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On bounded real lemma for fractional systems

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Abstract: Two state space "like" representation based methods for fractional systems L_2-gain computation are proposed in this paper. The first is based on an approach already presented in the literature and leads to a new theorem. The theorem is based on the location of the eigenvalues of a matrix issued from the state space "like" representation and is then converted using Riccati theory into an LMI constraint to give the second theorem. Its formulation is similar to the well known bounded real lemma whereas it does not guarantee stability. The theorems are finally applied to car suspension analysis for the computation of modulus margins. Prospects of this study are in the fields covered by the usual bounded real lemma such as H_∞ control, thus aiming at straightforward extension to fractional systems.

1. INTRODUCTION

Fractional differentiation is now a well known tool for controller synthesis (Xue and Chen, 2002). Several presentations and applications of the fractional PID controller (Podlubny, 1999), (Monje *et al.*, 2004), (Caponetto *et al.*, 2004), (Chen *et al.*, 2004) and of CRONE control (Oustaloup and Mathieu, 1999) demonstrate their efficiency. Fractional differentiation also permits a simple representation of some high order complex integer systems (Battaglia *et al.*, 2001). Consequently, basic properties of fractional systems have been investigated these last ten years and criteria and theorems are now available in the literature concerning stability (Matignon, 1996), observability, and controllability (Matignon and D'Andrea-Novel, 1996) of fractional systems.

Lyapunov based methods have also been developed for stability analysis and control law synthesis of integer linear systems, and for more complex systems such as nonlinear, time-varying, and LPV systems (Biannic, 1996). This has been possible, thanks to the development of efficient numerical methods to solve convex optimization problems (Boyd and Vandenberghe, 2004), by resolving Lyapunov stability conditions or quadratic robust control problems (Balakrishnan and Kashyap, 1999) (Balakrishnan, 2002) defined by Linear Matrix Inequalities (LMI).

Paradoxically, only few studies deal with Lyapunov based control laws synthesis for fractional systems. The most advanced method for such purposes consists in controlling an integer approximation while considering the remaining fractional part as perturbation (Hotzel, 1998). As analytical impulse response energy computation of fractional systems becomes available (Malti *et al.*, 2002), methods considering the whole behaviour of fractional systems are now to be developed.

In this paper, we propose two tools for fractional systems L_2-gain computation. The first one is based on a frequency analysis to obtain a condition on the location of the eigenvalues of a matrix issued from the state space "like" representation of the system. This approach is presented in section 3. The resulting condition is then converted into an LMI constraint to give a second condition, presented in section 4. This condition is based on a lemma whose proof investigates the relation between Riccati equality, Ricatti inequality and the location of the eigenvalues of a complex hamiltonian matrix. Both theorems are finally applied to car suspension analysis, for the computation of modulus margins.

2. NOTATIONS AND DEFINITIONS

2.1 Fractional calculus

Riemann-Liouville fractional differentiation is used and the fractional integral of a function $f(t)$ is defined by

$$I_0^\nu f(t)=\frac{1}{\Gamma(\nu)}\int_0^t (t-\xi)^{\nu-1} f(\xi)d\xi\,,\quad \Gamma(\nu)=\int_0^\infty e^{-x}x^{\nu-1}dx \qquad (1)$$

where $\nu\in\mathbb{R}^+$ denotes the fractional integration order.

Using (1), the fractional derivative of order $\nu\in\mathbb{R}^+$ of a function $f(t)$ is defined by (Miller and Ross, 1993)

$$D^\nu f(t)=D^m[I^{m-\nu} f(t)]\,, \qquad (2)$$

where m is the smallest integer that exceeds ν.

2.2 Fractional systems

Let us consider a stable Multi-Input, Multi-Output (MIMO) Linear Time-Invariant (LTI) fractional system G whose input $u(t)\in\mathbb{R}^{n_u}$ and output $y(t)\in\mathbb{R}^{n_y}$ are linked by the fractional differential equation:

$$\sum_{i=0}^{N} A_i\left(D^\nu\right)^{k_{yi}} y(t)=\sum_{i=0}^{N} B_i\left(D^\nu\right)^{k_{ui}} u(t)\,. \qquad (3)$$

In relation (3), A_i and B_i are real matrices of appropriate dimension, k_{yi} and k_{ui} are positive integers. Note that all the differentiation orders are multiples of commensurate order ν.

It is also assumed that system G is relaxed at $t=0$, so the Laplace transforms of $D^\alpha u(t)$ and of $D^\alpha y(t)$ are respectively given by $s^\alpha U(s)$ and $s^\alpha Y(s)$ for any $\alpha\in\mathbb{R}$.

Given commensurate order hypothesis, system G also admits the state-space "like" representation (Cois et al., 2001) (Miller and Ross, 1993):

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$$\begin{cases} D^\nu x(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}, \quad (4)$$

where $\nu \in \mathbb{R}$ denotes the fractional order of the system, and $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times n_u}$, $C \in \mathbb{R}^{n_y \times n}$, $D \in \mathbb{R}^{n_y \times n_u}$.
Based on this representation, transfer matrix $G(s)$ is given by

$$G(s) = C\left((s)^\nu I - A\right)^{-1} B + D \ . \quad (5)$$

For simplicity, the form (A, B, C, D, ν) is used in the paper to refer to description (4).

2.3 L_2-gain of LTI systems

The L_2-gain γ_2 of a continuous, LTI system whose transfer function is $G(s)$, can be defined through the H_∞ norm defined in the frequency domain as

$$\gamma_2 = \|G\|_\infty = \sup_\omega \overline{\sigma}(G(j\omega)), \ \omega \in \mathbb{R}, \quad (6)$$

where $\overline{\sigma}$ denotes the maximum singular value.

2.4 Notations

For a complex number λ, $\overline{\lambda}$ denotes its conjugate. Complex matrix A also admits a conjugate $B = \overline{A}$, whose elements b_{ij} are the conjugate of the elements a_{ij} of A. Conjugate transpose matrix C of A is denoted $C = A^*$, and its elements c_{ij} are the conjugate of the elements of A^T, such that $c_{ij} = \overline{a}_{ji}$.
For hermitian matrix A, the notation $A > 0$ means that A is positive definite, such that all its eigenvalues are strictly positive real.
The notation $\mathbb{C}_0$ denotes the set of purely imaginary numbers. This set can be decomposed into $\mathbb{C}_0^+$ and $\mathbb{C}_0^-$ which denote the sets of purely imaginary numbers with respectively positive and negative imaginary part.
Additional notation is standard or otherwise discussed where used.

3. FREQUENCY DOMAIN APPROACH FOR L_2-GAIN COMPUTATION

In this section a fractional system L_2-gain computation method which uses a bisection algorithm is presented. It is based on the existence of purely imaginary eigenvalues of a matrix associated to the system state space "like" representation. This approach was initially described in (Sabatier *et al.*, 2005). It is extended here and a new formulation is proposed, in order to introduce section 4.

3.1 LMI condition

From (6), the L_2-gain of fractional system $G = (A, B, C, D, \nu)$ described by (4) is bounded by

$$\gamma > \overline{\sigma}(D), \ \ \gamma \in \mathbb{R}^+, \quad (7)$$

if and only if (Alazard *et al.*, 1999)

$$\forall \omega \in \mathbb{R}, \ \sup_\omega \ \overline{\sigma}(G(j\omega)) < \gamma, \quad (8)$$

where $G(j\omega)$ is the transfer matrix evaluated at frequency ω, such that

$$G(j\omega) = C\left((j\omega)^\nu I - A\right)^{-1} B + D. \quad (9)$$

Equation (8) can be rewritten

$$\forall \omega \in \mathbb{R}, \ \forall i = 1, ..., \inf(n_u, n_y), \ \ \sigma_i(G(j\omega)) < \gamma, \quad (10)$$

or equivalently,

$$\forall \omega \in \mathbb{R}, \ \forall i = 1, ..., \inf(n_u, n_y), \ \ \sqrt{\lambda_i\left(G(j\omega)^* G(j\omega)\right)} < \gamma. \quad (11)$$

Due to eigenvalues properties, (11) can be rewritten as

$$\forall \omega \in \mathbb{R}, \ \forall i = 1, ..., \inf(n_u, n_y), \ \lambda_i\left(\gamma^2 I - G(j\omega)^* G(j\omega)\right) > 0, \quad (12)$$

or, noting that $G(j\omega)^* = G(-j\omega)^T$,

$$\forall \omega \in \mathbb{R}, \ \forall i = 1, ..., \inf(n_u, n_y),$$
$$\lambda_i\left(\gamma^2 I - G(-j\omega)^T G(j\omega)\right) > 0, \quad (13)$$

which is equivalent to the infinite dimensional LMI:

$$\forall \omega \in \mathbb{R}, \quad \gamma^2 I - G(-j\omega)^T G(j\omega) > 0. \quad (14)$$

3.2 Finite dimensional condition

As

$$\lim_{\omega \to \infty} \ \gamma^2 I - G(-j\omega)^T G(j\omega) = \gamma^2 I - D^T D, \quad (15)$$

it comes from (7) that (14) is met if and only if

$$\forall \omega \in \mathbb{R}, \ \gamma^2 I - G(-j\omega)^T G(j\omega) \text{ is non-singular}, \quad (16)$$

namely if and only if

$$H(j\omega) = \left(\gamma^2 I - G(-j\omega)^T G(j\omega)\right)^{-1} \text{ exists for all real } \omega. \quad (17)$$

From (9),

$$G(-j\omega)^T = (-1)^{-\nu} B^T \left((j\omega)^\nu I - (-1)^{-\nu} A^T\right)^{-1} C^T + D^T \quad (18)$$

at which can be associated the fractional system

$$G' = \left((-1)^{-\nu} A^T, (-1)^{-\nu} C^T, B^T, D^T, \nu\right). \quad (19)$$

A representation of fractional system whose frequency response is given in (17) is thus

$$H_\gamma = (A_H, B_H, C_H, D_H, \nu), \quad (20)$$

where

$$A_H = \begin{pmatrix} A + BRD^T C & BRB^T \\ (-1)^{-\nu} C^T\left(I + DRD^T\right)C & (-1)^{-\nu}\left(A^T + C^T DRB^T\right) \end{pmatrix}, \quad (21)$$

$$R = \left(\gamma^2 I - D^T D\right)^{-1}, \quad (22)$$

B_H, C_H and D_H being omitted here for brevity.

From condition (17), the L_2-gain of fractional system G is bounded by γ defined by (7) if and only if A_H has no eigenvalues on

$$\mathbb{C}_{\nu 0} = \left\{ \ (j\omega)^\nu = \omega^\nu e^{\nu j\pi/2}, \omega \in \mathbb{R} \ \right\}. \quad (23)$$

Let

$$\mathbb{C}_{\nu 0} = \mathbb{C}_{\nu 0}^- \cup \mathbb{C}_{\nu 0}^+, \quad (24)$$

where

$$\mathbb{C}_{\nu 0}^{-} = \left\{ (j\omega)^{\nu}, \omega \in \mathbb{R}^{-} \right\}, \text{ and } \mathbb{C}_{\nu 0}^{+} = \left\{ (j\omega)^{\nu}, \omega \in \mathbb{R}^{+} \right\}. \qquad (25)$$

The L$_2$-gain of fractional system G is thus bounded by γ defined by (7) if and only if A_H has no eigenvalues on $\mathbb{C}_{\nu 0}^{+}$ (case 1), and A_H has no eigenvalues on $\mathbb{C}_{\nu 0}^{-}$ (case2).

Case 1 Using the exponential form $-1 = \exp(-j\pi)$ and multiplying by $\exp((1-\nu)\pi/2)$, matrix A_H has no eigenvalue on $\mathbb{C}_{\nu 0}^{+}$ if and only if

$$A_\gamma = \begin{pmatrix} e^{(1-\nu)j\frac{\pi}{2}}\left(A + BRD^T C\right) & e^{(1-\nu)j\frac{\pi}{2}} BRB^T \\ e^{(1+\nu)j\frac{\pi}{2}} C^T\left(I + DRD^T\right)C & e^{(1+\nu)j\frac{\pi}{2}}\left(A^T + C^T DRB^T\right) \end{pmatrix}$$

has no eigenvalues on $\mathbb{C}_0^{+}$. (26)

Case 2 Using now the exponential form $-1 = \exp(j\pi)$ and multiplying by $\exp(-(1-\nu)\pi/2)$, matrix A_H has no eigenvalue on $\mathbb{C}_{\nu 0}^{-}$ if and only if

$$A'_\gamma = \begin{pmatrix} e^{-(1-\nu)j\frac{\pi}{2}}\left(A + BRD^T C\right) & e^{-(1-\nu)j\frac{\pi}{2}} BRB^T \\ e^{-(1+\nu)j\frac{\pi}{2}} C^T\left(I + DRD^T\right)C & e^{-(1+\nu)j\frac{\pi}{2}}\left(A^T + C^T DRB^T\right) \end{pmatrix}$$

has no eigenvalues on $\mathbb{C}_0^{-}$. (27)

As $A'_\gamma = \overline{A_\gamma}$, it follows that matrices A_γ and A'_γ have conjugate eigenvalues. A'_γ has thus an eigenvalue on $\mathbb{C}_{\nu 0}^{-}$ if and only if A_γ has an eigenvalue on $\mathbb{C}_{\nu 0}^{+}$. Condition (26) and condition (27) are thus equivalent, and condition (26) only is sufficient and necessary.

Defining

$$H = T A_\gamma T^{-1}, \quad T = \begin{pmatrix} I & 0 \\ 0 & e^{(1-\nu)j\pi/2} I \end{pmatrix}, \qquad (28)$$

implies that A_γ and

$$H = \begin{pmatrix} e^{(1-\nu)j\frac{\pi}{2}}\left(A + BRD^T C\right) & BRB^T \\ -C^T\left(I + DRD^T\right)C & e^{(1+\nu)j\frac{\pi}{2}}\left(A^T + C^T DRB^T\right) \end{pmatrix} \qquad (29)$$

have the same eigenvalues. The number of eigenvalues of A_γ on $\mathbb{C}_0^{+}$ is thus equal to the number of eigenvalues of H on $\mathbb{C}_0^{+}$.

Furthermore the relation

$$JH = (JH)^*, \text{ with } J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \qquad (30)$$

permits to infer that the eigenvalues are symmetric about the origin. Such a matrix is referred to as a (complex) hamiltonian matrix.

Theorem 1. The L$_2$-gain of fractional system (A, B, C, D, ν) is bounded by γ if and only if hamiltonian matrix H given by (29) has no eigenvalue on $\mathbb{C}_0^{+}$. □

4. EXTENDED BOUNDED REAL LEMMA

4.1 Hamiltonian matrix and Riccati inequality

Lemma 1. The Hamiltonian matrix

$$\widetilde{H} = \begin{pmatrix} \widetilde{A} & \widetilde{R} \\ -\widetilde{Q} & -\widetilde{A}^* \end{pmatrix}, \qquad (31)$$

where $\widetilde{A} = \widetilde{A}^* \in \mathbb{C}^{n\times n}$, $\widetilde{R} = \widetilde{R}^T \in \mathbb{R}^{n\times n}$, $Q = Q^T \in \mathbb{R}^{n\times n}$, $\widetilde{R} > 0$, has no purely imaginary eigenvalues if and only if the Riccati inequality

$$\widetilde{A}^* X + X\widetilde{A} + X\widetilde{R}X + \widetilde{Q} < 0, \qquad (32)$$

has one solution $P = P^* \in \mathbb{C}^{n\times n}$. □

See section 4.3 for a proof.

4.2 Bounded real lemma for fractional systems

From lemma 1, with

$$\widetilde{A} = e^{(1-\nu)j\frac{\pi}{2}}\left(A + B\left(\gamma^2 I - D^T D\right)^{-1} D^T C\right), \qquad (33)$$

$$\widetilde{R} = B\left(\gamma^2 I - D^T D\right)^{-1} B^T, \qquad (34)$$

and

$$\widetilde{Q} = C^T\left(I + D\left(\gamma^2 I - D^T D\right)^{-1} D^T\right)C, \qquad (35)$$

in conjunction with theorem 1, the L$_2$-gain of fractional system (A, B, C, D, ν) is bounded by γ if there exists a matrix $P = P^* \in \mathbb{C}^{n\times n}$ such that

$$e^{(\nu-1)j\frac{\pi}{2}}\left(A^T + C^T RB^T\right)P + Pe^{(1-\nu)j\frac{\pi}{2}}\left(A + BRD^T C\right) + PBRB^T P + C^T\left(I + DRD^T\right)C < 0. \quad (36)$$

Relation (36) leads to the LMI

$$e^{(\nu-1)j\frac{\pi}{2}} A^T P + Pe^{(1-\nu)j\frac{\pi}{2}} A + C^T C + \left(PB + e^{(\nu-1)j\frac{\pi}{2}} C^T D\right) R \left(B^T P + e^{(1-\nu)j\frac{\pi}{2}} D^T C\right) < 0, \quad (37)$$

whose first terms can be seen as a Schur complement to obtain

$$\begin{pmatrix} e^{(\nu-1)j\frac{\pi}{2}} A^T P + Pe^{(1-\nu)j\frac{\pi}{2}} A + C^T C & \left(PB + e^{(\nu-1)j\frac{\pi}{2}} C^T D\right) \\ \left(B^T P + e^{(1-\nu)j\frac{\pi}{2}} D^T C\right) & -\left(\gamma^2 I - D^T D\right) \end{pmatrix} < 0 \qquad (38)$$

which can be rewritten

$$\begin{pmatrix} e^{(\nu-1)j\frac{\pi}{2}}A^T P + Pe^{(1-\nu)j\frac{\pi}{2}}A & PB \\ B^T P & -\gamma^2 I \end{pmatrix} + \begin{pmatrix} e^{(\nu-1)j\frac{\pi}{2}}C^T \\ D^T \end{pmatrix}\begin{pmatrix} e^{(1-\nu)j\frac{\pi}{2}}C & D \end{pmatrix} < 0, \quad (39)$$

or

$$\begin{pmatrix} e^{(\nu-1)j\frac{\pi}{2}}A^T P + Pe^{(1-\nu)j\frac{\pi}{2}}A & PB & e^{(\nu-1)j\frac{\pi}{2}}C^T \\ B^T P & -\gamma^2 I & D^T \\ e^{(1-\nu)j\frac{\pi}{2}}C & D & -I \end{pmatrix} < 0\,. \quad (40)$$

Theorem 2. The L_2-gain of fractional system (A, B, C, D, ν) is bounded by γ if there exists a matrix $P = P^* \in \mathbb{C}^{n\times n}$ such that (40) holds. □

Proof. Theorem 2 results directly from section 4.2 analysis.

Theorem 2 is an extension of the bounded real lemma for fractional systems. It enables computation of a fractional system L_2-gain from its state-space "like" representation. However, stability is not guaranteed with this theorem. Note that if constraint on the positiveness of P is added and the system under consideration is integer, theorem 2 matches the well known bounded real lemma, that also ensures stability.
Note also that stability of fractional system can be inferred by adding one LMI constraint, such as those described in (Moze et al., 2005), to the LMI (40).
Whereas theorem 2 is only sufficient due to the extension of theorem 1 to the whole imaginary axis, it has been noticed that its application leads to accurate results in general. Future work will however focus on a sufficient and necessary condition.

4.3 *Proof of lemma 1*

The proof is largely inspired from (Scherer and Weiland, 2005) and corresponds to an extension to complex matrices of the results of Scherer and Weiland. It uses the following lemma to demonstrate the equivalence of both statements of lemma 1.

Lemma 2. Hamiltonian matrix (31) has no purely imaginary eigenvalues if and only if the Riccati equality

$$\tilde{A}^* X + X\tilde{A} + X\tilde{R}X + \tilde{Q} = 0\,, \quad (41)$$

has one stabilizing solution $P_- = P_-^* \in \mathbb{C}^{n\times n}$. □

Proof. The sufficiency is presented first, with some characteristics of the solution. The necessity is then proven.

Proof of sufficiency As matrix $\tilde{H}$ given by (31) is hamiltonian, its eigenvalues are symmetric about the origin. Hence $\tilde{H}$ has no purely imaginary eigenvalue if and only if there exist full rank matrix $Z \in \mathbb{C}^{2n\times n}$ and matrix $M \in \mathbb{C}^{n\times n}$, the later being stable, such that

$$\tilde{H}Z = ZM\,. \quad (42)$$

Let $Z = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}$, $\quad Z_1,\ Z_2 \in \mathbb{C}^{n\times n}$, $\quad (43)$

then condition (42) becomes

$$\tilde{A}Z_1 + \tilde{R}Z_2 = Z_1 M \text{ and } -\tilde{Q}Z_1 - \tilde{A}^* Z_2 = Z_2 M \quad (44)$$

or assuming that Z_1^{-1} exists,

$$\tilde{A}^* Z_2 + Z_2 Z_1^{-1}\tilde{A}Z_1 + Z_2 Z_1^{-1}\tilde{R}Z_2 + \tilde{Q}Z_1 = 0\,. \quad (45)$$

Right multiplying by Z_1^{-1} permits to infer that

$$X = Z_2 Z_1^{-1} \quad (46)$$

is a solution of Riccati equation (41).

Solution (46) is stabilizing Right multiplying by Z_1^{-1} the first equation in (44) gives

$$\tilde{A} + \tilde{R}X = Z_1 M\, Z_1^{-1}\,, \quad (47)$$

from which can be inferred that $\tilde{A} + \tilde{R}X$ and M have the same eigenvalues. As M is stable, the eigenvalues of $\tilde{A} + \tilde{R}X$ are all located in the left half complex plane. X is then a stabilizing solution.

The solution is hermitian As $\tilde{H}$ is hamiltonian, the relation

$$J\tilde{H} = \left(J\tilde{H}\right)^*,\ J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \quad (48)$$

holds.
Left and right multiplying (48) respectively by Z^* and Z gives

$$Z^* J\tilde{H}Z = \left(Z^* J\tilde{H}Z\right)^*. \quad (49)$$

Left multiplying (42) by $Z^* J$ and considering (49) shows that $Z^* JZM$ is hermitian, that is such that

$$Z^* JZM - M^* Z^* J^* Z = 0\,, \quad (50)$$

or, taking into account that $J^* = -J$, such that

$$Z^* JZM + M^* Z^* JZ = 0\,. \quad (51)$$

Hence,

$$Z^* JZ = 0\,, \quad (52)$$

or, considering (43),

$$Z_1^* Z_2 = Z_2^* Z_1\,. \quad (53)$$

Left and right multiplying (53) by respectively $\left(Z_1^*\right)^{-1} = \left(Z_1^{-1}\right)^*$ and Z_1^{-1} gives

$$Z_2 Z_1^{-1} = \left(Z_1^{-1}\right)^* Z_2^*\,, \quad (54)$$

which shows that matrix X is hermitian ($X = X^*$).

The stabilizing solution is unique Two solutions of Riccati equality (41), denoted $X_- = X_-^*$ and $X = X^*$ are related by

$$\left(\tilde{A}^* X + X\tilde{A} + X\tilde{R}X\right) - \left(\tilde{A}^* X_- + X_-\tilde{A} + X_-\tilde{R}X_-\right) = 0\,, \quad (55)$$

or, X, X_- and $\tilde{R}$ being hermitian, by

$$\left(\tilde{A} + \tilde{R}X_-\right)^*(X - X_-) + (X - X_-)\left(\tilde{A} + \tilde{R}X_-\right) + (X - X_-)\tilde{R}(X - X_-) = 0\,. \quad (56)$$

As $\tilde{R} > 0$,

$$\left(\tilde{A} + \tilde{R}X_-\right)^*(X - X_-) + (X - X_-)\left(\tilde{A} + \tilde{R}X_-\right) < 0\,, \quad (57)$$

and considering that X_- is stabilizing ($\tilde{A} + \tilde{R}X_-$ is stable),

$$X > X_- \,. \tag{58}$$

Riccati equality (41) has thus only one stabilizing solution. For all non stabilizing solution, (58) holds.

Existence of the stabilizing solution The solution properties have been derived assuming that Z_1^{-1} exists. Noticing that it does not exist if and only if there exists $q \in \mathbb{C}^n$, $q \neq 0$, such that

$$Z_1 q = 0 \,, \tag{59}$$

or, considering the first equation in (44), such that

$$\tilde{R} Z_2\, q = Z_1 M\, q \,. \tag{60}$$

Left multiplying by $q^* Z_2^*$ gives

$$q^* Z_2^* \tilde{R} Z_2\, q = q^* Z_2^* Z_1 M\, q \,, \tag{61}$$

which becomes, considering (53),

$$q^* Z_2^* \tilde{R} Z_2\, q = q^* Z_1^* Z_2 M\, q \,, \tag{62}$$

or, taking the conjugate transpose of (59),

$$q^* Z_2^* \tilde{R} Z_2\, q = 0 \,. \tag{63}$$

As $\tilde{R} > 0$,

$$Z_2 q = 0 \,. \tag{64}$$

Hence (59) implies $Zq = 0$, which contradicts the fact that Z has full rank and Z_1^{-1} does exist.

Proof of necessity As

$$\tilde{H}\begin{pmatrix} I & 0 \\ X & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ X & I \end{pmatrix} T \,, \tag{65}$$

where

$$T = \begin{pmatrix} \tilde{A} + \tilde{R}X & \tilde{R} \\ -\left(\tilde{A}^* X + X\tilde{A} + X\tilde{R}X + \tilde{Q}\right) & -\left(\tilde{A} + \tilde{R}X\right) \end{pmatrix} , \tag{66}$$

the eigenvalues of $\tilde{H}$ are the eigenvalues of T. Thus if X is a solution of the riccati equation (41), the eigenvalues of $\tilde{H}$ are those of both $\tilde{A} + \tilde{R}X$ and $-\left(\tilde{A} + \tilde{R}X\right)^*$. If X is a stabilizing solution, $\tilde{H}$ has no purely imaginary eigenvalue.

□

Lemma 1 is proven by first considering the implication. The converse is then proven.

Proof of sufficiency Suppose that matrix $\tilde{H}$ given by (31) has no purely imaginary eigenvalues. From eigenvalues properties, there exists $\varepsilon \in \mathbb{R}^+$ such that

$$\tilde{H}_\varepsilon = \begin{pmatrix} \tilde{A} & \tilde{R} \\ -\tilde{Q} - \varepsilon I & -\tilde{A}^* \end{pmatrix} \tag{67}$$

has no purely imaginary eigenvalues.
Then from lemma 2, there exists a matrix X_ε such that

$$\tilde{A}^* X + X\tilde{A} + X\tilde{R}X + \tilde{Q} = -\varepsilon I \,, \tag{68}$$

namely such that

$$\tilde{A}^* X + X\tilde{A} + X\tilde{R}X + \tilde{Q} < 0 \,. \tag{69}$$

Proof of necessity Considering a matrix $Y = Y^*$ that satisfies (32) such that

$$\tilde{A}^* Y + Y\tilde{A} + Y\tilde{R}Y + \tilde{Q} = P \,,\ P = P^* < 0 \,, \tag{70}$$

there exists a matrix $X = X^*$ such that the Riccati equality

$$\tilde{A}^* X + X\tilde{A} + X\tilde{R}X + \tilde{Q} = 0 \tag{71}$$

holds if there exists $\Delta = X - Y$, such that

$$\left(\tilde{A} + \tilde{R}Y\right)^* \Delta + \Delta\left(\tilde{A} + \tilde{R}Y\right) + \Delta\tilde{R}\Delta + P = 0 \,,\ \tilde{A} + \tilde{R}Y < 0 \,. \tag{72}$$

From lemma 2, (72) holds if and only if Hamiltonian matrix

$$H_\Delta = \begin{pmatrix} \tilde{A} + \tilde{R}Y & \tilde{R} \\ -P & -\left(\tilde{A} + \tilde{R}Y\right)^* \end{pmatrix} \tag{73}$$

has no purely imaginary eigenvalues.
The value $\lambda = j\omega$, $\omega \in \mathbb{R}$, is an eigenvalue of matrix H_Δ if and only if there exists a vector $V = \left(x^T\ y^T\right)^T$, $V \neq 0$, $x, y \in \mathbb{C}^n$ such that

$$\left(H_\Delta - j\omega I\right)V = 0 \,, \tag{74}$$

that is such that inequalities

$$\left(\tilde{A} + \tilde{R}Y\right)x + \tilde{R}y = 0 \ \text{and}\ -Px - \left(\tilde{A} + \tilde{R}Y\right)^* y = 0 \tag{75}$$

hold. Left multiplying by y^* and x^* respectively leads to

$$y^*\left(\tilde{A} + \tilde{R}Y\right)x + y^*\tilde{R}y = 0 \tag{76}$$

$$\text{and}\ -x^* Px - x^*\left(\tilde{A} + \tilde{R}Y\right)^* y = 0 \,. \tag{77}$$

Using the conjugate transpose of (77) in (76) gives the condition:

$$x^* Px = y^* \tilde{R}y \,. \tag{78}$$

As $P < 0$ and $\tilde{R} > 0$, condition (78) never holds and matrix H_Δ has no purely imaginary eigenvalue. Condition (72) thus holds, therefore $\tilde{H}$ has no purely imaginary eigenvalue.

5. APPLICATION

In (Moreau, 1995), car suspension design is presented as a robust controller synthesis problem, without consideration of the underlying technological aspect. This approach leads to the CRONE suspension (Moreau *et al.* 2002), its design relying on a CRONE controller design (Oustaloup and Mathieu, 1999).

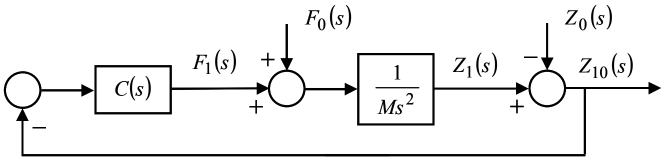

Fig. 1. Functional diagram associated with car suspension

The functional diagram associated with this approach is shown on Fig. 1, where $z_0(s)$ and $z_1(s)$ are respectively the vertical displacement of the road and of the car, $F_1(s)$ and $F_2(s)$ are respectively the load shift applied and the force due to the suspension. The feedback system then appears to regulate the suspension deflection $Z_{10}(s) = Z_1(s) - Z_0(s)$ around a null reference signal. The associated plant $G(s) = 1/\left(Ms^2\right)$ then appears to be only function of the mass M of the car. In (Ramus-Serment, 2001) the fractional CRONE controller $C(s)$ is given by

$$C(s) = C_0 \left(\frac{1+\frac{s}{\omega_b}}{\frac{s}{\omega_b}} \right) \left(\frac{1+\left(\frac{s}{\omega_b}\right)^{0.5}}{1+\left(\frac{s}{\omega_h}\right)^{0.5}} \right) \left(\frac{1}{1+\frac{s}{\omega_h}} \right), \quad (79)$$

where $\omega_b = 3.82\, rad/s$, $\omega_h = 3438\, rad/s$, and $C_0 = 9.95$.

The aim of this section is to obtain the modulus margin Δ_{mod} of the system for $M = 150\, kg$. As the modulus margin is the inverse of the L_2-gain of sensitivity function $S(s) = 1/(C(s)G(s))$, theorems 1 and 2 are thus successively applied to its fractional state space "like" representation:

$$\begin{cases} x^{(0.5)}(t) = A\,x(t) + B\,z_0(t) \\ z_{10}(t) = C\,x(t) + D\,z_0(t) \end{cases}, \quad (80)$$

where

$$A = \left(\begin{array}{c|c} 0 & -C_0\omega_h^{1.5}\omega_b/M \\ \hline & -C_0\omega_h^{1.5}\omega_b^{0.5}/M \\ & -C_0\omega_h^{1.5}/M \\ & -C_0\omega_h^{1.5}/\omega_b^{0.5}/M \\ I & 0 \\ & 0 \\ & -\omega_h^{1.5} \\ & -\omega_h \\ & -\omega_h^{0.5} \end{array} \right), \quad B = \begin{pmatrix} -C_0\omega_h^{1.5}\omega_b/M \\ -C_0\omega_h^{1.5}\omega_b^{0.5}/M \\ -C_0\omega_h^{1.5}/M \\ -C_0\omega_h^{1.5}/\omega_b^{0.5}/M \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad (81)$$

$$C = (0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ -1) \text{ and } D = -1. \quad (82)$$

Table 1 gives the results obtained with theorems 1 and 2 conjointly with a bisection algorithm. Last column shows the result obtained measuring the peak of the gain Bode diagram of $S(s)$. Results shown attest the efficiency of the theorems.

Table 1. Modulus margin computed using theorem 1, theorem 2 and through a graphical prospect.

	Theorem 1	Theorem 2	Graphic
$\|S\|_\infty$	1.131653	1.131653	1.1317
Δ_{mod}	0.8366	0.8366	0.8363

6. CONCLUSIONS

Fractional PID regulators and CRONE robust regulators are now well known in the field of fractional differentiation applications in control theory. Synthesis of these two classes of regulators is usually done in the frequency domain and is mainly based on the application of Nyquist criterion and its extensions. Paradoxically, no method based on more powerful tools such as Lyapunov stability or small gain theorem has been investigated for fractional systems. However, such methods are now essential for the extension of the existing control methods to time-varying or/and nonlinear fractional systems. In order to develop control methods for more complex fractional systems than the linear ones, this paper proposes two theorems for the computation of a fractional system L_2-gain. The first one is based on a frequency analysis and is easy to implement as it relies on the location of the eigenvalues of a matrix issued from the system state-space "like" representation. Using Riccati theory, the condition involved is then converted into an LMI constraint to give the second theorem. Relations between Riccati equality, Ricatti inequality and the location of the eigenvalues of a complex hamiltonian matrix are investigated for the proof of theorem 2.
Short term prospects of this study are in the fields covered by the usual bounded real lemma for integer systems such as $H\infty$ control, thus aiming at its extension to fractional systems.

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