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Nonlinear Multi-Agent System Consensus with Time-Varying Delays ⋆

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Abstract: Most consensus protocols for Multi-Agent Systems (MAS) presented in the past do not consider communication constraints such as delays in the exchange of information between the agents. In this paper, we provide conditions for a nonlinear, locally passive MAS with time-varying communication delays to reach a consensus, using a recently presented method based on an invariance principle for Lyapunov-Razumikhin functions. We consider both the cases of fixed and switching topologies. In the case of a fixed topology, the underlying directed graph has to contain a spanning tree. In the case of a switching topology, only the union graph of all graphs that persist over time is required to contain a spanning tree.

Keywords: Multi-agent systems, nonlinear consensus, time-varying delays, switched systems.

1. INTRODUCTION

Multi-Agent Systems (MAS) have attracted more and more interest in recent years. They represent a general description for large-scale systems consisting of small subunits, called agents. The behavior of MAS is particularly interesting because the agents may fulfill certain tasks as a group, even in the individual agent does not know about the overall task. Many examples come from nature, such as schooling fishes or fireflies flashing in unison, see, e.g., Strogatz [2003]. Clearly, this collective behavior is also interesting for engineers when solving problems such as flocking [Olfati-Saber, 2006, Fax and Murray, 2004, Jadbabaie et al., 2003], or synchronization [Jadbabaie et al., 2004, Strogatz, 2000]. Recent reviews on consensus and cooperation are given in Olfati-Saber et al. [2007] and Ren et al. [2007].

Most publications on MAS consider only linear subsystems and ideal communication channels without delay. However, many systems, such as for instance the well-known Kuramoto oscillator [Kuramoto, 1984], exhibit nonlinear, locally passive dynamics as discussed in Papachristodoulou and Jadbabaie [2006]. Nonlinear consensus problems without delay have been previously studied in Lin et al. [2007], Bauso et al. [2006], Moreau [2005], Qu et al. [2007]. Furthermore, many realistic communication networks exhibit delays as studied for example in Lee and Spong [2006], Bliman and Ferrari-Trecate [2005]. Another interesting issue is switching network topologies that can be used to model the loss and establishment of new communication links between agents as these move in space [Moreau, 2005]. However, there are very few publications that deal with both switching topologies and delayed communication: Olfati-Saber and Murray [2004] presented a delay-dependent result for MAS with switching topologies with identical, fixed delays in all channels. In Ghabcheloo et al. [2007], Papachristodoulou and Jadbabaie [2006, 2005], a synchronization problem with switching topology has been considered, but for constant delays.

In this paper, we present a continuous-time consensus protocol for nonlinear, locally passive MAS with delayed exchange of information. Locally passive means that the nonlinear dynamics $g_{ji}(x_i - x_j)$, which describe the influence of agent j on agent i, satisfy $y g_{ji}(y) > 0$ for $y \in [-\gamma_{ji}^-, \gamma_{ji}^+] \setminus \{0\}$ with $\gamma_{ji}^-, \gamma_{ji}^+ > 0$. The delay may result from a digital communication network between the agents or from other propagation processes that are used to exchange information between the agents, e.g., sonar for autonomous submarines. The delay is not fixed, but rather depends on the workload of the communication network or the distance between the two agents. For this reason, we assume a continuous, time-varying delay to capture the unsteadiness in the communication delay. This model differs from our previous work [Münz et al., 2008, 2007] where the communication channels were modeled as distributed delays. We consider both fixed and switching network topologies. The only requirement for the consensus set to be asymptotically attracting in the case of a fixed topology is that the underlying graph contains a spanning tree. For the switching topology case, only the union graph of all subgraphs that persist over time has to contain a spanning tree. The methodology we use is based on an invariance principle for Lyapunov-Razumikhin functions. The main ideas of the proof are based on recent results

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in Münz et al. [2008]. This underlines that the applied methods can also be used for other MAS problems with delays.

The paper is structured as follows: Some background information on time-delay systems and algebraic graph theory is given in Section 2. The problem is posed in Section 3. In Section 4, we present conditions for consensus over fixed topologies. In Section 5, these results are extended to switching topologies. The paper is concluded in Section 6.

2. PRELIMINARIES

In this section, we review briefly some stability results for functional differential equations using Lyapunov-Razumikhin functions as well as some tools and notation from Algebraic Graph Theory.

2.1 Stability of Functional Differential Equations

This subsection gives a brief summary of stability results for functional differential equations. The interested reader is referred to Hale and Lunel [1993] and Haddock and Terjéki [1983] for more details.

Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space with the standard norm $|\cdot|$. Let $\mathcal{C}([a,b],\mathbb{R}^n)$ denote the Banach space of continuous functions mapping the interval $[a,b] \subset \mathbb{R}$ into $\mathbb{R}^n$ with the topology of uniform convergence. Given $\mathcal{T} > 0$, we define $\mathcal{C} = \mathcal{C}([-\mathcal{T},0],\mathbb{R}^n)$. The norm on $\mathcal{C}$ is defined as $\|\varphi\| = \sup_{-\mathcal{T}\leq s\leq 0}|\varphi(s)|$. Let $\rho \geq 0$ and $x \in \mathcal{C}([-\mathcal{T},\rho],\mathbb{R}^n)$, then for any $t \in [0,\rho]$, define a segment $x_t \in \mathcal{C}$ by $x_t(s) = x(t+s), s \in [-\mathcal{T},0]$.

Let Ω be a subset of $\mathcal{C}$, $f : \Omega \to \mathbb{R}^n$ a given function, and '' represent the right-hand Dini derivative, then we call

$$\dot{x}(t) = f(x_t) \tag{1}$$

an autonomous Retarded Functional Differential Equation (RFDE) on Ω. Given an initial condition $\varphi \in \Omega$ and $\rho > 0$, a function $x(\varphi) : [-\mathcal{T},\rho] \to \mathbb{R}^n$ is said to be a solution to (1), if $x_t(\varphi) \in \Omega$, $x(\varphi)(t)$ satisfies (1) for $t \in [0,\rho)$, and $x_0(\varphi) = \varphi$. Such a solution exists and is unique if f is continuous and $f(\varphi)$ is Lipschitzian in each compact set in Ω. Note that $x_t(\varphi)(s) = x(\varphi)(t+s)$ for $s \in [-\mathcal{T},0]$.

An element $\phi \in \mathcal{C}$ is called a steady-state or equilibrium of (1) if $x_t(\phi) = \phi$ for all $t \geq 0$. Without loss of generality we assume that $\phi = 0$ is an equilibrium of (1). The stability of (1) around such a steady-state is defined in a way similar to the stability of nonlinear Ordinary Differential Equations (ODE) using an ϵ-δ argument, see Hale and Lunel [1993].

There are two types of Lyapunov theorems for stability of equilibria of RFDE, namely Lyapunov-Krasovskii and Lyapunov-Razumikhin. Lyapunov-Krasovskii is the natural extension of Lyapunov's theorem from ODEs to RFDEs. It is based on non-increasing Lyapunov-Krasovskii-functionals. In this work, we will be applying Lyapunov-Razumikhin-type theorems to prove consensus, which uses functions instead of functionals.

Let $D \subseteq \mathbb{R}^n$. By a *Lyapunov-Razumikhin Function* $V = V(x)$, we mean a continuous function $V : D \to \mathbb{R}$. The upper right-hand Dini derivative of V with respect to (1) is defined by

$$\dot{V}(\varphi) = \limsup_{h\to 0^+}\frac{1}{h}(V(\varphi(0)+hf(\varphi)) - V(\varphi(0))).$$

With this definition, we have the following Lyapunov-Razumikhin theorem:

Theorem 1. Suppose $f : \Omega \to \mathbb{R}^n$ maps bounded subsets of Ω into bounded sets of $\mathbb{R}^n$ and consider (1). Suppose $v, w : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous, non-decreasing functions, $v(s)$ positive for $s > 0$, $v(0) = 0$. If there is a Lyapunov-Razumikhin Function $V : D \to \mathbb{R}$ such that:

$$V(x) \geq v(|x|) \qquad \text{for } x \in D, \text{ and}$$
$$\dot{V}(\varphi(0)) \leq -w(\varphi(0)) \quad \text{if } V(\varphi(0)) = \max_{-\mathcal{T}\leq s\leq 0} V(\varphi(s)),$$

then the equilibrium $x = 0$ of (1) is stable.

Note that the function V in Razumikhin's theorem need not be non-increasing along the system trajectories, but may indeed increase within a delay interval. The proof of Razumikhin's theorem is based on the fact that

$$\overline{V}(\varphi) = \max_{-\mathcal{T}\leq s\leq 0} V(\varphi(s)) \tag{2}$$

is a Lyapunov-Krasovskii functional that is non-increasing along the system trajectories. This is an important fact that we will be using in our proofs.

In this paper, we have to prove the attractivity of a subspace of $\mathbb{R}^n$. Therefore, we will make repeated use of an invariance principle for RFDEs. For this, we need to define ω-limit sets of solutions and provide LaSalle-type theorems for RFDEs.

Definition 2. A set $M \subseteq \Omega$ is said to be positively invariant *with respect to (1) if, for any $\varphi \in M$, there is a solution $x(\varphi)$ of (1) that is defined on $[-\mathcal{T},\infty)$ such that $x_t(\varphi) \in M$ for all $t \geq 0$ and $x_0 = \varphi$.*

Definition 3. Let $\varphi \in \Omega$. An element ψ of Ω is in $\omega(\varphi)$, the ω-limit set of φ, *if $x(\varphi)$ is defined on $[-\mathcal{T},\infty)$ and there is a sequence $\{t_n\}$ of non-negative real numbers satisfying $t_n \to \infty$ and $\|x_{t_n}(\varphi) - \psi\| \to 0$ as $n \to \infty$.*

If $x(\varphi)$ is a solution of (1) that is defined and bounded on $[-\mathcal{T},\infty)$, then the orbit through φ, i.e., the set $\{x_t(\varphi) : t \geq 0\}$ is precompact, $\omega(\varphi)$ is non-empty, compact, connected, and invariant, and $x_t(\varphi) \to \omega(\varphi)$ as $t \to \infty$.

For a given set $\Omega \subset \mathcal{C}$, define

$$E_V = \Big\{\varphi \in \Omega : \max_{s\in[-\mathcal{T},0]} V(x_t(\varphi)(s)) = \max_{s\in[-\mathcal{T},0]} V(\varphi(s)),\ \forall t \geq 0\Big\} \tag{3}$$

$$M_V : \quad \text{Largest set in } E_V \text{ that is invariant wrt. (1).} \tag{4}$$

Here, E_V is the set of functions $\varphi \in \Omega$ which can serve as initial conditions for (1) such that $x_t(\varphi)$ satisfies

$$\max_{s\in[-\mathcal{T},0]} V(x_t(\varphi)(s)) = \max_{s\in[-\mathcal{T},0]} V(\varphi(s))$$

for all $t \geq 0$. Note that the above condition guarantees that $\overline{V}$ defined in (2) satisfies $\dot{\overline{V}}(\varphi) = 0$. In particular, for a Lyapunov-Razumikhin function V and for any $\varphi \in E_V$, we have $\dot{V}(x_t(\varphi)) = 0$ for any $t > 0$ such that $\max_{s\in[-\mathcal{T},0]} V(x_t(\varphi)(s)) = V(x_t(\varphi)(0))$.

We then have the following theorem:

Theorem 4. Suppose there exists a Lyapunov-Razumikhin function $V = V(x)$ and a closed set Ω that is positively invariant with respect to (1) such that

$$\dot{V}(\varphi) \leq 0 \ \forall \varphi \in \Omega \ s.t. \ V(\varphi(0)) = \max_{s \in [-\mathcal{T},0]} V(\varphi(s)). \quad (5)$$

Then, for any $\varphi \in \Omega$ *such that* $x(\varphi)$ *is defined and bounded on* $[-\mathcal{T}, \infty)$, $\omega(\varphi) \subseteq M_V \subseteq E_V$, *and we have*

$$x_t(\varphi) \to M_V \ \text{as} \ t \to \infty.$$

Theorem 4 will be used extensively in our work. It proves the attractivity of invariant subsets M_V of Ω for the solutions of RFDE (1).

2.2 Algebraic Graph Theory

The topology of the communication network between the agents is represented by a graph. A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists of a set of vertices (nodes) $\mathcal{V} = \{v_i\}, i \in \mathcal{I} = \{1, \ldots, N\}$, which represent the agents, and a set of edges (links) $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, which represent the communication channels between the agents. If $v_i, v_j \in \mathcal{V}$ and $e_{ij} = (v_i, v_j) \in \mathcal{E}$, then there is an edge (a directed arrow) from node v_i to node v_j, i.e., agent j can receive data from agent i. In this paper, we assume that the graph $\mathcal{G}$ is *directed*, i.e., $e_{ij} \in \mathcal{E}$ does not necessarily imply that $e_{ji} \in \mathcal{E}$. We also assume that the network topology does not contain self-loops, i.e., $e_{ii} \notin \mathcal{E}$. The *graph adjacency matrix* $A = [a_{ij}]$, $A \in \mathbb{R}^{N \times N}$, is such that $a_{ij} = 1$ if $e_{ij} \in \mathcal{E}$ and $a_{ij} = 0$ if $e_{ij} \notin \mathcal{E}$. If $e_{ji} \in \mathcal{E}$, then v_j is a *parent* of v_i. The number of parents of agent i, also called the *in-degree* of vertex v_i, is denoted by $d_i = \sum_{j=1}^{N} a_{ji}$.

A *directed path* from v_i to v_j is a sequence of edges out of $\mathcal{E}$ that takes the following form $(v_i, v_{i_1}), (v_{i_1}, v_{i_2}), \ldots, (v_{i_p}, v_j)$. A *directed cycle* is a directed path that starts and ends at the same vertex. A *directed tree* is a directed graph where every vertex has exactly one parent except for one node, the so-called *root* v_R. Clearly, there is a directed path from v_R to all other nodes of the directed tree.

A *subgraph* $(\tilde{\mathcal{V}}, \tilde{\mathcal{E}})$ of $\mathcal{G}$ is a graph with $\tilde{\mathcal{V}} \subseteq \mathcal{V}$ and $\tilde{\mathcal{E}} \subseteq \mathcal{E}$. If there exists a subgraph $(\mathcal{V}, \tilde{\mathcal{E}})$ of $\mathcal{G}$ that is a directed tree, then we say that $\mathcal{G}$ contains a *directed spanning tree*. Hence, a graph $\mathcal{G}$ contains a directed spanning tree if and only if it contains at least one *root*, i.e., one node with a directed path to all other vertices. We denote the set of all roots of $\mathcal{G}$ as $\mathcal{I}_R$ and $\mathcal{I}_{\overline{R}} = \mathcal{I} \setminus \mathcal{I}_R$. In the following, we also say *spanning tree* when referring to a *directed spanning tree*. The *union graph* of a set of P graphs $\{(\mathcal{V}, \mathcal{E}_p)\}, p \in \mathcal{P} = \{1, \ldots, P\}$, is $\left(\mathcal{V}, \bigcup_{p \in \mathcal{P}} \mathcal{E}_p\right)$. More details on algebraic graph theory can be found for example in Godsil and Royle [2000].

3. PROBLEM SETUP

Consider N agents with nonlinear, locally passive dynamics and delayed exchange information. The delay τ_{ji} when agent j transmits its state to agent i is continuous, bounded, and time-varying: $\tau_{ji} : \mathbb{R}_+ \to [0, \mathcal{T}], \mathcal{T} \in \mathbb{R}$. The scalar state x_i of agent i is updated continuously by comparing its own state with the states of its parent agents $x_j, e_{ji} \in \mathcal{E}$. This is summarized in the following RFDE:

$$\dot{x}_i(t) = -k_i \sum_{j=1}^{N} a_{ji} g_{ji} \left(x_i(t) - x_j(t - \tau_{ji}(t)) \right), \ i \in \mathcal{I}, \quad (6)$$

where $k_i > 0$ is the coupling gain and a_{ji} are the elements of the adjacency matrix A of the underlying graph. We assume that the delays τ_{ji} are sufficiently heterogeneous so that (6) does not have a limit cycle. The nonlinear dynamics g_{ji} satisfy the following assumption:

Assumption 5. The continuous functions $g_{ji} : \mathbb{R} \to \mathbb{R}$ *are* locally passive, *i.e.,*

$$g_{ji}(0) = 0 \ \text{and} \ y g_{ji}(y) > 0 \ \text{for all} \ y \in [-\gamma_{ji}^-, \gamma_{ji}^+] \setminus \{0\},$$

with $\gamma_{ji}^-, \gamma_{ji}^+ > 0$.

This model extends the standard linear MAS, studied for example in Jadbabaie et al. [2003], with nonlinear dynamics g_{ji} and time-varying delays τ_{ji}. We considered a similar model with distributed delays in Münz et al. [2008].

The main result of our work is to provide conditions for MAS (6) to reach consensus asymptotically, i.e., all agents eventually converge to the same state $x_i = x_j$ for all $i, j \in \mathcal{I}$. For some $x^* \in \mathbb{R}$, a *consensus point* $\phi_{x^*} \in \mathcal{C}$ is such that all components of ϕ_{x^*} satisfy $\phi_{x^*,i}(\eta) = x^*, i \in \mathcal{I}$, for all $\eta \in [-\mathcal{T}, 0]$. The *consensus set* Θ of the MAS (6) is

$$\Theta = \bigcup_{x^* \in \mathbb{R}} \{\phi_{x^*}\}, \quad (7)$$

It can be easily checked that any element of the consensus set is a steady-state of (6). We investigate both fixed and switching topologies in Section 4 and 5, respectively.

4. CONSENSUS OVER FIXED TOPOLOGIES

We have to prove that the consensus set Θ (7) is asymptotically attracting for appropriate initial conditions $\varphi \in \mathcal{C}_{\mathbb{D}} = \mathcal{C}([-\mathcal{T}, 0], \mathbb{D})$. The *region of attraction* $\mathbb{D}$ is

$$\mathbb{D} = \left\{ x \in \mathbb{R}^N : \ |x_i| \leq \frac{\gamma}{2} \right\} \quad (8)$$

with $\gamma = \min_{i,j \in \mathcal{I}} \{\gamma_{ij}^-, \gamma_{ij}^+\}$, where $\gamma_{ij}^-, \gamma_{ij}^+$ are the bounds of the locally passive functions g_{ij}, cf. Assumption 5. Note that $\mathbb{D} = \mathbb{R}^n$ if g_{ji} are globally passive, e.g., linear. For $\mathcal{C}_{\mathbb{D}}$, we have the following result:

Lemma 6. If Assumption 5 holds, then $\mathcal{C}_{\mathbb{D}} = \mathcal{C}([-\mathcal{T}, 0], \mathbb{D})$ *is a positively invariant set of* (6).

Proof. Consider the Lyapunov-Razumikhin function candidate

$$V(x(t)) = \frac{1}{2} \max_{i \in \mathcal{I}} x_i^2(t).$$

We denote I the index that satisfies $x_I^2(t) = \max_{i \in \mathcal{I}} x_i^2(t)$. If there are several possible indices, we choose that one which the maximal modulo of the derivative $|\dot{x}_I(t)|$. If there are still several possible indices, we choose any one of them but fix the index I as long as it satisfies the maximum conditions. With this notation, the upper right-hand Dini derivative of V along solutions of (6) is

$$\dot{V}(x_t) = -k_I \sum_{j=1}^{N} a_{jI} \, x_I(t) g_{jI} \left(x_I(t) - x_j(t - \tau_{jI}(t)) \right). \quad (9)$$

The condition on $|\dot{x}_I(t)|$ is necessary in order to guarantee that (9) is indeed the upper right-hand derivative.

Following Theorem 1, we consider the behavior of $\dot{V}$ if $V(x(t)) = \max_{\eta \in [0, \mathcal{T}]} V(x(t - \eta))$, i.e., $|x_I(t)| =$

$\max_{\eta\in[0,\mathcal{T}]}\max_{j\in\mathcal{I}}|x_j(t-\eta)|$. We conclude with Assumption 5 that

$$x_I(t)g_{jI}\left(x_I(t)-x_j(t-\tau_{jI}(t))\right)\geq 0$$

for all j with $e_{jI}\in\mathcal{E}$ and all $x_t\in\mathcal{C}_{\mathbb{D}}$. Hence, $\dot{V}\leq 0$ if $V(x(t))=\max_{\eta\in[0,\mathcal{T}]}V(x(t-\eta))$ and consequently $x_t(\varphi)\in\mathcal{C}_{\mathbb{D}}$ for all $t\geq 0$ if $\varphi\in\mathcal{C}_{\mathbb{D}}$. □

With this result, we prove that the consensus set Θ is asymptotically attracting for any initial condition $\varphi\in\mathcal{C}_{\mathbb{D}}$, as long as the directed interaction graph contains a spanning tree.

Theorem 7. Given a MAS consisting of N agents with dynamics (6)*, where g_{ji} satisfy Assumption 5, and with initial condition $\varphi\in\mathcal{C}_{\mathbb{D}}$, as well as an underlying network topology of a directed graph with a spanning tree, then the consensus set Θ of this MAS is asymptotically attracting.*

Proof. Consider the Lyapunov-Razumikhin function candidates

$$V_1=\max_{i\in\mathcal{I}}x_i(t),$$
$$V_2=-\min_{i\in\mathcal{I}}x_i(t).$$

As in the former proof, we denote I and J the indices that satisfy $x_I(t)=\max_{i\in\mathcal{I}}x_i(t)$ and $x_J(t)=\min_{i\in\mathcal{I}}x_i(t)$, respectively. If there are several such indices, we choose those with the maximal derivative $\dot{x}_I(t)$ and minimal derivative $\dot{x}_J(t)$, respectively. If there are still several possible indices, we choose any one of them but fix the indices I and J as long as they satisfy the extremum conditions. Using this notation, the right-hand Dini derivatives of V_1 and V_2 along solutions of (6) is

$$\dot{V}_1(x_t)=-k_I\sum_{j=1}^{N}a_{jI}\,g_{jI}\left(x_I(t)-x_j(t-\tau_{jI}(t))\right),$$
$$\dot{V}_2(x_t)=k_J\sum_{j=1}^{N}a_{jJ}\,g_{jJ}\left(x_J(t)-x_j(t-\tau_{jJ}(t))\right).$$

Following Theorem 4, we are interested in the behavior of $\dot{V}_k, k=1,2$, whenever $V_k(x(t))=\max_{\eta\in[0,\mathcal{T}]}V_k(x(t-\eta))$, i.e., $x_I(t)=\max_{\eta\in[0,\mathcal{T}]}\max_{j\in\mathcal{I}}x_j(t-\eta)$ and $x_J(t)=\min_{\eta\in[0,\mathcal{T}]}\min_{j\in\mathcal{I}}x_j(t-\eta)$, respectively. A similar argument as in the proof of Lemma 6 shows that $\dot{V}_k\leq 0, k=1,2$. Hence, condition (5) in Theorem 4 is fulfilled.

Next, we have to find the sets E_{V_k} and $M_{V_k}, k=1,2$, according to (3) and (4). For every $\varphi\in E_{V_k}$, there is an $x_k^*\in\mathbb{R}$ such that $\max_{\eta\in[0,\mathcal{T}]}V_k(x(\varphi)(t-\eta))=x_k^*$ for all $t\geq 0$. Moreover, any $\varphi\in E_{V_k}$ satisfies $\dot{V}_k(x_t(\varphi))=0$ for any $t\geq 0$ whenever $V_k(x(\varphi)(t))=\max_{\eta\in[0,\mathcal{T}]}V_k(x(\varphi)(t-\eta))$, see Haddock and Terjéki [1983]. For V_1, this transforms into $\dot{V}_1=0$ if $x_I(t)=\max_{\eta\in[0,\mathcal{T}]}\max_{j\in\mathcal{I}}x_j(t-\eta)=x_1^*$. Furthermore, $\dot{V}_1=0$ if $x_I(t)=x_j(t-\tau_{jI}(t))$ for all j with $e_{jI}\in\mathcal{E}$. Hence, all parents v_j of v_I must fulfill $x_j(t-\tau_{jI}(t))=x_1^*$. Since g_{ji} and τ_{ji} are continuous, all $x_i, i\in\mathcal{I}$, are differentiable. As x_1^* is the maximum of all states and of all times t, this requires that $\dot{x}_j(t-\tau_{jI}(t))=0$. Thus, all the parents of the parents of v_I also satisfy $x_\kappa(t-\tau_{jI}(t)-\tau_{\kappa j}(t))=x_1^*$ for all $e_{jI}, e_{\kappa j}\in\mathcal{E}$. This can be continued up to any root of the underlying graph of the MAS. There exists at least one root because the graph contains a spanning tree. Since the delays τ_{ji} are sufficiently heterogeneous to avoid persisting oscillatory behavior, we have

$$E_{V_1}=\bigcup_{x_1^*\in\mathbb{R}}\left\{\varphi\in\mathcal{C}_{\mathbb{D}}:\begin{cases}\varphi_i(\eta)=x_1^*,\ \forall\, i\in\mathcal{I}_R\\ \varphi_i(\eta)\leq x_1^*,\ \forall\, i\in\mathcal{I}_{\overline{R}}\end{cases}\forall\,\eta\in[-\mathcal{T}^+,0]\right\}. \quad (10)$$

With similar arguments for E_{V_2}, we get

$$E_{V_2}=\bigcup_{x_2^*\in\mathbb{R}}\left\{\varphi\in\mathcal{C}_{\mathbb{D}}:\begin{cases}\varphi_i(\eta)=x_2^*,\ \forall\, i\in\mathcal{I}_R\\ \varphi_i(\eta)\leq x_2^*,\ \forall\, i\in\mathcal{I}_{\overline{R}}\end{cases}\forall\,\eta\in[-\mathcal{T}^+,0]\right\}. \quad (11)$$

where $x_2^*=\min_{\eta\in[0,\mathcal{T}]}\min_{j\in\mathcal{I}}x_j(t-\eta)$ for all $t\geq 0$.

Since both Lyapunov functions V_1 and V_2 satisfy the conditions of Theorem (4), we conclude that the consensus set $\Theta=E_{V_1}\cap E_{V_2}$ is asymptotically attracting to all solutions $x_t(\varphi)$ of (6) with $\varphi\in\mathcal{C}_{\mathbb{D}}$. □

5. CONSENSUS WITH SWITCHING TOPOLOGY

We now turn to MAS with dynamic topologies. Therefore, we assume a finite set of directed graphs $\{\mathcal{G}_p\}$ with $p\in\mathcal{P}=\{1,\ldots,P\}$. At any time t, one of the graphs $\mathcal{G}_p$ represents the topology of the communication network between the agents. The switching signal $\sigma:[0,\infty)\to\mathcal{P}$ determines the index of the active graph at time t. σ is piecewise constant from the right and non-chattering, i.e., there is a dwell-time $h>0$ between any two switching instants $t_{l+1}-t_l\geq h$ for all $l=1,2,\ldots$. We assume that the topology switches infinitely often because otherwise this problem could be solved as in Section 4 considering only the last active graph. We denote all switching times when graph p becomes active $t_{p_\nu},\ t_{p_{\nu+1}}>t_{p_\nu}$, i.e., $\sigma(t)=p$ for $t\in[t_{p_\nu},t_{p_\nu+1})$ with $\nu=1,2,\ldots$. The set $\mathcal{P}_\infty\subseteq\mathcal{P}$ is such that every graph $\mathcal{G}_p,\ p\in\mathcal{P}_\infty$, is infinitely often active, i.e., there are infinitely many switching times t_{p_ν}. Finally, we define the set of graphs that *persist over time* as the union graph $\mathcal{G}_\infty=\left(\mathcal{V},\bigcup_{p\in\mathcal{P}_\infty}\mathcal{E}_p\right)$.

The dynamics of the MAS with N agents and switching topology are

$$\dot{x}_i(t)=-k_i^{(\sigma)}\sum_{j=1}^{N}a_{ji}^{(\sigma)}g_{ji}\left(x_i(t)-x_j(t-\tau_{ji}(t))\right), \quad (12)$$

for all $i\in\mathcal{I}$ and with $k_i^{(p)}>0$ for all $i\in\mathcal{I}$ and all $p\in\mathcal{P}$. $A^{(p)}=[a_{ij}^{(p)}]$ is the adjacency matrix of graph $\mathcal{G}_p$. The initial condition is $x_0=\varphi$. The consensus set of the MAS (12) is Θ as defined in (7).

Theorem 8. Given a MAS consisting of N agents with dynamics (12)*, where g_{ji} satisfy Assumption 5, and with initial condition $\varphi\in\mathcal{C}_{\mathbb{D}}$, as well as an underlying switched network topology of directed graphs, such that the union graph $\mathcal{G}_\infty$ has a spanning tree, then the consensus set Θ of this MAS is asymptotically attracting.*

Proof. The proof is based on a common Lyapunov function argument to allow for the arbitrary switching and on Theorem 4 to prove attractivity of the consensus set Θ. Note first, that $\mathcal{C}_{\mathbb{D}}$ is positively invariant with respect to an arbitrarily switching system (12) because it is invariant with respect to every subsystem. Next, we briefly outline the remainder of the proof. In Part (i), we define two common Lyapunov Razumikhin function candidates V_1 and V_2 and prove that they satisfy condition (5). In order to determine the sets E_{V_k} and $M_{V_k}, k = 1, 2$, we adopt an invariance principle from Hespanha et al. [2005]. This requires the definition of two functionals $\overline{V}_k, k = 1, 2$, which are based on V_1 and V_2. Since the derivatives of $\overline{V}_k$ cannot be determined easily, we just distinguish between the two important cases $\dot{\overline{V}}_k = 0$ and $\dot{\overline{V}}_k < 0$ through simple rules in Part (ii) of the proof. Using these conditions, we apply the result from Hespanha et al. [2005] and determine the sets E_{V_k} and M_{V_k} in Part (iii).

Part (i): We consider the functions V_1 and V_2 from the proof of Theorem 7 as common Lyapunov-Razumikhin function candidates. The indices I and J are defined as in Theorem 7 to be the maximal and minimal states over all agents. The right-hand Dini derivatives of V_1 and V_2 along solutions of (12) are

$$\dot{V}_1^{(\sigma)}(x_t) = -k_I^{(\sigma)} \sum_{j=1}^{N} a_{jI}^{(\sigma)}\, g_{jI}\left(x_I(t) - x_j(t - \tau_{jI}(t))\right),$$

$$\dot{V}_2^{(\sigma)}(x_t) = k_J^{(\sigma)} \sum_{j=1}^{N} a_{jJ}^{(\sigma)}\, g_{jJ}\left(x_J(t) - x_j(t - \tau_{jJ}(t))\right).$$

Following the proof of Theorem 7, we know that $\dot{V}_k^{(p)} \leq 0, k = 1, 2$, for all $p \in \mathcal{P}$, whenever $V_k(x(t)) = \max_{\eta\in[0,\mathcal{T}]} V_k(x(t-\eta))$. We conclude that condition (5) is satisfied and that the functionals

$$\overline{V}_k(x_t) = \max_{\eta\in[0,\mathcal{T}]} V_k(x(t-\eta)),$$

$k = 1, 2$, are nonincreasing.

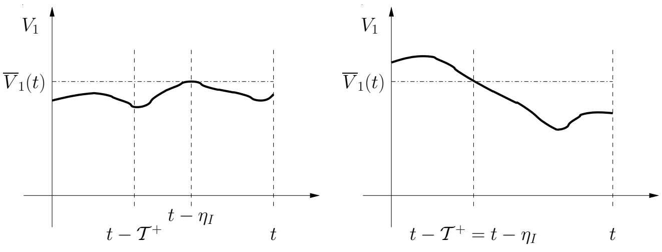
Part (ii): It would be quite difficult to calculate the right-hand Dini derivatives of $\overline{V}_k$ along solutions of (12). Instead, we determine simple rules to distinguish the two main cases $\dot{\overline{V}}_k = 0$ and $\dot{\overline{V}}_k < 0$. Therefore, we use the following notation: Let the values I_η and J_η indicate the indices at each time t that satisfy $x_{I_\eta}(t-\eta) = \max_{i\in\mathcal{I}} x_i(t-\eta)$ and $x_{J_\eta}(t-\eta) = \min_{i\in\mathcal{I}} x_i(t-\eta)$, respectively. If there are several possible indices, we chose any one of them. Let $\eta_I, \eta_J \in [0, \mathcal{T}]$ be such that

$$x_{I_{\eta_I}}(t-\eta_I) = \max_{\eta\in[0,\mathcal{T}]} \max_{i\in\mathcal{I}} x_i(t-\eta), \qquad (13)$$

$$x_{J_{\eta_J}}(t-\eta_J) = \min_{\eta\in[0,\mathcal{T}]} \min_{i\in\mathcal{I}} x_i(t-\eta), \qquad (14)$$

i.e., $\overline{V}_1(x_t) = x_{I_{\eta_I}}(t-\eta_I)$ and $\overline{V}_2(x_t) = -x_{J_{\eta_J}}(t-\eta_J)$. Clearly, η_I and η_J are changing with time and there might be several values $\eta_I, \eta_J \in [0, \mathcal{T}]$ that satisfy (13) and (14), respectively. Now, we can state the following about the derivatives of $\overline{V}_k, k = 1, 2$, along solutions of (12):

- $\dot{\overline{V}}_k^{(p)}(x_t) \leq 0$ for all $p \in \mathcal{P}$.
- $\dot{\overline{V}}_1^{(p)}(x_t) = 0$ if and only if there exists an $\eta_I \in [0, \mathcal{T})$ that satisfies (13), see Figure 1(a).
- $\dot{\overline{V}}_1^{(p)}(x_t) < 0$ if and only if $\eta_I = \mathcal{T}$ satisfies (13) and there does not exist an $\eta_I^* \in [0, \mathcal{T})$ that satisfies (13), see Figure 1(b).



(a) (b)

Fig. 1. Exemplary Lyapunov function V_1 and exemplary Lyapunov functional $\overline{V}_1$, see text for details

- $\dot{\overline{V}}_2^{(p)}(x_t) = 0$ if and only if there exists an $\eta_J \in [0, \mathcal{T})$ that satisfies (14).
- $\dot{\overline{V}}_2^{(p)}(x_t) < 0$ if and only if $\eta_J = \mathcal{T}$ satisfies (14) and there does not exist an $\eta_J^* \in [0, \mathcal{T})$ that satisfies (14).

Part (iii): With these conditions, we can now turn to an invariance principle for switched RFDEs. Recall the definition of the switching times t_{p_ν} given above, such that $\sigma(t) = p$ for $t \in [t_{p_\nu}, t_{p_\nu+1})$ with $\nu = 1, 2, \ldots$. Since two switching times t_l are separated by a dwell time h, we have

$$\overline{V}_k(x(t_l)) - \overline{V}_k(x(0)) = \sum_{p=1}^{P} \sum_{\nu=1}^{\nu_p^*} \int_{t_{p_\nu}}^{t_{p_\nu+1}} \dot{\overline{V}}_k^{(p)}(x_t)dt, \quad (15)$$

with ν_p^* such that $t_{p_{\nu_p^*}+1} \leq t_l$ and $t_{p_{\nu_p^*+1}} > t_l$. From our former arguments, we know that $\dot{\overline{V}}_k^{(p)}(x_t) \leq 0$. Moreover, we know that the left hand side of (15) converges to a finite value for $t_l \to \infty$ because $\overline{V}_k$ is nonincreasing and bounded from below. Following the proof of Theorem 7 in Hespanha et al. [2005], we conclude that $\dot{\overline{V}}_k^{(p)}(x_t) \to 0$ as $t \to \infty$ for all $p \in \mathcal{P}_\infty$ and $k = 1, 2$. Clearly. $\dot{\overline{V}}_k^{(p^*)}(x_t) \to 0$ is not necessary for those $p^* \in \mathcal{P}\backslash\mathcal{P}_\infty$ because these graphs are only active a finite number of times, i.e., $\nu_{p^*}^*$ does not go to infinity for $t_l \to \infty$.

Now, we have to determine the sets

$$E_{V_k} = \left\{\varphi \in \mathcal{C}_{\mathbb{D}} : \dot{\overline{V}}_k^{(p)}(x_t(\varphi)) = 0 \quad \forall\, p \in \mathcal{P}_\infty,\ t \geq 0\right\},$$

$k = 1, 2$, in order to conclude that $x_t \to E_{V_k}$ as $t \to \infty$. We first consider E_{V_1}. For every $\varphi \in E_{V_1}$, there exists an $x_1^* \in \mathbb{R}$ such that $\overline{V}_1(x_t(\varphi)) = x_1^*$, i.e., $x_1^* = \max_{\eta\in[0,\mathcal{T}]} \max_{i\in\mathcal{I}} x_i(\varphi)(t-\eta)$, for all $t \geq 0$. Following our former arguments, we know that $\dot{\overline{V}}_1^{(p)}(x_t) = 0$ if and only if there exists an $\eta_I \in [0, \mathcal{T})$ that satisfies (13). Hence, we know that the right-hand Dini derivatives of the Razumikhin candidate V_1 satisfy $\dot{V}_1^{(p)}(x(t-\eta_I)) = 0$ (see Figure 1(a)) and this requires that

$$x_{I_{\eta_I}}(t-\eta_I) - x_j(t-\eta_I-\tau_{jI_{\eta_I}}(t-\eta_I)) = 0, \qquad (16)$$

for all j with $e_{jI_{\eta_I}} \in \mathcal{E}_p$. Since $x_{I_{\eta_I}}(t-\eta_I) = x_1^*$ and since x_1^* is the maximum of all states at all times, we conclude that $\dot{x}_j(t-\eta_I-\tau_{jI_{\eta_I}}(t-\eta_I)) = 0$ for all $e_{I_{j\eta_I}} \in \mathcal{E}_p$. The same arguments hold for all $p \in \mathcal{P}_\infty$. Note that x_1^* depends on φ but not on p. Following the proof of Theorem 7, we conclude that, if the union graph $\mathcal{G}_\infty$ has at least one root $v_i, i \in \mathcal{I}_R$, then E_{V_1} is given by (10). We know that $\mathcal{G}$ has at least one root because it contains a spanning tree. We use similar arguments for E_{V_2} and obtain (11). As in Theorem 7, we conclude that $x_t \to \Theta \subseteq E_{V_1} \cap E_{V_2}$ for $t \to \infty$. □

This result shows that consensus is reached among nonlinear, locally passive MAS even if they exchange information over communication networks with an arbitrarily switching network topology. The assumption that $\mathcal{G}_\infty$ has a spanning tree resembles the *connected-over-time* assumption in previous works, e.g., Jadbabaie et al. [2003], Moreau [2005]. We have shown in Münz et al. [2008] that the new condition is not more restrictive than the former conditions. Note that both Theorem 7 and 8 apply also for MAS with a leader following the same arguments as in Münz et al. [2008].

6. CONCLUSIONS

Theorem 7 and 8 provide conditions for a class of nonlinear, locally passive multi-agent systems with time-varying communication delays to reach consensus. These conditions hold for for arbitrary delay sizes and variations as well as both for fixed and switching network topologies. We only require that the underlying graph contains a spanning tree for the fixed topology case; and in the case of switching graphs, we require that the union graph of the set of graphs that persist over time contains a spanning tree. These results are obtained using an invariance principle for Lyapunov-Razumikhin functions.

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