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Guaranteed bounds for robust LMI problems with polynomial parameter dependence

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Abstract: Many problems from control theory can be stated as so-called robust LMI problems. In this paper, the theorem of Ehlich and Zeller, a powerful tool for analyzing polynomials and rational functions, will be presented and applied to the robust LMI problems which are used to analyze uncertain systems.

Keywords: Robust linear matrix inequalities, Relaxations, convex optimization

1. INTRODUCTION

Linear matrix inequalities (LMI), which have the form

$$F(\lambda) = F_0 + \lambda_1 F_1 + ... + \lambda_h F_h < 0,$$

where the F_i are constant symmetric $s \times s$ matrices and the λ_i are decision variables, are used to describe many problems stemming from control theory (cf. Boyd et al. (1994)). The task of solving an LMI is to find a point in the solution set $\Lambda := \{\lambda | F(\lambda) < 0\}$.

A typical LMI problem is the Lyapunov inequality: If for a linear, time-invariant, autonomous system with a constant system matrix $A \in \mathbb{R}^{s\times s}$ a symmetric, positive definite matrix $P \in \mathbb{R}^{s\times s}$ exists which fulfills the inequality

$$A^T P + PA > 0,$$

we know that the system is asymptotically stable. For problem statements of this kind powerful approaches are available (cf. Nemirovski and Gahinet (1994), Nesterov and Nemirovski (1994)). Unfortunately sometimes not all parameters of an LMI are known. There are several reasons, e.g. sensor and implementation errors, calculation (floating point) errors or an approximation of nonlinear systems by uncertain linear systems. Let $\delta = [\delta_1, \cdots, \delta_q]$ denote the vector of parameter uncertainties, which is bounded by an interval vector $\Delta \in \mathbb{R}^q$. Now we can formulate the problem statement with h uncertain parameters as

$$F(\lambda, \delta) = F_0(\delta) + \lambda_1 F_1(\delta) + \cdots + \lambda_h F_h(\delta) \quad (1)$$

with the solution set

$$\Omega = \left\{\lambda \in \mathbb{R}^h | F(\lambda, \delta) < 0 \quad \forall \delta \in \Delta\right\}.$$

This is called the robust LMI problem.

Solving such a robust LMI problem is NP-hard, because (1) leads to an infinite number of conventional LMIs to solve.

In this paper we pay special attention to optimization problems under robust LMI conditions. In the past, several asymptotically exact approaches have been proposed to relax robust LMI problems with polynomial parameter dependence. Bliman (2006) and Ohara and Sasaki (2001) proposed methods based on the Kalman-Yakubovich lemma. In Scherer (2005), an approach based on Polya's theorem is presented. If we define the polynomial $p(\delta)$ to be a quadratic form corresponding to the robust LMI (1), i.e.

$$p(\delta) := z^T F(\lambda, \delta) z, \quad \forall z \in \mathbb{R}^s, \quad (2)$$

the theory of sum-of-squares (cf. Parillo (2000), Lasserre (2001)) can be applied to robust LMIs, which was, for example, shown in Scherer and Hol (2006). For robust stability analysis, see Oliveira and Peres (2005) and its bibliography.

In this paper we present the theorem of Ehlich and Zeller, which can be used to analyze the positivity (and negativity) of polynomials and rational functions on a compact interval. In the past we applied this method to find guaranteed bounds for the domain of attraction of dynamical polynomial systems (cf. Tibken et al. (1999)) or in the context of polynomial positivity (cf. Tibken and Dilaver (2003), Tibken and Dilaver (2004), and Tibken and Dilaver (2005)). In this publication we apply this theorem to a scalarized robust LMI problem and verify this method on several examples.

2. THEOREM OF EHLICH AND ZELLER

This section will closely follow the corresponding sections in Tibken et al. (1999), Tibken and Dilaver (2003), Tibken and Dilaver (2004), and Tibken and Dilaver (2005). In the following $\Delta = [a, b]$ denotes a nonempty compact interval with $\Delta \subset \mathbb{R}$. For an algebraic variable $\delta \in \Delta$ we define the set of $N(\in \mathbb{N})$ Chebychev points in Δ as

$$X(N, \Delta) := \left\{\delta^{(j)}, j = 1, ..., N\right\}, \text{ where}$$

$$\delta^{(j)} = \frac{a+b}{2} + \frac{b-a}{2} \cos\left(\frac{(2j-1)\,N}{2N}\right).$$

For any continuous function f defined on a set I the norm

$$\|f\|^I := \max_{\delta \in I} |f(\delta)|,$$

is the usual maximum norm. Let p_m be the set of polynomials p in one variable with $\deg(p) = m$. Then the following equality

$$\|p\|^{\Delta} = C\left(\frac{m}{N}\right)||p||^{X(N,\Delta)} \text{ with } N > m \text{ and} \tag{3}$$

$$C\left(\frac{m}{N}\right) := \left[\cos\left(\frac{m\pi}{2N}\right)\right]^{-1}$$

is valid for every $p \in p_m$ and every nonempty compact interval Δ. This result was given by Ehlich and Zeller (1964). For the minimum and maximum of a polynomial p in the set I we use the notation

$$p^I_{\min} := \min_{\delta \in I} p(\delta) \text{ and } p^I_{\max} := \max_{\delta \in I} p(\delta),$$

respectively. Using (3), the following inequalities

$$\begin{aligned} p^{\Delta}_{\max} \le \frac{1}{2}\Big\{&\left(C\left(\frac{m}{N}\right)+1\right)p^{X(N,\Delta)}_{max} \\ &-\left(C\left(\frac{m}{N}\right)-1\right)p^{X(N,\Delta)}_{min}\Big\} \end{aligned} \tag{4}$$

$$\begin{aligned} p^{\Delta}_{\min} \ge \frac{1}{2}\Big\{&\left(C\left(\frac{m}{N}\right)+1\right)p^{X(N,\Delta)}_{min} \\ &-\left(C\left(\frac{m}{N}\right)-1\right)p^{X(N,\Delta)}_{max}\Big\}, \end{aligned} \tag{5}$$

which are valid for every $p \in p_m$ and $N > m$, are given by Ruttmann (1982).

The inequalities (3), (4) and (5) are valid for polynomials in one variable. They can be extended to polynomials of q variables if we use the following replacements. First of all, we replace the interval Δ by $\hat{\Delta} = [a_1, b_1] \times ... \times [a_q, b_q]$, which represents a hyperrectangle or an interval vector. We introduce the abbreviation m_h for the degree of p with respect to the h-th variable δ_h and define the set of Chebychev points by

$$X\left(\hat{N}, \hat{\Delta}\right) := X(N_1, [a_1, b_1]) \times ... \times X(N_q, [a_q, b_q])),$$

where N_h is the number of Chebychev points for the h-th variable δ_h in the interval $[a_h, b_h]$. Then the inequalities

$$p^{\hat{\Delta}}_{\max} \le \frac{1}{2}\left\{(K+1)\,p^{X(\hat{N},\hat{\Delta})}_{max} - (K-1)\,p^{X(\hat{N},\hat{\Delta})}_{min}\right\} \tag{6}$$

$$p^{\hat{\Delta}}_{\min} \ge \frac{1}{2}\left\{(K+1)\,p^{X(\hat{N},\hat{\Delta})}_{min} - (K-1)\,p^{X(\hat{N},\hat{\Delta})}_{max}\right\} \tag{7}$$

with

$$K = \prod_{h=1}^{q} C\left(\frac{m_h}{N_h}\right)$$

under the conditions $N_h > m_h, h = 1, ..., q$ are valid.

The results above can be applied to rational functions $r(\delta)$ with

$$r(\delta) = \frac{p(\delta)}{q(\delta)}$$

and $p, q \in p_m$, $q(\delta) \neq 0 \quad (\delta \in \Delta)$. If we define

$$\kappa := \frac{|q|^{X(N,\Delta)}_{\max}}{|q|^{X(N,\Delta)}_{\min}}$$

and replace

$$K := \frac{C\left(\frac{m}{N}\right)+1+\left(C\left(\frac{m}{N}\right)-1\right)\kappa}{C\left(\frac{m}{N}\right)+1-\left(C\left(\frac{m}{N}\right)-1\right)\kappa}$$

and the inequalities

$$C\left(\frac{m}{N}\right)+1-\left(C\left(\frac{m}{N}\right)-1\right)\kappa > 0$$

and $N > m$ are fulfilled , then the inequalities

$$r^{\hat{\Delta}}_{\max} \le \frac{1}{2}\left\{(K+1)\,p^{X(\hat{N},\Delta)}_{max} - (K-1)\,p^{X(\hat{N},\Delta)}_{min}\right\} \tag{8}$$

and

$$r^{\hat{\Delta}}_{\min} \ge \frac{1}{2}\left\{(K+1)\,p^{X(\hat{N},\Delta)}_{min} - (K-1)\,p^{X(\hat{N},\Delta)}_{max}\right\} \tag{9}$$

are valid.

Remark 1. With the theorem of Ehlich and Zeller it is possible to make a statement about the positivity (or negativity) of polynomials on an interval Δ since $\min_{\delta\in\Delta} p(\delta) > 0$ is a necessary and sufficient condition for $p(\delta) > 0 \quad \forall \delta \in \Delta$ and $\max_{\delta\in\Delta} p(\delta) < 0$ is a necessary and sufficient condition for $p(\delta) < 0 \quad \forall \delta \in \Delta$, respectively. Hence if the inequalities

$$(K+1)\,p^{X(\hat{N},\hat{\Delta})}_{min} - (K-1)\,p^{X(\hat{N},\hat{\Delta})}_{max} > 0$$

and $N > m$, as defined above, are valid we know that the positivity of $p(\delta)$ is guaranteed on the complete interval Δ.

3. RELAXATION METHODS

If we define the polynomial $p(\delta)$ to be a quadratic form corresponding to the robust LMI (1), i.e.

$$p(\delta) := z^T F(\lambda, \delta) z, \tag{10}$$

we can apply the theorem of Ehlich and Zeller. Furthermore we define

$$p\left(\delta^{(i)}\right) := z^T F(\lambda, \delta^{(i)}) z \tag{11}$$

as the polynomial $p(\delta)$ at the Chebychev point $\delta^{(i)} \in X(N, \Delta)$.

3.1 Method 1

Theorem 1. If all inequalities

$$(K+1)F(\lambda, \delta^{(j)}) - (K-1)F(\lambda, \delta^{(k)}) < 0 \tag{12}$$

are satisfied for all $i, j = 1, ..., N$, the robust LMI problem $F(\lambda, \delta) < 0$ is satisfied $\forall \delta \in \Delta$ as well. In other words, (12) is a sufficient condition for $F(\lambda, \delta) < 0$.

Proof. Since

$$\begin{aligned} \max_{\delta\in\Delta} p(\delta) \le \frac{1}{2}\Big[&(K+1)\max_{i=1...N} p(\delta^{(i)}) \\ &-(K-1)\min_{i=1...N} p(\delta^{(i)})\Big] \end{aligned} \tag{13}$$

is a result of the theorem of Ehlich and Zeller, we achieve

$$(K+1)\max_{i=1...N} p(\delta^{(i)}) - (K-1)\min_{i=1...N} p(\delta^{(i)}) < 0 \tag{14}$$

as a sufficient condition for $\max_{\delta\in\Delta} p(\delta) < 0$.

Due to the fact that inequality (14) is no LMI condition, we have to replace it by N^2 inequalities of the form

$$(K+1)p(\delta^{(j)}) - (K-1)p(\delta^{(k)}) < 0 \quad j, k = 1, ..., N.$$

Using definition (11) we get

$$(K+1)z^T F\left(\lambda, \delta^{(j)}\right) z - (K-1)z^T F\left(\lambda, \delta^{(k)}\right) z < 0,$$
$$j, k = 1, ..., N, \forall z \in \mathbb{R}^s,$$

which we can rewrite as

$$(K+1)F\left(\lambda, \delta^{(j)}\right) - (K-1)F\left(\lambda, \delta^{(k)}\right) < 0.$$

Remark 2. Please note that the number of conventional LMIs used to represent one robust LMI depends quadratically on the number of chosen Chebychev points.

Remark 3. This method is a direct implementation of the theorem of Ehlich and Zeller. Since the bounds given by the theorem of Ehlich and Zeller converge against the exact minimum/maximum of the polynomial, the bounds of this method will converge as well.

3.2 Method 2

If we introduce two new symmetric matrices $P_{\mathrm{T}}, P_{\mathrm{B}} \in \mathbb{R}^{s\times s}$ as decision variables in the form

$$\max_{i=1\dots N} p(\delta^{(i)}) \le z^T\ P_{\mathrm{T}}\ z \quad \forall\ z\in\mathbb{R}^s$$
$$\min_{i=1\dots N} p(\delta^{(i)}) \ge z^T\ P_{\mathrm{B}}\ z \quad \forall\ z\in\mathbb{R}^s,$$

we can propose another method to solve robust LMI problems.

Theorem 2. If all inequalities

$$F\left(\lambda,\delta^{(i)}\right) \le P_{\mathrm{T}} \tag{15}$$
$$F\left(\lambda,\delta^{(i)}\right) \ge P_{\mathrm{B}} \tag{16}$$
$$(K+1)P_{\mathrm{T}} - (K-1)P_{\mathrm{B}} < 0 \tag{17}$$

are satisfied $\forall i = 1,...,N$, the robust LMI problem $F(\lambda,\delta) < 0$ is satisfied $\forall\delta\in\Delta$ as well.

Proof. With (11) follows

$$\begin{aligned}\max_{i=1\dots N} z^T F\left(\lambda,\delta^{(i)}\right) z \le z^T\ P_{\mathrm{T}}\ z \quad \forall z\in\mathbb{R}^s\\ \min_{i=1\dots N} z^T F\left(\lambda,\delta^{(i)}\right) z \ge z^T\ P_{\mathrm{B}}\ z \quad \forall z\in\mathbb{R}^s\end{aligned} \tag{18}$$

which we can be rewritten as

$$\left.\begin{aligned}F\left(\lambda,\delta^{(i)}\right) \le P_{\mathrm{T}}\\ F\left(\lambda,\delta^{(i)}\right) \ge P_{\mathrm{B}}\end{aligned}\right.\quad i = 1,...,N. \tag{19}$$

If we utilize (11) in (13), we get

$$(K+1)\ z^T\ P_{\mathrm{T}}\ z - (K-1)z^T\ P_{\mathrm{B}}\ z < 0,$$

which is equivalent to

$$(K+1)P_{\mathrm{T}} - (K-1)P_{\mathrm{B}} < 0.$$

Remark 4. With Method 2, we need $2N+1$ conventional LMIs to solve one robust LMI instead of N^2 with the first method.

Remark 5. The first method is a direct implementation of the theorem of Ehlich and Zeller, whereas in the second method piecewise quadratic functions are approximated by quadratic functions (18). Thus we expect that the second method will yield more conservative results than the first one.

3.3 Quality control

With the methods proposed above we get an inner boundary of our solution set Ω, which we will call Ω_{i}. This is because our two methods use sufficient conditions.

If we verify our robust LMI at the Chebychev points only, i.e. without using the theorem of Ehlich and Zeller, we fulfill the necessary conditions, so that we get an outer boundary of Ω, which we will call Ω_{o}.

If the goal is to solve an optimization problem of the form

$$\gamma = \min_{F(\lambda,\delta)<0,\ \forall\delta\in\Delta} g(\lambda) \tag{20}$$

for a performance function $g(\lambda)$ it is clear that we get a lower bound for γ, if we solve the problem

$$\gamma_l = \min_{F(\lambda,\delta^{(j)})<0,\ j=1,\dots,N} g(\lambda) \tag{21}$$

and an upper bound, if we solve the problem

$$\gamma_u = \min_{(12)} g(\lambda) \tag{22}$$

or

$$\gamma_u = \min_{(15)-(17)} g(\lambda). \tag{23}$$

Thus, the exact value is bounded as follows:

$$\gamma_L \le \gamma \le \gamma_u$$

and we have a direct quality control. We observe that the bounds can be improved by increasing the number of Chebychev points. A second important observation is that the problems (21), (22) and (23) are optimization problems with a finite number of LMIs as constraints and thus can be solved with existing software.

4. EXAMPLES

4.1 Stability analysis

As mentioned in the introduction, the asymptotic stability of a linear, time-invariant, autonomous system with a constant system matrix $A\in\mathbb{R}^{s\times s}$ can be verified if a symmetric, positive definite matrix $P\in\mathbb{R}^{s\times s}$ exists which fulfills the inequality

$$A^TP + PA > 0.$$

If some of the parameters of the system matrix A are dependent on an uncertain but constant parameter $\delta = [\delta_1,\cdots,\delta_q]\in\Delta$ we have to rewrite our problem statement as

$$A^T(\delta)P(\delta) + P(\delta)A(\delta) > 0 \quad P > 0. \tag{24}$$

The problem is discussed further in Chesi et al. (2005).

For this example we use the approach that $P(\delta)$ depends linearly on the uncertain parameters, i.d.

$$P(\delta) = P_0 + \delta_1P_1 + \cdots + \delta_qP_q.$$

To verify that the inequality $P(\delta) > 0$ is fulfilled, it is sufficient to verify this inequality only at the 2^q corners of the hypercube Δ. See Schwenk and Tibken (2008) for a more efficient relaxation than vertexization.

Let us consider a simple system with quadratic parameter dependence given by

$$\dot{x}(t) = \begin{pmatrix} -2+\delta^2 & 0\\ 2 & -2\end{pmatrix} x(t).$$

We have one uncertain parameter δ and it is obvious that the system is asymptotically stable if $|\delta| < \sqrt{2}$. If we apply our methods we get the following results:

Number of Chebychev points N	$\lvert\delta\rvert$ (method 1)	$\lvert\delta\rvert$ (method 2)
10	1.390971	1.390971
20	1.408676	1.408676
50	1.413339	1.413339

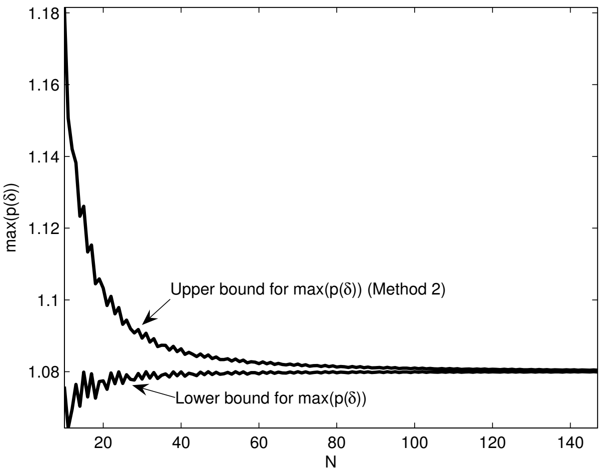

Fig. 1. Upper and lower bound for $\max p(\delta)$.

Two things are noteworthy: Firstly, there does not seem to be a difference between the results of our two proposed methods. Secondly, with an increasing number of Chebychev points the results converge against the exact result.

4.2 Optimization of polynomials

As shown in Oishi (2006), the maximization of a scalar polynomial over a set Δ can be written as

$$\begin{array}{ll} \min & x \\ \text{subject to} & x - p(\delta) \geq 0 \quad (\forall \delta \in \Delta), \end{array} \tag{25}$$

which is a robust LMI minimization problem with the decision variable x and the uncertain parameter δ. We will consider the polynomial

$$p(\delta) = -5\delta_1^2\delta_2 - 5\delta_1\delta_2^2 + 9\delta_1\delta_2$$

on the interval $\delta = [0,1] \times [0,1]$. With classic methods we get the unique maximum $p(0.6, 0.6) = 1.08$. With our second method and $N = (150)^2 = 22500$ Chebychev points we get an upper bound for the maximum of $x_i = 1.0805$, which is a good approximation of the exact maximum. As can be seen in Fig. 1, the bounds for the maximum of $p(\delta)$ do not monotonically increase or decrease, but we expect better results for larger values of N.

4.3 L^2 gain analysis for linear systems with scheduling parameter

In Azuma et al. (2000) an LMI approach to analyze linear systems of the form

$$\Phi: \begin{array}{rl} \dot{x}(t) &= A(\delta(t))x(t) + B(\delta(t))w(t), \quad x(0) = 0, \\ y(t) &= C(\delta(t))x(t) \end{array}$$

which depend polynomial on a scheduling parameter $\delta(t)$, was proposed. It is assumed that $\delta(t) \in [0,1]$ and $\left|\dot{\delta}(t)\right| \leq v_{max} \quad \forall t \in [0,\infty)$ with $v_{max} > 0$. $x(t) \in \mathbb{R}^s$ is the state, $w(t) \in \mathbb{R}^u$ the disturbance and $y(t) \in \mathbb{R}^l$ the observed output of Φ. For an internally stable linear system Φ with scheduling parameter , the L^2 gain of Φ was given by

$$G(\Phi) := \sup_{w \in L^2, w \neq 0} \frac{\|y\|_{L^2}}{\|w\|_{L^2}}.$$

Furthermore a theorem about the internal stability and the L^2 gain was proposed.

Theorem 3. (Azuma et al. (2000)). The system Φ is internally stable and $G(\Phi)$ is less than γ if there exists a matrix function $P(\delta)$ defined on $[0,1]$ such that

$$P(\delta) > 0 \tag{26}$$

$$\frac{dP}{d\delta}(\delta) < 0 \tag{27}$$

$$\begin{bmatrix} \begin{pmatrix} A^T(\delta)P(\delta) + P(\delta)A(\delta) \\ +\sigma v_{max}\dfrac{dP}{dt}(\delta) \end{pmatrix} & C^T(\delta) & P(\delta)B(\delta) \\ C(\delta) & -I & 0 \\ B^T(\delta)P(\delta)) & 0 & -\gamma^2 I \end{bmatrix} < 0 \tag{28}$$

for all $\delta \in [0,1]$ and $\sigma = \pm 1$.

In Azuma et al. (2000) a model of a gasturbine engine with the following parameters was considered :

$$\begin{aligned} \dot{x}(t) &= \left[A_0 + \delta A_1 + \delta^2 A_2\right] x(t) \\ &+ \left[B_0 + \delta B_1 + \delta^2 B_2\right] w(t) \\ y(t) &= Cx(t) \end{aligned}$$

$$A_0 = \begin{bmatrix} -4.3650 & -.67230 & -.33630 \\ 7.0880 & -6.5570 & -4.6010 \\ -2.4100 & 7.5840 & -14.310 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} -.56081 & .85534 & .58923 \\ .66981 & -1.3750 & -.99093 \\ -3.1917 & 1.7971 & -.58870 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} .66981 & -1.3750 & -.99093 \\ -2.8963 & 1.5292 & 10.516 \\ -3.5777 & 2.8389 & 1.9087 \end{bmatrix},$$

$$B_0 = \begin{bmatrix} 2.374 & .7485 \\ 1.366 & 3.444 \\ .9461 & 9.619 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} -.16023 & -.35209 \\ .11622 & -2.4839 \\ -.11058 & -4.6057 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} .15623 & .13063 \\ -.49582 & 4.0379 \\ -.030616 & .89473 \end{bmatrix}, \; C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The optimization problem is to find the minimum γ as an upper bound for the L^2 gain. It can be formulated as

$$\min_{(26)-(28)} \gamma.$$

We will examine this problem for the case where $P(\delta)$ depends quadratically on δ, i.e.

$$P(\delta) = P_0 + \delta P_1 + \delta^2 P_2$$

and $v_{\max} = 10$.

Condition (26) leads with our first method to N^2 conventional LMIs, with the second one to $2N + 1$ conditions. It is quite obvious that $m_1 := \deg P(\delta) = 2$.

Since condition (27) is linear in δ, it is sufficient to verify this LMI at the bounds of the interval $\Delta = [0,1]$, i.e.

$$\begin{array}{ll} P_1 & < 0 \\ P_1 + 2P_2 & < 0. \end{array}$$

So we need to verify only two conditions, independently of the chosen method and the number of Chebychev points.

The two conditions in (28) lead with our first method to $2N^2$, with the second method to $4N+2$ conventional LMIs. The degree is $m_2 = 4$.

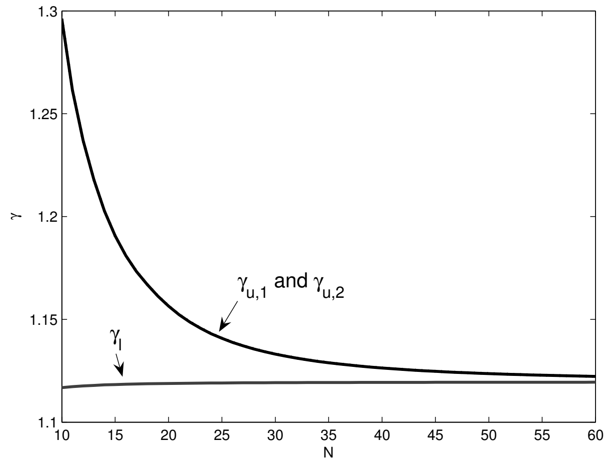

Fig. 2. Guaranteed bounds for γ.

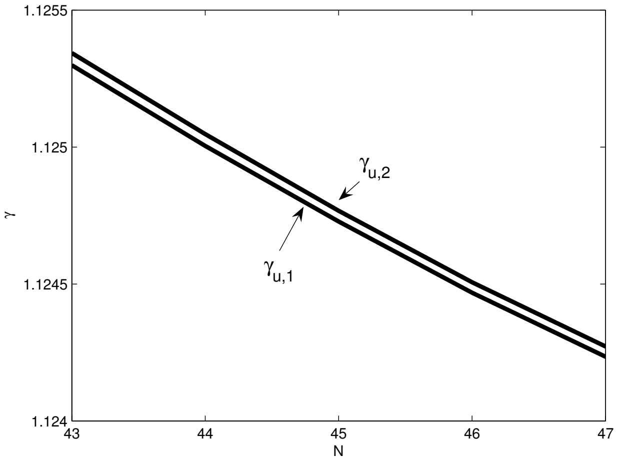

Fig. 3. Difference between our two methods.

In summary, we need to verify with the first method $3N^2 + 2$, with the second one $6N + 5$ conventional LMIs. This means for $N = 50$ Chebychev points, that the number of conditions to solve with the first method is 18.5 times larger than with the second method. We get the following bounds for γ:

$$\begin{aligned} \gamma_{u,1} &\leq 1.12361726 \\ \gamma_{u,2} &\leq 1.12365007 \\ \gamma_{l} &\geq 1.11940421 \end{aligned}$$

As can be seen, the difference between the upper and lower bound is less than 10^{-2}. What attracts more attention is the fact that the difference between $\gamma_{u,1}$ and $\gamma_{u,2}$, which are the optimization results of our two methods, is less than 10^{-4}. So we can suggest that the conservatism of Method 2 compared to the first method does not legitimate the higher complexity (extra memory, computation time) of the first method.

As can be seen in Fig. 2, the difference between the result of our two methods compared to the result using the sufficient conditions is not visible. But it is still existent, as shown in Fig.3.

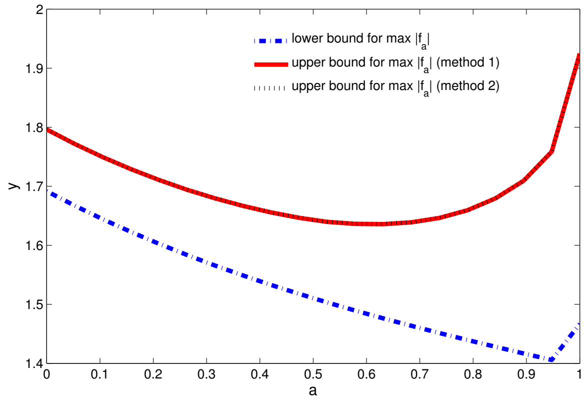

Fig. 4. Upper and lower bounds for the absolute value of the supremum of f_a on Δ (10 Chebychev points)

4.4 Rational functions

In Scherer and Hol (2006) the absolute value of the supremum of a rational function $f_a(\delta)$ on a compact set Δ is computed by minimizing the decision variable y in the LMI

$$\begin{pmatrix} y & f_a(x) \\ f_a(x) & y \end{pmatrix} > 0 \text{ for all } x \in \Delta.$$

Then the optimal value is equal to $\sup_{x\in\Delta} |f_a(\delta)|$. We will analyze the rational function

$$\begin{aligned} f_a(\delta_1, \delta_2) = &- \frac{a^2\delta_1^2\delta_2^2 + 2a^2\delta_2^2 + a\delta_1^2\delta_2^2 + 2a\delta_1^2\delta_2}{2 - 2a^2\delta_2^2 - \delta_1^2 + a^2\delta_1^2\delta_2^2} \\ &- \frac{2a\delta_2^2 + \delta_1^2\delta_2 + \delta_1^2 - 2\delta_2 - 2}{2 - 2a^2\delta_2^2 - \delta_1^2 + a^2\delta_1^2\delta_2^2} \end{aligned}$$

on the compact set

$$\Delta := \left\{(\delta_1, \delta_2) \in \mathbb{R}^2 | -0.8 \leq \delta_1 \leq 0.7, -0.65 \leq y \leq 0.7\right\}$$

for 20 equidistant values of $a \in [0, 1]$. This function is taken from Scherer (2005).

If we apply both methods with $N = 10$ Chebychev points, we get upper and lower bounds for the absolute value of the supremum of f_a on Δ as shown in Fig. 4. In contrast to the first example we can see here a small difference between the results of the two methods.

If we use $N = 100$ Chebychev points we get an enclosure, as shown in Fig. 5. This solution can only be found by the second method, because for the first method we would need to verify 10^8 LMIs which is not solvable with the LMI toolbox from MATLAB due to the large size of the problem.

For the case $a = 0.9$ and $N = 100$ we get the following values for the *min* and *max* of the denominator

$$\begin{aligned} |q|_{\max}^{X(N,\Delta)} &= -0.8598 \text{ and} \\ |q|_{\min}^{X(N,\Delta)} &= 1.4292, \end{aligned}$$

which yield the correction factor

$$K = 1.0024 \text{ with } \kappa = 2.4375.$$

With these factors we get an upper bound

$$y_{\mathrm{u}} = 1.4320$$

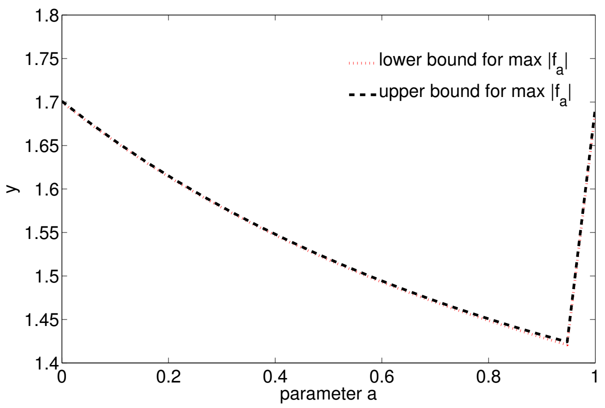

Fig. 5. Upper and lower bounds for the absolute value of the supremum of f_a on Δ (100 Chebychev points)

and a lower limit

$$y_l = 1.4292$$

for the absolute value of the supremum of f_a on Δ.

5. CONCLUSIONS

The theorem of Ehlich and Zeller was used to get guaranteed information about the positivity of a polynomial or a rational function on a compact interval. We applied a method based on this theorem to robust LMI problems which are known to have a wide range of applications in stability and performance analysis of uncertain systems.

The difference between the calculated bounds and the exact solution and the complexity both depend on the number of Chebychev points. Thus it is possible to balance between conservatism and the calculation effort. Even with a small number of Chebychev points our method delivers guaranteed bounds for the exact value.

Because the first method is a direct implementation of the theorem of Ehlich and Zeller, we know that for $N \to \infty$ our results will converge asymptotically against the exact solution. The second method seems to converge as well, but future work will be necessary to verify and prove it mathematically. Since the first method depends quadratically on the number of Chebychev points and delivers only minimally better results than the second method, one should prefer the second method which depends only linearly on the number of Chebychev points even though proof of the convergence is outstanding.

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