$ is of feedback type, or a dynamic one, in case it is not. This property is fundamental in dealing with the problem of decoupling the output of the system from a disturbance input (see Conte and Perdon [1995]).

Given a system Σ, with coefficient in R, of the form (1), let us consider the submodule $\mathcal{V}^*$ of its state module. In order to deal with all possible situations at the same time, let us consider a system $\Sigma_e = (A_e, B_e, C_e, X_e)$ that, in case $\mathcal{V}^*$ is not of feedback type, is an extension of Σ constructed as in Section 5 and that, in case $\mathcal{V}^*$ is of feedback type, coincides with Σ. In both cases we have the submodules $\mathcal{V}_e$ and $\mathcal{R}_e$, which, in particular, when Σ_e coincides with Σ, coincide with $\mathcal{V}^*$ and $CL(\mathcal{R}^*)$. Now, let us assume that the zero dynamics of Σ is defined and that $\mathcal{V}_e$ is a direct summand of $\mathcal{X}_e$ (as we have seen in Section 5, this is true by construction if Σ_e is actually an extension of Σ, but it must be assumed explicitly in the other situation). In case Σ_e coincides with Σ, the existence of the zero dynamics implies, in particular, that $\mathcal{R}^*$ is closed, that is $\mathcal{R}^* = CL(\mathcal{R}^*)$. In both situations, we can write $X_e = X \oplus R^r = \mathcal{R}_e \oplus W_1 \oplus W_2$ for some submodules W_1 and W_2, such that $\mathcal{V}_e = \mathcal{R}_e \oplus W_1$ and $X_e = \mathcal{V}_e \oplus W_2$. Writing A_e and B_e in that basis and partitioning accordingly, we get

$$A_e = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}; B_e = \begin{bmatrix} B_1 \\ 0 \\ B_3 \end{bmatrix}. \qquad (10)$$

The dynamic matrix $A_c = (A_e + B_e F)$ of the compensated system, for any friend $F_e = [\, F_1 \; F_2 \; F_3 \,]$ of $\mathcal{V}_e$, takes therefore the form

$$A_c = \begin{bmatrix} A_{11} + B_1F_1 & A_{12} + B_1F_2 & A_{13} + B_1F_3 \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} + B_3F_3 \end{bmatrix} \qquad (11)$$

showing that the dynamics of the block A_{22} remains fixed for any choice of F_e and that $(A_e + B_eF_e)|_{\mathcal{V}_e/\mathcal{R}_e} = A_{22}$.

We can recall now the abstract notion of stability for systems with coefficients in a ring R, by introducing the concept of Hurwitz set (see e.g. Habets [1994]).

Definition 8. Given a ring R, a subset $S \subseteq R[z]$ of polynomials with coefficients in R in the indeterminate z is said an Hurwitz set if (i) it is multiplicatively closed, (ii) it contains at least an element of the form $z - \alpha$, with $\alpha \in R$, (iii) it contains all monic factors of all its elements.

Given a Hurwitz set S, a system Σ of the form (1) with coefficients in R is said S-stable if $det(zI - A)$ belongs to S. If $R = I\!R[\Delta]$, then $R[z] = I\!R[z, \Delta]$ and, letting

$$\begin{aligned} S = \{&p(z,\Delta) \in R[z], \text{ such that } p\,(\gamma, e^{-\gamma h}) \neq 0 \\ &\text{for all complex number } \gamma \text{ with Re } \gamma \geq 0\}, \end{aligned} \qquad (12)$$

we have that stability of a system Σ_d in the time-delay framework corresponds to S-stability of the associated system Σ in the ring framework.

When the zero dynamics is defined, together with the notion of stability we have recalled above, it allows us to give the following Definition.

Definition 9. Let Σ be a system with coefficients in the ring R and assume that the zero dynamics of Σ is defined and that it is given by (R^r, Z). Then, Σ is said to be S-minimum phase, where S in an Hurwitz set, if its zero dynamics is S-stable.

In case Σ is associated to the time-delay system Σ_d and S is the Hurwitz set defined by (12), Σ_d is said to be minimum phase if Σ is S-minimum phase.

For invertible systems, the notion of minimum phase is related the existence of stable inverses, due to the relation between zeros of the system and poles of the inverse. Various aspects of this situation, following the lines of Wyman and Sain [1981] have been dealt with in Conte and Perdon [1984], Conte and Perdon [2000b] and, more recently, Conte and Perdon [2007].

Example 1. Let us consider the time-delay system Σ_d described by the equations

$$\Sigma_d = \begin{cases} \dot{x}_1(t) = x_3(t-h) + u(t-h) \\ \dot{x}_2(t) = x_1(t) + x_3(t-h) + u(t-h) \\ \dot{x}_3(t) = x_2(t) + x_3(t-h) \\ \;\; y(t) = x_3(t) \end{cases}$$

and the associated system $\Sigma = (A, B, C, X)$ with coefficients in $R = I\!R[\Delta]$ and matrices

$$A = \begin{bmatrix} 0 & 0 & \Delta \\ 1 & 0 & \Delta \\ 0 & 1 & \Delta \end{bmatrix}; B = \begin{bmatrix} \Delta \\ \Delta \\ 0 \end{bmatrix}; C = [0 \; 0 \; 1].$$

Computations in the ring framework show that $\mathcal{V}^*$ is the submodule of R^3 spanned by the vector $V = (\Delta \; 0 \; 0)^T$ and that $\mathcal{R}^* = \{0\}$. $\mathcal{V}^*$ is not of feedback type (because it is not closed), so we consider the extension $\Sigma_e = (A_e, B_e, C_e, X_e)$, where

$$A_e = \begin{bmatrix} 0 & 0 & \Delta & 0 \\ 1 & 0 & \Delta & 0 \\ 0 & 1 & \Delta & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; B_e = \begin{bmatrix} \Delta & 0 \\ \Delta & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}; C_e = [0 \; 0 \; 1 \; 0]^T.$$

We have, now, that $\mathcal{V}_e$ is the submodule of R^4 spanned by the vector $[V^T \; 1]^T = (\Delta \; 0 \; 0 \; 1)^T$ and $\mathcal{R}_e = \{0\}$. The zero dynamics of Σ, as well as that of Σ_d, is defined and can be explicitly evaluated. A friend F_e of $\mathcal{V}_e$ is given, for instance, by $F_e = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$. Therefore, the dynamic matrix $A_c = (A_e + B_eF)$ of the compensated system is $A_c = \begin{bmatrix} 0 & 0 & \Delta & -\Delta \\ 1 & 0 & \Delta & -\Delta \\ 0 & 1 & \Delta & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$. Since $A_c[V^T \; 1]^T = -[V^T \; 1]^T$, the zero dynamics turns out to be given by $(R, [-1])$ or, in other terms by

$$\begin{array}{c} \xi(t+1) = -\xi(t) \\ (\dot{\xi}(t) = -\xi(t) \text{ in the time-delay framework}) \end{array} \tag{13}$$

with $\xi \in R$ (respectively $\xi \in I\!R$).

Taking the Hurwitz set S as in (12), Σ is S-minimum phase (respectively, Σ_d is minimum phase).

The transfer function matrix of Σ is $G(z) = \frac{\Delta z+\Delta}{z^3-\Delta z^2-\Delta z-\Delta}$ and then, by Proposition 1, the zero module $\mathcal{Z}$ of Σ can be viewed, through the injective $R[z]$-morphism α, as a submodule of $Tor(\Omega\mathcal{Y}/(\Delta z+\Delta)\Omega\mathcal{U})$. We have, in our case, $Tor(\Omega\mathcal{Y}/(\Delta z+\Delta)\Omega\mathcal{U}) = I\!R[\Delta, z]/(\Delta z+\Delta)I\!R[\Delta, z]$ and, since any element $p(\Delta, z) \in I\!R[\Delta, z]$ can be written in a unique way as $p(\Delta, z) = \Delta p'(\Delta, z) + p''(z) = \Delta[(z+1)q(\Delta, z) + q'(\Delta)] + p''(z)$, we can say that, denoting equivalence classes by brackets, any element $[p(\Delta, z)] \in Tor(\Omega\mathcal{Y}/(\Delta z+\Delta)\Omega\mathcal{U})$ can be written in a unique way as $[p(\Delta, z)] = [\Delta q'(\Delta)] + [p''(z)]$, for suitable polynomials $q'(\Delta) \in I\!R[\Delta]$ and $p''(z) \in I\!R[z]$. It turns out that $Im(\alpha)$ coincides with the submodule of $I\!R[\Delta, z]/(\Delta z+\Delta)I\!R[\Delta, z]$ consisting of all elements of the form $[\Delta q'(\Delta)]$ with $q'(\Delta) \in I\!R[\Delta]$. Inspection shows that, for any element of that kind, $z[\Delta q'(\Delta)] = [z\Delta q'(\Delta)] = [-\Delta q'(\Delta)]$ and therefore we can conclude that the $R[z]$-module $\mathcal{Z}$ can be viewed as defined by the pair $(R, [-1])$, in accordance with what expressed by equation (13).

Example 2. Let us consider the time-delay system Σ_d described by the equations

$$\Sigma_d = \begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = x_1(t) + u_1(t-h_1) + u_2(t-h_2) \\ y(t) = x_2(t) \end{cases}$$

and the associated system $\Sigma = (A, B, C, X)$ with coefficients in $R = I\!R[\Delta_1, \Delta_2]$ and matrices

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; B = \begin{bmatrix} 0 & 0 \\ \Delta_1 & \Delta_2 \end{bmatrix}; C = [0 \ \ 1]\,.$$

Computations in the ring framework show that $\mathcal{V}^*$ is the submodule of R^2 given by $span\{(\Delta_1 \ 0)^T, (\Delta_2 \ 0)^T\}$ and that $\mathcal{R}^* = 0$. $\mathcal{V}^*$ is not of feedback type (because it is not closed), so we should consider an extension Σ_e as described in Section 5. However, we know that $\mathcal{V}_e/\mathcal{R}_e = \mathcal{V}^*/\mathcal{R}^* = \mathcal{V}^*$ is not free, since it has a minimal set of generators which are not linearly independent. Therefore, in this case, the zero dynamics is not defined.

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