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Solving The Optimal PWM Problem for Odd Symmetry Waveforms ⋆

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Abstract: Optimal pulse width modulation (PWM) problem is an established method of generating PWM waveforms with low baseband distortion. In this paper we focused on computation of optimal switching angles of a PWM waveform for generating general odd symmetric waveforms with applications in control. We introduce an exact and fast algorithm with the complexity of only $\mathcal{O}(n^2)$ arithmetic operations. This algorithm is based on transformation of appropriate trigonometric equations for each harmonics to a polynomial system of equations that is transferred to a special system of power sums. The solution of this system is carried out by modification of Newton's identity via Padé approximation and orthogonal polynomials theory and property of symmetric polynomials. Finally, the optimal switching sequence is obtained by computing the zeros of two orthogonal polynomials in one variable.

Keywords: Polynomial methods; optimal PWM; selected harmonics elimination; Newton identities; Padé approximation; orthogonal polynomials; composite power sums.

1. INTRODUCTION

The problem of optimal PWM waveform (or some times called selected harmonic elimination (SHE) problem) is tackled in this paper. The main aim is to compute switching angles for a odd symmetry PWM waveform so as the arbitrary required output waveform is obtained after its filtration. The odd symmetry T periodic waveform (or function) $f(t)$ is defined as

$$f(t) = -f(-t) \qquad \ldots \text{ odd symmetry,} \tag{1a}$$
$$f(t) = f(t+T) \qquad \ldots \text{ periodicity.} \tag{1b}$$

Lets see the concrete example of function $f(t) = 2\sin t - \sin 2t$ whose figure is depicted in (Fig. 1). We focused

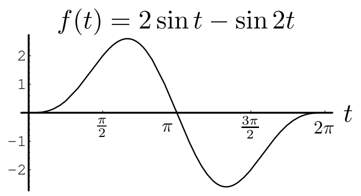

Fig. 1. Example of odd symmetry 2π periodic function $f(t) = 2\sin t - \sin 2t$.

on finding the switching $\alpha_1, \alpha_2, \ldots, \alpha_n$ for odd symmetry periodic PWM waveform $p(t)$ see (Fig. 2) so that its baseband frequency is equal to frequency spectrum of $f(t)$ and several following higher harmonics are equal

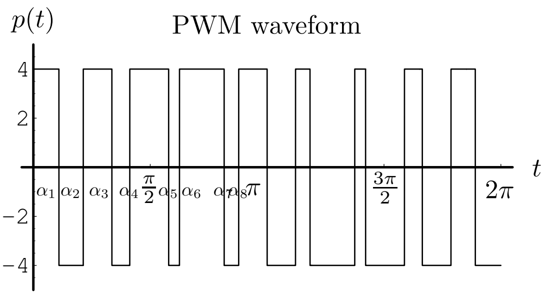

Fig. 2. Optimal odd PWM waveform for generation of odd symmetry periodic signal $f(t) = 2\sin t - \sin 2t$.

to zero see (Fig. 3). Amplitude frequency spectrum of

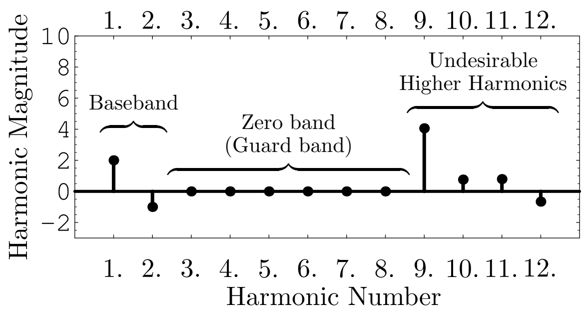

Fig. 3. Amplitude frequency spectrum $p(t)$.

signal $f(t)$ is $\{a_1 = 2, a_2 = -1\}$ (i.e. the first and the second harmonics; the following harmonics are zero). The

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baseband frequency spectrum for $p(t)$ is chosen the same like the frequency spectrum $f(t)$ and then following six harmonics (from the third to the eight) are put equal to zero. Finally, it is required that $\{b_1 = 2, b_2 = -1, b_3 = \cdots = b_8 = 0\}$ holds for the first eight harmonics of $p(t)$. The other higher harmonics $p(t)$, that are impossible to change by computing of switching angels $\alpha_1, \ldots, \alpha_8$ are $\{b_9 = 4.06, b_{10} = .76, b_{11} = .79, b_{12} = -.66, \ldots\}$. These higher harmonics are necessary to be cancelled by suitable filter. It is clear, that the wider band of zero harmonics is, the better result of filtration will be.

1.1 Other Methods

A lot of methods for the optimal PWM problem were developed up to now. The most effective method for single-phase quarter-symmetry inverter is described in Czarkowski et al. [2002], Chudnovsky and Chudnovsky [1999]. This method is based on trigonometric identity where original trigonometric system is transformed to the polynomial system of specific structure leading to the odd power sums system. The problem result in construction special set of one variable polynomials and computing their zeros. It is noticeable that these polynomials are orthogonal and recurrence formula is derived for them. The solution is based on diagonal Padé approximation. In case of single-phase inverter for given modulation index only one or none solution exists.

2. OPTIMAL PWM PROBLEM SOLUTION

In this study we restrict ourselves to computation of the optimal switching sequence of single-phase odd symmetry periodic PWM waveform $p(t)$ for generally odd symmetry periodic waveforms $f(t)$. Each T periodic waveform can be described using sine Fourier series

$$f(t) \sim \sum_{k=1}^{\infty} a_k \sin k\omega t. \tag{2}$$

The same for odd symmetry T periodic PWM waveform $p(t)$ that is depicted in (Fig. 2). Its Fourier series is also odd

$$p(t) \sim \sum_{k=1}^{\infty} b_k \sin k\omega t, \tag{3}$$

where

$$b_k = \frac{4}{T}\int_0^{\frac{T}{2}} p(t) \sin k\omega t \; \mathrm{d}t, \; k = 1, 2, \ldots \tag{4}$$

and $\omega = 2\pi/T$. For $p(t)$ according to (Fig. 2) is

$$b_k(\alpha_1, \ldots, \alpha_n) = \frac{4A}{k\pi}\left(o_{n+k} + \sum_{i=1}^{n}(-1)^i \cos \omega k \alpha_i\right),$$

where

$$0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n < T/2 \tag{5}$$

are switching times and n is number of pulses in half-period. To simplify the notation, we used and now define the following symbol as the *test of odd parity*

$$o_m = \frac{1-(-1)^m}{2} = \begin{cases} 0 & \text{for even } m, \\ 1 & \text{for odd } m. \end{cases}$$

Without loss of generality we restricted only to period 2π because of more simple notation. All the results α_i are possible to resolve to origin period according to $\frac{T}{2\pi}\alpha_i$.

Then the solution of optimal PWM problem is given by the following set of equations

$$b_k(\overline{\alpha}) = \begin{cases} a_k & \text{for } k = 1, \ldots, n_C \\ 0 & \text{for } k = n_C + 1, \ldots, n, \end{cases} \tag{6}$$

where $n = n_C + n_E$ is number of pulses in half period, $\overline{\alpha} = (\alpha_1, \ldots, \alpha_n)$ are unknown variables, n_C is a number of terms a_k describing a function $f(t)$ and generating a baseband (controlled harmonics). The last n_E equations describe zero band of higher harmonics (eliminated harmonics). The solution certainly must respect condition (5).

2.1 Polynomial Equations

The system of equations (6) is nonlinear trigonometric. Finding its solution is possible only with using numerical iterative methods Sun and Grotstollen [1994] and their convergence is considerably dependent on initial iteration. Regarding these facts let us do simplifying by converting to polynomial system by substitution for Chebyshev polynomials of first kind. This substitution for cosine was used for example in Czarkowski et al. [2002] for solution of optimal single-phase quarter-wave three-level PWM waveform. The substitution is done according to following trigonometric identity (multiple-angle formula)

$$\cos(n\alpha) = T_n(\cos(\alpha)),$$

where T_n is n-degree Chebyshev polynomial of first kind and it is hold $T_n(x) = t_{n,0} + t_{n,1}x + \cdots + t_{n,n}x^n$. Then let us convert k-th harmonics $b_k(\overline{\alpha})$ to polynomial

$$b_k(\overline{x}) = \frac{4A}{\pi k}\left(o_{n+k} + \sum_{i=1}^{n}(-1)^i T_k(x_i)\right) \tag{7}$$

in variables $\overline{x} = (x_1, \ldots, x_n)$. Now the trigonometric system (6) can be transform to polynomial system

$$b_k(\overline{x}) = \begin{cases} a_k & \text{for } k = 1, \ldots, n_C \\ 0 & \text{for } k = n_C + 1, \ldots, n. \end{cases} \tag{8}$$

This system is already possible to solve using other methods. For example Groebner basis method, or eliminating method based on solving resultant and a lot of others. The origin unknowns α_i from x_i are obtained by

$$\alpha_i = \arccos x_i, \quad i = 1, \ldots, n. \tag{9}$$

If inequality (5) is given for α_i, then for x_i must hold

$$-1 < x_n < \cdots < x_2 < x_1 < 1. \tag{10}$$

Now, the polynomial system (8) is possible to transforming by substitution for power sums p_i to the following linear system

$$b_{2i-1}(\overline{p}) = \frac{4A}{(2i-1)\pi}\left(o_{n+2i-1} - \sum_{j=1}^{i} t_{2i-1,2j-1}p_{2j-1}\right) = a_{2i-1}$$

$$i = 1, 2, \ldots, \left\lceil \frac{n_C}{2} \right\rceil,$$

$$b_{2i}(\overline{p}) = \frac{4A}{2i\pi}\left[o_{n+2i} - \left((-1)^i o_n + \sum_{j=1}^{i} t_{2i,2j}p_{2j}\right)\right] = a_{2i}$$

$$i = 1, 2, \ldots, \left\lfloor \frac{n_C}{2} \right\rfloor, \tag{11}$$

$$b_{2i-1}(\overline{p}) = o_{n+2i-1} - \sum_{j=1}^{i} t_{2i-1,2j-1} p_{2j-1} \qquad = 0$$
$$i = \left\lceil \frac{n_C}{2} \right\rceil + 1, \ldots, \left\lceil \frac{n}{2} \right\rceil,$$
$$b_{2i}(\overline{p}) = o_{n+2i} - \left((-1)^i o_n + \sum_{j=1}^{i} t_{2i,2j} p_{2j} \right) = 0$$
$$i = \left\lfloor \frac{n_C}{2} \right\rfloor + 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor,$$

where $\overline{p} = (p_1, \ldots, p_n)$ are special power sums

$$p_i = \sum_{j=1}^{n} (-1)^{j+1} x_j^i = x_1^i - \cdots + (-1)^{n+1} x_n^i. \tag{12}$$

It is clear that this system of equations is in lower triangular form and variables with odd and even indexes are independent on each other. Therefore the system can be solved like as two separate systems of linear equations. In addition this system is special, terms are coefficient of Chebyshev polynomials $t_{n,i}$. Therefore, let us solve this problem without generating the coefficients of Chebyshev polynomials and sequentially solving the system (11) using standard algorithm for triangular matrices. The solution is

$$p_{2i} = o_n - 2^{-2i} \frac{\pi}{A} \sum_{k=1}^{i} \binom{2i}{i-k} k \; a_{2k}, \quad i = 1, 2, \ldots, \left\lfloor \frac{n_c}{2} \right\rfloor,$$
$$p_{2i} = o_n - 2^{-2i} \frac{\pi}{A} \sum_{k=1}^{\lfloor \frac{n_c}{2} \rfloor} \binom{2i}{i-k} k \; a_{2k}, \quad i = \left\lfloor \frac{n_c}{2} \right\rfloor + 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor$$
$$p_{2i-1} = o_{n+1} - 2^{-2i} \frac{\pi}{A} \sum_{k=1}^{i} \binom{2i-1}{i-k} (2k-1) \; a_{2k-1}, \tag{13}$$
$$i = 1, 2, \ldots, \left\lceil \frac{n_c}{2} \right\rceil,$$
$$p_{2i-1} = o_{n+1} - 2^{-2i} \frac{\pi}{A} \sum_{k=1}^{\lceil \frac{n_c}{2} \rceil} \binom{2i-1}{i-k} (2k-1) \; a_{2k-1},$$
$$i = \left\lceil \frac{n_c}{2} \right\rceil + 1, \left\lceil \frac{n_c}{2} \right\rceil + 2, \ldots, \left\lceil \frac{n}{2} \right\rceil.$$

(We proof this according to the Gauss-Banachiewitz decomposition for Chebyshev polynomials of the first kind.) Thus the solution in unknown p_i is easily obtained in $\mathcal{O}(nn_C)$ steps of operations. The solution of modified power sums system (12), where p_i is according to (13), leads back to unknowns x_i.

2.2 Modified power sums

The solution of optimal PWM problem for odd waveforms dependence only on computing the modified power sums system (12) where the right hand sides are solved according to (13). This system is very similar to standard power sums $\sum_{i=1}^{n} x_i^k = p_k, \quad k = 1, \ldots, n$ that is easily solvable by Newton's identity.

For following steps it is better to solve this type of modified system of power sums

$$p_j(y_1, \ldots, y_n) = \sum_{i=1}^{k} y_i^j - \sum_{i=k+1}^{n} y_i^j, \quad j = 1, \ldots, n, \tag{14}$$

where $k \leq \lfloor \frac{n}{2} \rfloor$. When $k > \lfloor \frac{n}{2} \rfloor$, we can multiply the equation system (14) by -1 and convert it to the case $k < \lfloor \frac{n}{2} \rfloor$. The system (14) can be also easily obtained by reassorting variables from the system (12).

Generally, the total number of solutions (14) is $k!(n-k)!$. They all are combinations of two sets that origins by permutation of elements of vectors $y^+ = \{y_1, \ldots, y_k\}$ and $y^- = \{y_{k+1}, \ldots, y_n\}$. Therefore, for modified power sums (14), where $\sum_{i=1}^{n} y_i^j$ and $\sum_{i=k+1}^{n} y_i^j$ are symmetric polynomials, is given

$$p_j(y_1, \ldots, y_k, y_{k+1}, \ldots, y_n) =$$
$$p_j(y_{\pi_1(1)}, \ldots, y_{\pi_1(k)}, y_{\pi_2(k+1)}, \ldots, y_{\pi_2(n)}).$$

$\{y_{\pi_1(1)}, \ldots, y_{\pi_1(k)}\}$ is arbitrary permutation of the set y^+ and $\{y_{\pi_2(k+1)}, \ldots, y_{\pi_2(n)}\}$ is arbitrary permutation of the set y^-.

2.3 Converting to optimal PWM problem

The equation (12) is converted to (14) by the following way. If n is even number then $n/2$ variables with sign plus and the same number with sign minus are in system (12). Therefore, converting to (14) is easily done by introducing new variables according to

$$y^+ = \{y_i = x_{2i-1}, \; i = 1, \ldots, k\} \tag{15}$$
$$y^- = \{y_{i+n/2} = x_{2i}, \; i = 1, \ldots, k\}$$

and $k = \frac{n}{2}$. If n is odd number then $\lceil \frac{n}{2} \rceil$ variables with sign plus and $\lfloor \frac{n}{2} \rfloor$ variables with sign minus are in the system (12). Therefore, converting similar to case with even n leads to $k > \lfloor \frac{n}{2} \rfloor$ that is not in agreement with condition $k \leq \lfloor \frac{n}{2} \rfloor$ of equation (14). Therefore, each equation is necessary to multiply by -1. Then this substitution can be done

$$y^+ = \left\{ y_i = x_{2i}, \; i = 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \right\} \tag{16}$$
$$y^- = \left\{ y_{i+\lfloor n/2 \rfloor} = x_{2i-1}, \; i = 1, \ldots, \left\lceil \frac{n}{2} \right\rceil \right\}$$

with $k = \lfloor \frac{n}{2} \rfloor$. The signs of right sides must be changed $p_i := -p_i$. The solution $x_1, \ldots, x_n$ of optimal PWM problem is obtained as follow. From all the solution only one is chosen. The one that is in agreement with the condition (10), it means that all terms y^+ and y^- are real numbers belonging to interval $(-1, 1)$. As all terms y^+ and y^- can be permutated, then terms y^+ and y^- are sorted in descending order, it is $y^+ := \{1 > y_1 > \cdots > y_k > -1,\}$, $y^- := \{1 > y_{k+1} > \cdots > y_n > -1\}$. Therefore we get $x = \{x_1, \ldots, x_n\} = \{y_1, y_{k+1}, y_2, y_{k+2}, \ldots, y_n, y_k\}$ by reassorting variables $x_1, \ldots, x_n$ according to (15) for even n. For odd n the solution $x_1, \ldots, x_n$ we get by reassorting (16) that is $x = \{x_1, \ldots, x_n\} = \{y_{k+1}, y_1, y_{k+2}, y_2, \ldots, y_k, y_n\}$. Finally the condition (10) for x must hold.

2.4 Solving Modified Power Sums

In this section the algorithm for solution the modified system of power sums (14) is described. The solution is derived from Padé approximation Chudnovsky and Chudnovsky [1999], Czarkowski et al. [2002], Brezinski [2002] and Baker and Graves-Morris [1996]. The problem of this system was solved in Wu and Hadjicostis [2005] but the authors did not use Padé approximation and theory of orthogonal polynomials that play a crucial role in solution of the whole problem.

We will find the solution as the zeros of polynomials

$$P_k(y) = \prod_{i=1}^{k}(y-y_i) = y^k + p_{k,k-1}y^{k-1} + \cdots + p_{k,0}, \tag{17}$$

$$Q_{n-k}(y) = \prod_{i=1}^{n-k}(y-y_{i+k}) = y^{n-k} + q_{n-k,n-k-1}y^{n-k-1} + \dots \cdots + q_{n-k,0}. \tag{18}$$

Then let us do logarithmic derivative

$$\frac{P_k(y)}{Q_{n-k}(y)} = \frac{\prod_{i=1}^{k}(y-y_i)}{\prod_{i=1}^{n-k}(y-y_{i+k})}$$

and we get

$$\frac{P_k'(y)}{P_k(y)} - \frac{Q_{n-k}'(y)}{Q_{n-k}(y)} = \sum_{i=1}^{k}\frac{1}{y-y_i} - \sum_{i=1}^{n-k}\frac{1}{y-y_{i+k}}.$$

The series $\frac{1}{y-z}$ at $y=\infty$ is $\sum_{j=0}^{\infty}\frac{z^j}{y^{j+1}}$. Then

$$\frac{P_k'(y)}{P_k(y)} - \frac{Q_{n-k}'(y)}{Q_{n-k}(y)} = \sum_{j=0}^{\infty}\frac{p_j^+}{y^{j+1}} - \sum_{j=0}^{\infty}\frac{p_j^-}{y^{j+1}}, \tag{19}$$

where $p_j^+ = \sum_{i=1}^{k} y_i^j,\quad p_j^- = \sum_{i=1}^{n-k} y_{i+k}^j$ with $p_j = p_j^+ - p_j^-$. Therefore the previous equation (19) is in the form

$$\frac{P_k'(y)}{P_k(y)} - \frac{Q_{n-k}'(y)}{Q_{n-k}(y)} = \sum_{j=0}^{\infty}\frac{p_j}{y^{j+1}}. \tag{20}$$

By integrating (20) we have

$$\frac{P_k(y)}{Q_{n-k}(y)} = y^{2k-n}\exp(-\sum_{j=1}^{\infty}\frac{p_j}{jy^j}). \tag{21}$$

If we solve the series of right side (21) at $y=\infty$ then the problem leads to the Padé approximation.

2.5 SHE problem and Padé approximation

Now we must solve the unknown coefficients of polynomials $P_k(y)$ and $Q_{n-k}(y)$ usig Padé approximation of function

$$y^{2k-n}\exp(-\sum_{j=1}^{\infty}\frac{p_j}{jy^j}) = y^{2k-n}\exp\sum_{j=1}^{\infty}v_j y^{-j},$$

where $v_j = \frac{p_j}{j},\ j=1,2,\dots$.Therefore it is necessary to solve the series expansion of function

$$g(y) = \exp\sum_{j=1}^{\infty}v_j y^{-j} = \sum_{i=0}^{\infty}g_i y^{-i} \text{ at } y=\infty. \tag{22}$$

The solution is carried out according to Knuth [1997] and it is $g_0 = 1,\ g_k = -\frac{1}{k}\sum_{j=1}^{k}p_j g_{k-j},\ k=1,\dots,\infty$, that is computation of elementary symmetric polynomials according to Newton's formula.

Detail form of equation (21) considering (22) leads to

$$\begin{aligned} P_k(y) &= Q_{n-k}(y)y^{2k-n}(g_0 + g_1y^{-1} + g_2y^{-2} + \dots) = \\ &= (y^{n-k} + q_{n-k,n-k-1}y^{n-k-1} + \cdots + q_{n-k,0})\ y^{2k-n}\cdot \\ &\quad\cdot(g_0 + g_1y^{-1} + g_2y^{-2} + \dots) = \\ &= y^k + p_{k,k-1}y^{k-1} + \cdots + p_{k,0}. \end{aligned}$$

First, let us consider the case when

n odd number and $k = \lfloor \frac{n}{2} \rfloor$: The Padé approximation $[k, k+1]_{y^{-1}g}(y) = \frac{P_k(y)}{Q_{k+1}(y)}$ is solved. The multiplying of the previous form and comparing the coefficients of the same powers (the coefficients with negative power are equal zero) leads to the following system

$$\left[\begin{array}{cccc|c} g_0 & g_1 & \cdots & & g_{k+1} \\ g_1 & & & \iddots & \vdots \\ \vdots & \iddots & & & g_{2k+1} \\ \hline g_{k+1} & \cdots & & g_{2k+1} & g_{2(k+1)} \end{array}\right] \cdot \left[\begin{array}{c} q_{k+1,0} \\ \vdots \\ q_{k+1,k} \\ \hline q_{k+1,k+1} \end{array}\right] = \left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ \hline K_k \end{array}\right], \tag{23}$$

where $q_{k+1,k+1}$ is leading coefficient of polynomial $Q_{k+1}(y)$ and equal to one 1. It is normalization condition, in that case, the polynomials are called monic. Then the last equation of previous system is reduced. K_k is appropriate constant. Therefore we solve the system

$$\begin{bmatrix} g_0 & g_1 & \cdots & g_k \\ g_1 & & \iddots & \vdots \\ \vdots & \iddots & & g_{2k} \\ g_k & \cdots & g_{2k} & g_{2k+1} \end{bmatrix} \cdot \begin{bmatrix} q_{k+1,0} \\ q_{k+1,1} \\ \vdots \\ q_{k+1,k} \end{bmatrix} = \begin{bmatrix} -g_{k+1} \\ -g_{k+2} \\ \vdots \\ -g_{2k+1} \end{bmatrix}. \tag{24}$$

It is a linear system with Toeplitz structure (Hankel matrix) in size $[k+1\times k+1]$. For solution it is possible to use special fast Levinson - Durbin algorithm with complexity $\mathcal{O}(n^2)$ (for more details see Golub and Loan [1996])) or even super fast algorithm with complexity $\mathcal{O}(n\ln n^2)$ (see Bini and Pan [1994]). Standard algorithms for general linear systems need $\mathcal{O}(n^3)$ complexity operations.

Unknown polynomial coefficients $P_k(y)$ are easily obtained from known polynomial coefficients $Q_{k+1}(y)$ as follows

$$\begin{bmatrix} p_{k,k-1} \\ p_{k,k-2} \\ \vdots \\ p_{k,0} \end{bmatrix} = \begin{bmatrix} 0 & \dots & 0 & g_0 \\ \vdots & \iddots & \iddots & g_1 \\ 0 & \iddots & \iddots & \vdots \\ g_0 & g_1 & \dots & g_{k-1} \end{bmatrix} \cdot \begin{bmatrix} q_{k+1,1} \\ q_{k+1,2} \\ \vdots \\ q_{k+1,k} \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_k \end{bmatrix}. \tag{25}$$

Again, we have system with special structure. It is lower triangular Toeplitz matrix and its multiplying is easily done by fast algorithms with complexity $\mathcal{O}(n\log n)$ (for more details see Bini and Pan [1994].)

n even number and $k = \frac{n}{2}$: The procedure is similar to previous section. Lets derive appropriate equations for computing the coefficients of polynomials $P_k(y)$ and $Q_k(y)$. In this case it is $y^{2k-n} = 1$, e.i. Padé approximation is solved $[k,k]_g(y) = \frac{P_k(y)}{Q_k(y)}$. The coefficients of polynomial $Q_k(y)$ is solved according to

$$\begin{bmatrix} g_1 & g_2 & \cdots & g_k \\ g_2 & & \iddots & \vdots \\ \vdots & \iddots & & g_{2k-1} \\ g_k & \cdots & g_{2k-1} & g_{2k} \end{bmatrix} \cdot \begin{bmatrix} q_{k,0} \\ q_{k,1} \\ \vdots \\ q_{k,k-1} \end{bmatrix} = \begin{bmatrix} -g_{k+1} \\ -g_{k+2} \\ \vdots \\ -g_{2k} \end{bmatrix} \tag{26}$$

and the coefficients of polynomial $P_k(y)$ are obtained as follows

$$\begin{bmatrix} p_{k,k-1} \\ p_{k,k-2} \\ \vdots \\ p_{k,0} \end{bmatrix} = \begin{bmatrix} 0 & \dots & 0 & g_0 \\ \vdots & \iddots & \iddots & g_1 \\ 0 & \iddots & \iddots & \vdots \\ g_0 & g_1 & \dots & g_{k-1} \end{bmatrix} \cdot \begin{bmatrix} q_{k,0} \\ q_{k,1} \\ \vdots \\ q_{k,k-1} \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_k \end{bmatrix}. \tag{27}$$

2.6 The SHE problem and orthogonal polynomials

The theory of orthogonal polynomials and Padé approximation leads to the fact that founded polynomials are orthogonal and other important formulas and theorems are characteristic for them, for more details see Brezinski [2002] or Bultheel and Van Barel [1997], Baker and Graves-Morris [1996]. Comparing the systems (23,24) and the system of moments for orthogonal polynomials (see Brezinski [2002]) determine that polynomials $Q_{n-k}(y)$ a $P_k(y)$ (see (17)) are orthogonal. Therefore the following formulas hold:

2.7 Three-term recurence formula for polynomials $Q(y)$ a $P(y)$

The three-term recurrence formula for odd n is

$$Q_k(y) = (y + b_k)Q_{k-1}(y) - c_kQ_{k-2}(y) \tag{28}$$

where

$$b_k = -\frac{\mathcal{L}[yQ_{k-1}^2(y)]}{K_{k-1}} \tag{29}$$

$$c_k = \frac{K_{k-1}}{K_{k-2}} \tag{30}$$

and where

$$K_{-1} = K_0 = 1,\ K_k = \sum_{i=0}^{k} g_{k+i}q_{k,i}. \tag{31}$$

The initial conditions are $Q_{-1}(y) = 0$, $Q_0(y) = 1$, and linear moment functional in (29) is $\mathcal{L}[y^k] = g_k$. The linear moment functional $\mathcal{L}$ of polynomial is defined follows $\mathcal{L}\Big[\sum_{i=0}^{n} l_iy^i\Big] = \sum_{i=0}^{n} l_i\mathcal{L}[y^i]$.

The polynomial $P_k(y)$ is associated polynomial to $Q_{k+1}(y)$ in Padé approximation $[k, k+1]_{y^{-1}g}(y) = \frac{P_k(y)}{Q_{k+1}(y)}$. The polynomial $P_k(y)$ is sometimes called the polynomial of second kind and undergoes to this recurrence formula

$$P_k(y) = (y + b_k)P_{k-1}(y) - c_kP_{k-2}(y) \tag{32}$$

with the different initial conditions $P_{-1}(y) = -1$, $P_0(y) = 0$ and b_k, c_k are according to (29,30) and (31).

The three-term recurrence formula for even n is similar previous patterns. According to equation (26) it is adjacent family of orthogonal polynomials in Padé approximation $[k, k]_g(y) = \frac{P_k(y)}{Q_k(y)}$ and is characterized by this recurrent formula

$$Q_k(y) = (y + b_k)Q_{k-1}(y) - c_kQ_{k-2}(y) \tag{33}$$

where b_k, c_k are according to (29,30) with

$$K_{-1} = 1, K_0 = g_1,\ K_k = \sum_{i=0}^{k} g_{k+i+1}q_{k,i} \tag{34}$$

and initial conditions are $Q_{-1}(y) = 0$, $Q_0(y) = 1$ and linear moment functional in (29) is $\mathcal{L}[x^k] = g_{k+1}$.

Finding the recurrent formula for polynomial $P_k(y)$ is more difficult due to the fact that $P_k(y)$ is not associated polynomial to the $Q_k(y)$ (because the degree of $P_k(y)$ is same as degree of $Q_k(y)$). It is hold

$$P_k(y) = Q_k(y) + \mathrm{assoc}(Q_k(y)).$$

Therefore it is

$$P_k(y) = (y + b_k)P_{k-1}(y) - c_kQ_{k-2}(y) \tag{35}$$

where b_k, c_k are according to (29,30) and (34) and initial conditions are

$$P_{-1}(y) = Q_{-1}(y) + \mathrm{assoc}\{Q_{-1}(y)\} = 0 + (-1) = -1,$$
$$P_0(y) = Q_0(y) + \mathrm{assoc}\{Q_0(y)\} = 1 + 0 = 1$$

and in (29) is $\mathcal{L}[x^k] = g_{k+1}$.

2.8 Determinantal formula

As solved polynomials are orthogonal, they undergo to other special patterns. These polynomials are possible to solve as determinants of special polynomial matrices.

Determinantal formula for odd n:

$$Q(y)_{k+1} = D_{q_{k+1}} \det \begin{bmatrix} g_0 & g_1 & \cdots & g_k & g_{k+1} \\ g_1 & & \iddots & \iddots & \vdots \\ \vdots & \iddots & \iddots & & g_{2k-1} \\ g_k & g_{k+1} & \cdots & g_{2k-1} & g_{2k} \\ 1 & y & \cdots & y^k & y^{k+1} \end{bmatrix} \tag{36}$$

and

$$P(y)_k = D_{p_k} \det \begin{bmatrix} g_0 & g_1 & \cdots & g_{k-1} & g_k \\ g_1 & & \iddots & \iddots & \vdots \\ \vdots & \iddots & \iddots & & g_{2k-2} \\ g_{k-1} & g_k & \cdots & g_{2k-2} & g_{2k-1} \\ 0 & 1 & \cdots & \sum_{i=0}^{k-1} g_iy^{k-i-1} & \sum_{i=0}^{k} g_iy^{k-i} \end{bmatrix} \tag{37}$$

where $k = \lfloor \frac{n}{2} \rfloor$ and $D_{q_{k+1}}, D_{p_k}$ are normalization factors so that $Q(y)_{k+1}, P(y)_k$ are monomials. Similar formulas can be derived for even n.

2.9 Eigenvalues computation

The zeros of $Q_{k+1}(y)$ and $P_k(y)$ it is possible to obtained as eigenvalues of special matrix

$$J_{k+1} = \begin{bmatrix} -b_1 & 1 & 0 & \cdots & 0 \\ c_2 & -b_2 & 1 & \ddots & \vdots \\ 0 & c_3 & -b_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & c_{k+1} & -b_{k+1} \end{bmatrix}. \tag{38}$$

It is hold for odd n that

$$\begin{aligned} Q_{k+1}(y) &= \det(yI_{k+1} - J_{k+1}) \\ P_k(y) &= \det(yI_k - J_k'), \end{aligned} \tag{39}$$

where J_k' is the matrix obtained by suppressing the first row and the first column of J_{k+1}. So, the zeros of $Q_{k+1}(y)$ are the eigenvalues of J_{k+1} and the zeros of $P_k(y)$ are the eigenvalues of J_k'. General patterns are described in Brezinski [2002]. These polynomials are also the solution of special differential equations.

The position of zeros of orthogonal polynomials is also very important. Each n-degree polynomial in an orthogonal sequence has all n of its roots real from interval $(-1, 1)$, distinct, and strictly inside the interval of orthogonality. The roots of each polynomial lie strictly between the roots of the next higher polynomial in the sequence. This important property can be used in numerical finding the zeros in iterative algorithm - the choice of the first iteration in Newton's method. The position of zeros of polynomial $P(y)$ and $Q(y)$ is depicted in (Fig. 4). The sequences of roots of polynomials $P(y)$ and $Q(y)$ in (Fig. 4) are generated from example in introduction section.

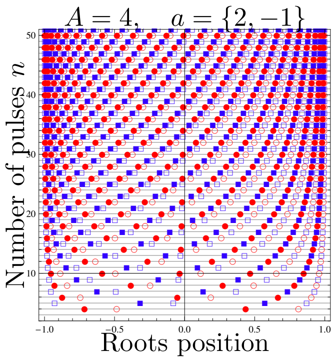

Fig. 4. The roots position of orthogonal polynomials for example in introduction section. ○, ● and □, ■ are the roots of $P(y)$ and $Q(y)$ for even respectively odd n.

3. CONCLUSION

This paper presents a contribution to theory of optimal PWM problem for odd symmetry waveforms and gives efficient algorithms for fast calculation of PWM switching sequence. Some new results or expansion of know results regarding the optimal odd PWM problem are derived in this paper. The results are summarized in the following part.

(1) After variables transformation the solution of the optimal PWM problem is given by the zeros of two polynomials $P(y)$ and $Q(y)$ that are suitable sorted.
(2) The polynomials $P(y)$ and $Q(y)$ are given by $\frac{P_k(y)}{Q_{n-k}(y)} = y^{2k-n}\exp(-\sum_{j=1}^{\infty}\frac{p_j}{jy^j})$, where $k = \lfloor \frac{n}{2} \rfloor$ and $p_j = \sum_{i=1}^{k} y_i^j - \sum_{i=k+1}^{n} y_i^j, \quad j = 1, \ldots, n$, where $y_i, \ i = 1, \ldots, k$ are the zeros of $Q_k(y)$ and $y_i, \ i = k+1, \ldots, n$ are the zeros of $P_{n-k}(y)$.
(3) The polynomials $Q(y)$ and $P(y)$ give also the solution of a Padé approximation and therefore constitute a set of orthogonal polynomials where the polynomial $P(y)$ is the associated polynomial (or polynomial of second kind) to $Q(y)$ for odd n. In case even n, the polynomials $Q(y)$ and $P(y)$ are adjacent family of orthogonal polynomials.
(4) The solution of optimal PWM problem can be obtained through
    (a) Hankel (Toeplitz) system (24, 25) for odd n and (26, 27) for even n. The complexity of a fast algorithm is $\mathcal{O}(n \log n^2)$.
    (b) The simple three term recurrence relationship hold (28, 32) for odd n and (33, 35) for n even. The complexity is $\mathcal{O}(n^2)$.
    (c) Determinants of special polynomial matrices (36, 37) for odd n.
    (d) Eigenvalues of special matrices (38) and (39) for odd n.

As shown above, the solution of optimal odd PWM waveform depend only on solving Toeplitz system which can be solved very efficiently. Also it is possible to use other mentioned methods such as three term recurrence formula, polynomial matrix determinant or eigenvalue computations. Therefore with high-performance DSP capabilities, it is possible of on-line construction of arbitrary waveforms. Areas of applications are active filters, digital audio amplifiers and other problems where selected harmonic elimination is needed.

Here, it is necessary to mentioned that the appropriate matrix is ill-conditioned when direct applying algorithm based on solving of the Hankel system. This problem complicates the solution for large n. In case of CAS it is important to extend precision of real numbers.

It is also important to say that our solution is consistent with the solution of Czarkowski et al. [2002] in case of quarter-wave symmetric waveforms, i.e. even harmonics are zero. In addition in this study, the polynomial $P(y)$ is solved. Its roots are always reversed to polynomial $Q(y)$ therefore there are no need to compute it.

Of course, the three- or multi-level pulse width modulated waveform can be converted to bi-level PWM waveform. The other interesting problem is to apply the theory of orthogonal polynomials and fast algorithms to the three-phase connection.

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