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Regulation of Linear Systems with Unknown Additive Sinusoidal Sensor Disturbances ⋆

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Abstract: Known linear stabilizable and detectable systems, which are allowed to be non minimum phase, are considered: the problem of tracking unknown output reference trajectories and rejecting unknown input disturbances when the output tracking error is affected by unknown additive sensor disturbances is addressed. All the exogenous signals to be tracked and/or to be rejected are assumed to be the sum of sinusoids: only upper bounds on their numbers are supposed to be known, along with a set in which the output disturbance frequencies may range. A constructive algorithm is proposed to drive the regulation error exponentially to zero. The regulation strategy includes an on-line detector of the number of excited frequencies and exponentially converging estimates of the exosystems parameters. An example containing a variable number of frequencies is worked out and simulated.

Keywords: Adaptive Regulation, Sinusoidal disturbances, Adaptive Observers.

1. INTRODUCTION AND PROBLEM STATEMENT

A major goal in feedback control system design is the rejection of unknown disturbances. The output regulation theory for linear systems establishes the internal model principle (see Francis and Wonham (1976)): the feedback control which rejects a disturbance modelled by a linear exosystem must incorporate the exosystem itself. The exosystems for sinusoidal disturbances contain their unknown frequencies: in this case adaptive internal models are to be resorted to, as shown in Nikiforov (1998). Typically periodic disturbances are acting on the system input (see Bodson (2005) for a recent survey) in applications such as active suspensions design in Landau et. al. (2005), disk drives speed regulation in Liu and Yang (2004), eccentricity compensation in De Wit and Praly (2000), active noise control in Kuo and Morgan (1996), Wu and Bodson (2003), feedback control vibrations in helicopters in Bittanti and Moiraghi (1994), Ariyur and Krstic (1999): in these cases they can be viewed as matching disturbances in the tracking error dynamics. A different control problem arises when sinusoidal sensor disturbances act additively on the measured output. The stabilization problem for linear systems in this case has been addressed and solved in Serrani (2006) and Marino, Santosuosso and Tomei (2008) (see also applications in Wu and Bodson (2003), Zarikian and Serrani (2007)).

The aim of this paper is to combine the two problems above: given a linear stabilizable and detectable system, which is allowed to be non minimum phase, we solve the regulation problem concerned with tracking unknown reference trajectories and rejecting unknown input disturbances in the presence of unknown disturbances acting additively on the measured output, when all these signals are generated by unknown exosystems with simple eigenvalues on the imaginary axis.

The class of linear time invariant systems

$$\begin{cases} \dot{x} &= Ax + bu + Pw_t;; \quad x(0) = x_0 \\ \dot{w}_t &= Rw_t;\ w_t \in \Re^r,\ w_t(0) = w_{t0} \\ e &= cx - qw_t; \end{cases} \tag{1}$$

is considered, with state $x \in \Re^n$, control input $u \in \Re$; output $e \in \Re$ to be regulated to zero; the signals qw_t and Pw_t, respectively to be tracked and to be rejected, are both generated by the unknown linear exosystem $\dot{w}_t = Rw_t$, with state $w_t \in \Re^r$. The measurable output

$$y(t) = e(t) + \delta(t),\ y \in \Re \tag{2}$$

is the sum of the output $e(t)$ of system (1) and the disturbance $\delta(t)$ generated by an unknown linear exosystem

$$\begin{aligned} \dot{w}_\delta &= R_\delta w_\delta,\ w_\delta \in \Re^{r_\delta}, \quad w_\delta(0) = w_{\delta 0}, \\ \delta &= q_\delta w_\delta. \end{aligned} \tag{3}$$

We address the regulation problem formulated as follows.

Problem 1.1. Consider the extended system (1)-(3), with known A, b, c, unknown P, q, q_δ, R, R_δ where the orders of R and R_δ are known and the spectra $\sigma(R)$, $\sigma(R_\delta)$ contain only simple eigenvalues on the imaginary axis. Design a dynamic feedback controller

$$\begin{cases} \dot{\Psi} &= F(\Psi, y) \\ u &= H(\Psi, y) \end{cases} \tag{4}$$

from the output y, with state $\Psi \in \Re^{n_c}$, suitable initial condition $\Psi(0) = \Psi_0 \in \Re^{n_c}$ and $F(\Psi, y)$, $H(\Psi, y)$ suitable functions, such that in the closed loop system (1)-(4), the output variable $e(t)$ of system (1) tends exponentially to

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zero as t goes to infinity and the state variables $(x(t), \Psi(t))$ are bounded for any $t \geq 0$, for any initial conditions $x(0) \in \Re^n$, $w_t(0) \in \Re^r$, $w_\delta(0) \in \Re^{r_\delta}$, of the extended system (1)-(3).

We assume the following hypotheses to hold:

(H1) *The pair (A, b) is stabilizable, i.e.: $rank(A - \lambda I_n, b) < n$ implies* $\mathrm{Re}(\lambda) < 0$.

(H2) *The pair (A, c) is detectable, i.e.: $rank\big(A^T - \lambda I_n, c^T\big) < n$ implies* $\mathrm{Re}(\lambda) < 0$.

(H3) *rank* $\begin{pmatrix} A - \lambda I_n & b \\ c & 0 \end{pmatrix} = n + 1$ *for any eigenvalue λ of the matrix R.*

The conditions (H1)-(H3), were established in Francis and Wonham (1976) to be necessary and sufficient for the solution of the regulator problem arising when $\delta = 0$, R and r are known. In particular, by virtue of (H1)-(H3) there exists a unique matrix $\Gamma \in \Re^n \times \Re^r$ and a unique vector $\gamma \in \Re^r$ which solve the regulator equations $\Gamma R = A\Gamma + b\gamma + P$ and $c\Gamma = q$. The pair (Γ, γ) generates the signals $x_r = \Gamma w_t$ and $u_r = \gamma w_t$ which are the references for x and u respectively, since $\dot{x}_r = Ax_r + bu_r$ and $cx_r = qw_t$. The transformation $\tilde{x} = x - \Gamma w_t$ yields an error system

$$\begin{cases} \dot{\tilde{x}} = A\tilde{x} + b\,(u - u_r) \\ y = c\tilde{x} + \delta \end{cases} \tag{5}$$

with a scalar reference input $u_r = \gamma w_t$ which satisfies the "matching condition" given by the reference exosystem $\dot{w}_t = Rw_t$ and a scalar output disturbance $\delta(t) = q_\delta w_\delta$ generated by the disturbance exosystem $\dot{w}_\delta = R_\delta w_\delta$.

(H4) *The set of the eigenvalues of R_δ is disjoint with respect the set of the eigenvalues of A.*

Condition (H4), introduced in Serrani (2006), is shown in Marino, Santosuosso and Tomei (2008) to be necessary along with (H1) and (H2) for the solution of the stabilization problem of the system (1) with a controller from the disturbed output $cx + \delta(t)$.

(H5) *The unknown eigenvalues of R_δ belong to the set $S_\delta = \{\pm j\omega : \varkappa_i \leq \omega \leq \bar{\varkappa}_i, i \in [1, N]\}$ where $0 \leq \varkappa_1 < \bar{\varkappa}_1 < \varkappa_2 < \ldots < \varkappa_N < \bar{\varkappa}_N \leq \infty$ are suitable known numbers and the eigenvalues of R do not belong to S_δ.*

In this paper we prove that hypotheses (H1)-(H5) are sufficient for the solution of Problem 1.1.

2. REGULATION ALGORITHM

In this section we construct a dynamic controller that drives regulation error $e(t)$ in (1) exponentially to zero. By virtue of (H5) the eigenvalue $\lambda = 0$ may belong either to R or to R_δ. For ease of exposition we assume that $\lambda = 0$ belongs to R. In this case let $\{0, \pm i\omega_1, \ldots, \pm i\omega_M\}$ and $\{\pm i\bar{\omega}_1, \ldots, \pm i\bar{\omega}_{\bar{M}}\}$ be the eigenvalues of R and R_δ respectively, with $2M + 1 \triangleq r$, $2\bar{M} \triangleq r_\delta$ (M, $\bar{M}$ are known integers), $\omega_h \in \Re^+$, $h \in [1, M]$, $\omega_j \in \Re^+$, $j \in [1, \bar{M}]$. Notice that $u_r(t)$ and $\delta(t)$ may contain respectively only m and $\bar{m}$ harmonics of the exosystems, with $0 \leq m \leq M$, and $0 \leq \bar{m} \leq \bar{M}$. If we assume (without loss of generality) that the first harmonics ω_h, $h = 1, 2 \ldots m$ and $\bar{\omega}_j$, $j = 1, 2 \ldots \bar{m}$ of the exosystems appear in u_r and δ respectively, by setting $\theta \triangleq [\theta_1, \theta_2, \ldots \theta_m]^T$ and $\bar{\theta} \triangleq [\bar{\theta}_1, \bar{\theta}_2, \ldots \bar{\theta}_{\bar{m}}]^T$ where $s^m + \theta_1 s^{m-1} + \ldots + \theta_m = \prod_{h=1}^{m}(s + \omega_h^2)$, and $s^{\bar{m}} + \bar{\theta}_1 s^{\bar{m}-1} + \ldots + \bar{\theta}_{\bar{m}} = \prod_{j=1}^{\bar{m}}(s + \bar{\omega}_j^2)$, respectively, then $u_r(t)$ and $\delta(t)$ can be expressed as the outputs of some reduced order (unknown) exosystems with states $w \in \Re^{2m+1}$, $\bar{w} \in \Re^{2\bar{m}}$, defined as

$$\begin{cases} \dot{w} = \left[A_{2m+1} - (0, \theta_1, \ldots, 0, \theta_m, 0)^T C_{2m+1}\right] w, \\ u_r = C_{2m+1} w, \quad w(0) \in \Re^{2m+1} \end{cases} \tag{6}$$

$$\begin{cases} \dot{\bar{w}} = \left[A_{2\bar{m}} - (0, \bar{\theta}_1, \ldots, 0, \bar{\theta}_{\bar{m}})^T C_{2\bar{m}}\right] \bar{w}, \\ \delta = C_{2\bar{m}} \bar{w}, \quad \bar{w}(0) \in \Re^{2\bar{m}} \end{cases} \tag{7}$$

where (as in the rest of the paper), given a positive integer j, the "canonical" matrix $A_j \in \Re^j \times \Re^j$, and the "canonical" row vector $C_j \in \Re^j$ are

$$A_j = \begin{bmatrix} 0 & I_{j-1} \\ 0 & 0 \end{bmatrix}_{j \times j}, \; C_j = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}_{1 \times j}, \; j \in \mathcal{Z}^+ \tag{8}$$

with I_{j-1} denoting the identity matrix of order $j - 1$. Let the rank of the observability matrix of the couple (A, c) be $v \leq n$; then there is a suitable known coordinate change

$$\begin{bmatrix} x_u \\ x_o \end{bmatrix} = T_{\mathcal{O}} \tilde{x}, \quad \begin{matrix} x_u \in \Re^{n-v} \\ x_o \in \Re^v \end{matrix} \tag{9}$$

operating a Kalman decomposition of the system $\dot{\tilde{x}} = A\tilde{x}$ with output $c\tilde{x}$ into the unobservable and observable parts with vector states $x_u \in \Re^{n-v}$, $x_o \in \Re^v$, respectively. The system (5) by virtue of (9), becomes

$$\dot{x}_u = A_u x_u + A_{uo} x_o + b_u\,(u - u_r) \tag{10}$$

$$\dot{x}_o = [A_v - aC_v]\, x_o + b_o\,(u - u_r)\,; \qquad y = C_v x_o + \delta \tag{11}$$

where system (10) with state $x_u \in \Re^{n-v}$, known matrices $A_{uo} \in \Re^{n-v} \times \Re^v$, $A_u \in \Re^{n-v} \times \Re^{n-v}$, ($A_u$ is Hurwitz since by (H2) the pair (A, c) is detectable) known vector $b_u \in \Re^{n-v}$ is unobservable from the output $y = C_v x_o + \delta$ and system (11), (6), (7) with state $\left[x_o^T, w^T, \ \bar{w}^T\right]^T \in \Re^v \times \Re^{2m+1} \times \Re^{2\bar{m}}$ by virtue of (H2)-(H5) is observable from the output $y = C_v x_o + \delta$. The row vectors C_v, C_r are defined as in (8) with v, r, respectively, in place of j and $a \in \Re^v$, $b_o \in \Re^v$ are known vectors. Consider the linear filter

$$\begin{cases} \dot{\bar{x}}_o = [A_v - fC_v]\, \bar{x}_o + (f - a)\, y + b_o u, \\ \tilde{y} = y - C_v \bar{x}_o \end{cases} \tag{12}$$

with state $\bar{x}_o \in \Re^v$, inputs $u \in \Re$, $y \in \Re$, available output $\tilde{y} = (y - C_v \bar{x}_o) \in \Re$, where $a = [a_1, a_2, \ldots a_v]$, $b_o = [b_1, b_2, \ldots b_v]$ are defined in (11) and $f = [f_1, \ f_2 \ , \ldots, \ f_v]^T$ with $f_i \in \Re^+$, $1 \leq i \leq v$ are design parameters such that

$$p_f(s) = s^v + f_1 s^{v-1} + \ldots + f_{v-1} s + f_v \tag{13}$$

has all its roots with negative real part. By setting $\tilde{x}_o = x_o - \bar{x}_o$, from (11), (12), we obtain the error dynamics

$$\begin{cases} \dot{\tilde{x}}_o = [A_v - fC_v]\, \tilde{x}_o + (a - f)\, \delta - b_o u_r \\ \tilde{y} = C_v \tilde{x}_o + \delta = y - C_v \bar{x}_o. \end{cases} \tag{14}$$

Defining $m_T \triangleq m + \bar{m}$, $v_T \triangleq v + 2\,(m + \bar{m}) + 1$, and $\vartheta = [\vartheta_1, \ \vartheta_2, \ \ldots, \vartheta_{m_T}]^T$ so that $s^{m_T} + \vartheta_1 s^{m_T - 1} + \ldots + \vartheta_{m_T} = \prod_{h=1}^{m}\left(s + \omega_h^2\right) \prod_{j=1}^{\bar{m}}\left(s + \bar{\omega}_j^2\right)$, the autonomous system (14), (6), (7) (with dimension v_T) is observable from the available output $\tilde{y} = C_v \tilde{x}_o + \delta$ by virtue of (H2)-(H5); hence it is transformed into the observer canonical form

$$\begin{cases} \dot{\zeta} = A_{v_T} \zeta - \bar{f}_0\,[m_T]\, \tilde{y} - \displaystyle\sum_{i=1}^{m_T} \vartheta_i \bar{f}_i\,[m_T]\, \tilde{y}; \\ \tilde{y} = C_{v_T} \zeta \end{cases} \tag{15}$$

with state $\zeta \in \Re^{v_T}$, constant vectors $\bar{f}_i\,[m_T] \in \Re^{v_T}$, $i \in [0, m_T]$ given by

$$\begin{cases} \bar{f}_0\,[m_T] = [f_1, ..., f_v, 0, 0, 0, ..., 0, 0, 0]^T, \\ \bar{f}_1\,[m_T] = [0, 1, f_1, ..., f_v, 0, 0, ..., 0, 0]^T, \\ \quad\vdots \quad \vdots \\ \bar{f}_m\,[m_T] = [0, 0, 0, ..., 0, 1, f_1, ..., f_v, 0]^T. \end{cases} \tag{16}$$

Let $\mathcal{O}_{m,\bar{m}}(\theta, \bar{\theta})$ be the observability matrix of system (14) (6), (7) and $\Theta_{m_T}(\vartheta)$ be the observability matrix of system (15); by hypotheses (H2)-(H5) they are both nonsingular. The coordinate transformation from $\left[\tilde{x}_o^T, w^T, \bar{w}^T\right]^T \in \Re^{v_T}$ to $\zeta \in \Re^{v_T}$ is

$$\zeta = \left[\Theta_{m_T}^{-1}(\vartheta)\mathcal{O}_{m,\bar{m}}(\theta, \bar{\theta})\right]\left[\tilde{x}_o^T, w^T, \bar{w}^T\right]^T \tag{17}$$

where the matrix $\left[\Theta_{m_T}^{-1}(\vartheta)\mathcal{O}_{m,\bar{m}}(\theta, \bar{\theta})\right]$ is well defined by construction. Set $\bar{d}\,[m_T] = [\bar{d}_1, \ldots, \bar{d}_{v_T-1}]^T \in \Re^{v_T-1}$ where $\bar{d}_i \in \Re^+$, $1 \le i \le v_T - 1$, are positive real numbers such that all the roots of $p_d(s) = s^{v_T-1} + \bar{d}_1 s^{v_T-2} + \cdots + \bar{d}_{v_T-1}$ have negative real part. Define as in Marino and Tomei (1992) the filters ($\bar{\xi}_i \in \Re^{v_T-1}$, $\bar{\mu}_i \in \Re$, $1 \le i \le m_T$)

$$\begin{aligned} \dot{\bar{\xi}}_i &= \left[A_{v_T-1} - \bar{d}\,[m_T]\,C_{v_T-1}\right]\bar{\xi}_i - [0,\ I_{v_T-1}]\,\bar{f}_i\,[m_T]\,\tilde{y}; \\ \bar{\mu}_i &= C_{v_T-1}\bar{\xi}_i, \quad 1 \le i \le m_T. \end{aligned} \tag{18}$$

According to Marino and Tomei (1992), the transformation $\bar{\zeta} = \zeta - \left[0,\ \sum_{i=1}^{m+\bar{m}}\left(\bar{\xi}_i\vartheta_i\right)^T\right]^T$, mapping the state vector $\zeta \in \Re^{v_T}$ into a new state vector $\bar{\zeta} \in \Re^{v_T}$, transforms system (15) into an "adaptive observer" form

$$\begin{aligned} \dot{\bar{\zeta}} &= A_{v_T}\bar{\zeta} - \bar{f}_0\,[m_T]\,\tilde{y} + \begin{bmatrix} 1 \\ \bar{d}\,[m_T] \end{bmatrix}\bar{\mu}^T\vartheta, \\ \tilde{y} &= C_{v_T}\bar{\zeta} \end{aligned} \tag{19}$$

where $\bar{\mu} = [\bar{\mu}_1,\ \bar{\mu}_2,\ \ldots\bar{\mu}_{m_T}]^T$; defining the vector function $z_{m,\bar{m}} \in \Re^{v+1}$ as

$$z_{m,\bar{m}}(\theta, \bar{\theta}, \bar{\xi}_1, \ldots, \bar{\xi}_{m_T}, \bar{\zeta}) =$$

$$[0, I_{v+1}, 0]\,Adj\left[\mathcal{O}_{m,\bar{m}}(\theta, \bar{\theta})\right]\Theta_{m_T}(\vartheta)\left(\bar{\zeta}(t) + \begin{bmatrix} 0 \\ \sum_{i=1}^{m+\bar{m}}\bar{\xi}_i\vartheta_i \end{bmatrix}\right) \tag{20}$$

the state $x_o \in \Re^v$ of system (11) along with the reference input $u_r = C_{2m+1}w$ can be expressed as

$$\begin{pmatrix} x_o \\ u_r \end{pmatrix} = \begin{pmatrix} \bar{x}_o \\ 0 \end{pmatrix} + \frac{z_{m,\bar{m}}(\theta, \bar{\theta}, \bar{\xi}_1 \ldots, \bar{\xi}_{m_T}, \bar{\zeta})}{\det \mathcal{O}_{m,\bar{m}}(\theta, \bar{\theta})}. \tag{21}$$

The equality above requires the inversion of a parameter dependent mapping, so that if m and $\bar{m}$ are unknown the online estimation of these integers along with the vectors θ, $\bar{\theta}$ is required; this is described in the following.

2.1 Identification of the adaptive observer form system.

In this subsection we describe an estimation strategy for the state $\bar{\zeta}$ and the parameters ϑ_j $1 \le j \le m_T$ of system (19) in "adaptive observer" form which is a function of the maximum number of excited sinusoids $M_T \triangleq M + \bar{M}$. We follow the strategy in Marino and Santosuosso (2007), pages 355, 356. First, three cascaded filters are introduced to detect the number of frequencies that are excited. The first filter with state $\bar{\eta} = [\bar{\eta}_1,\ \bar{\eta}_2,\ \ldots\bar{\eta}_{2M_T+2}]^T \in \Re^{2M_T+2}$, initial condition $\bar{\eta}(0) \in \Re^{2M_T+2}$, input $\tilde{y}(t)$ given in (14), output $\nu = [\nu_1,\ \nu_2,\ \ldots\nu_{M_T+1}]^T$ is

$$\begin{cases} \dot{\bar{\eta}}_j = \bar{\eta}_{j+1},\ 1 \le j \le 2M_T + 1, \\ \dot{\bar{\eta}}_{2M_T+2} = -\sum_{j=1}^{2M_T+2}\bar{\alpha}_{3-j+M_T}\bar{\eta}_j + \tilde{y} \\ \nu = [\bar{\eta}_{2M_T+2},\ \bar{\eta}_{2M_T},\ \ldots, \bar{\eta}_4,\ \bar{\eta}_2]^T \end{cases} \tag{22}$$

where the design parameters $\bar{\alpha}_i$, $1 \le i \le 2M_T + 2$, are such that the polynomial

$$p_\alpha(s) = s^{2M_T+2} + \bar{\alpha}_1 s^{2M_T+1} + \bar{\alpha}_2 s^{2M_T} ... + \bar{\alpha}_{2M_T+2} \tag{23}$$

has all its roots with negative real part. The vector $\nu(t) \in \Re^{M_T+1}$ is the input to the second filter

$$\begin{cases} \dot{\Omega} = -c_1\Omega + c_2\nu\nu^T,\ \Omega(0) \ge 0, \\ \varrho_i = \left|\det\left(\Omega_i\right)\right|^{1/i},\ 1 \le i \le M_T + 1 \end{cases} \tag{24}$$

with state $\Omega \in \Re^{M_T+1} \times \Re^{M_T+1}$, $1 \le i \le M_T + 1$, symmetric and positive semi-definite initial condition $\Omega(0) \ge 0$, outputs $\varrho_i(t)$, $1 \le i \le M_T + 1$, where $\Omega_i \in \Re^i \times \Re^i$ denotes the matrix collecting the first $i \times i$ entries of Ω and c_1, c_2 are positive design parameters. The outputs $\varrho_i(t)$ of filter (24) are the inputs of the third filter with state $\chi = (\chi_1, \ldots, \chi_{M_T})^T$, where

$$\dot{\chi}_j = -\left[\sigma_j\left(\varrho_j\right) + \psi(\chi_j)\right]\chi_j + \tilde{\sigma}_j\left(\varrho_{M_T+1}\right),\ j \in [1, M_T] \tag{25}$$

in which $\chi_j(0) > 0$; $\sigma_j\left(\varrho_j\right)$ and $\tilde{\sigma}_j\left(\varrho_{M_T+1}\right)$ with $1 \le j \le M_T$ are suitable class $\mathcal{K}$ functions. The function $\psi(\chi_j)$ depends on a design parameter $\chi_0 \in \Re^+$ and is defined as $\psi(\chi_j) = 0$ if $\chi_j \le \chi_0$; $\psi(\chi_j) = 4\left(\chi_j - \chi_0\right)^2/\chi_j^2$ if $\chi_0 \le \chi_j \le 2\chi_0$ and $\psi(\chi_j) = 1$ if $2\chi_0 \le \chi_j$. Let c_3 be a positive design parameter; define the residuals:

$$i \in [1, M_T]: \begin{cases} \beta_i = 1 & \text{if } \varrho_i > c_3\chi_i, \\ \beta_i = \left(\dfrac{\varrho_i}{c_3\chi_i}\right)^2 & \text{if } \varrho_i \le c_3\chi_i. \end{cases} \tag{26}$$

By the arguments in Marino and Santosuosso (2007) page 355, Lemma 3.1, it is shown that all the states of the filters (22), (24), (25) and the residuals (26) are bounded, and that $\lim_{t\to\infty}\beta_i(t) = 1$ exponentially for $1 \le i \le m_T$, while $\lim_{t\to\infty}\beta_i(t) = 0$ exponentially for $m_T + 1 \le i \le M_T$. Next we design an adaptive observer of system (19) that estimates the exosystem's parameters ϑ_j $1 \le j \le m_T$, according to the guidelines in Marino and Santosuosso (2007). To this purpose we define the diagonal matrix $\bar{U}(t) \in \Re^{v+2M_T} \times \Re^{v+2M_T}$ with entries $\bar{U}_{i,i}(t) = 1$ for $1 \le i \le v$ and $\bar{U}_{i,i}(t) = \beta_k(t)$, with $k = \left\lceil\dfrac{i-v}{2}\right\rceil$ for $v + 1 \le i \le v + 2M_T$. Consider the vectors $\bar{\beta}(t) = \left[\bar{\beta}_0(t), \ldots, \bar{\beta}_{M_T}(t)\right]$ and $\bar{d}_\beta(t)$ defined from (26) as

$$\begin{cases} \bar{\beta}_0(t) = (1 - \beta_1(t));\ \bar{\beta}_{M_T}(t) = \beta_{M_T}(t) \\ \bar{\beta}_i(t) = \beta_i(t)(1 - \beta_{i+1}(t)) \text{ for } 1 \le i \le M_T - 1 \end{cases} \tag{27}$$

$$\bar{d}_\beta(t) = \bar{\beta}_0(t)\begin{pmatrix} \bar{d}\,[0] \\ 0 \end{pmatrix} + \ldots + \bar{\beta}_{M_T}(t)\left(\bar{d}\,[M_T]\right) \tag{28}$$

where in (28) the entries of the constant vectors $\bar{d}\,[i] \in \Re^{v+2i}$, $0 \le i \le M_T$ are design parameters such that the polynomials $s^{v+2i} + \left[s^{v+2i-1},\ s^{v+2i-2},\ \ldots, 1\right]\bar{d}\,[i]$ are Hurwitz. The matrix $\bar{U}(t)$ and the vector $\bar{d}_\beta(t)$ are tools to construct a generalization of the filters (18) that are adaptive with respect to the unknown number m_T: they are defined as

$$\begin{cases} \dot{\widehat{\xi}}_i = \bar{U}\left\{\left(A_{v+2M_T} - \bar{d}_\beta(t)C_{v+2M_T}\right)\widehat{\xi}_i \right. \\ \qquad \left. - \left([0,\ I_{v+2M_T}]\,\bar{f}_i\,[M_T]\right)\tilde{y}\right\} - c_4\left(I_{v+2M_T} - \bar{U}\right)\widehat{\xi}_i; \\ \widehat{\mu}_i = \beta_i C_{v+2M_T}\widehat{\xi}_i;\ \ 1 \le i \le M_T \end{cases} \tag{29}$$

with state variables $\widehat{\xi}_i \in \Re^{v+2M_T}$, $1 \le i \le M_T$, arbitrary initial conditions $\widehat{\xi}_i(0) \in \Re^{v+2M_T}$, where $c_4 \in \Re^+$ is a positive design parameter and $\bar{f}_i\,[M_T]$, $1 \le i \le M_T$ are defined according to (16) with M_T in place of m_T. We consider now an observer for system (19) which is adaptive with respect to the unknown number m_T of exosystem's frequencies

$$\begin{cases} \dot{\widehat{\zeta}} = U\left[A_{v+2M_T+1}\widehat{\zeta} + \bar{k}\left(\tilde{y} - \widehat{\zeta}_1\right) - \bar{f}_0\,[M_T]\,\tilde{y} \right. \\ \qquad \left. + d_\beta \sum_{i=1}^{M_T} \widehat{\mu}_i \widehat{\theta}_i\right] - c_5\left\{I_{v+2M_T+1} - U\right\}\widehat{\zeta}; \\ \dot{\widehat{\vartheta}}_i = \gamma_i \widehat{\mu}_i\left(\tilde{y} - \widehat{\zeta}_1\right) - \bar{\gamma}_i\left(1 - \beta_i\right)\widehat{\vartheta}_i; \quad 1 \le i \le M_T \end{cases} \tag{30}$$

with state $\widehat{\zeta} \in \Re^{v+2M_T+1}$, $\widehat{\vartheta} = \left[\widehat{\vartheta}_1, \widehat{\vartheta}_2, \ldots, \widehat{\vartheta}_{M_T}\right]^T \in \Re^{M_T}$, arbitrary initial conditions $\widehat{\zeta}(0) \in \Re^{v+2M_T+1}$, $\widehat{\vartheta}(0) \in \Re^{M_T}$, in which $\widehat{\zeta}_1 = C_{v+2M_T+1}\widehat{\zeta}$ and $\bar{f}_0\,[M_T]$ is defined according to (16) with M_T in place of m_T. In system (30) $U(t) = \begin{bmatrix} 1 & 0 \\ 0 & \bar{U} \end{bmatrix}$, $d_\beta = \begin{pmatrix} 1 \\ \bar{d}_\beta \end{pmatrix}$, the vector $\bar{k}(t) \in \Re^{v+2M_T+1}$ defined as $\bar{k}(t) = \left(A_{v+2M_T+1} + \bar{\lambda} I_{v+2M_T+1}\right)\left[1, \bar{d}_\beta^T(t)\right]^T$ with $\bar{\lambda} \in \Re^+$ any positive design parameter and $c_5 \in \Re^+$, $\gamma_i, \bar{\gamma}_i \in \Re^+$, with $1 \le i \le M_T$, are any positive design parameters. Let j be an integer such that $j \in [0, M_T]$; consider the subvectors

$$\widehat{\xi}_i^{[j]} = [I_{v+2j}, 0]\,\widehat{\xi}_i, \; \widehat{\zeta}^{[j]} = [I_{v+2j+1}, 0]\,\widehat{\zeta}, \quad \widehat{\vartheta}^{[j]} = [I_j, 0]\,\widehat{\vartheta}. \tag{31}$$

By following the arguments in Marino and Santosuosso (2007), page 356, Lemma 3.2, it is shown that the states of the systems (29), (30) are bounded and the partition in (31) obtained by setting $j = m_T$ complies with the following properties.

Claim 1. (i) The vectors $\widehat{\xi}_i$ with $1 \le i \le M_T$, $\widehat{\zeta}$ and $\widehat{\vartheta}$ are bounded; (ii) The vectors $(\bar{\xi}_i - \widehat{\xi}_i^{[m_T]})$, with $1 \le i \le m_T$, $(\bar{\zeta} - \widehat{\zeta}^{[m_T]})$, $(\vartheta - \widehat{\vartheta}^{[m_T]})$, tend exponentially to zero; (iii) the last $2\,(M_T - m_T)$ entries of the vectors $\widehat{\xi}_i$ with $1 \le i \le M_T$, the last $2\,(M_T - m_T)$ entries of the vector $\widehat{\zeta}$, the last $M_T - m_T$ entries of the vector $\widehat{\vartheta}$ tend exponentially to zero.

2.2 Identification of the exosystems structure

In this section we describe an algorithm to obtain from $\widehat{\vartheta}(t)$ along with the residuals $\beta(t)$ suitable estimates of $\theta \in \Re^m$ and $\bar{\theta} \in \Re^{\bar{m}}$ along with their dimensions m and $\bar{m}$ respectively. To this purpose, given $\widehat{\vartheta} = \left[\widehat{\vartheta}_1, \widehat{\vartheta}_2, \ldots, \widehat{\vartheta}_{M_T}\right]^T$ in (30), consider the polynomial

$$s^{M_T} + \widehat{\vartheta}_1(t)s^{M_T-1} + \ldots + \widehat{\vartheta}_{M_T}(t) = \prod_{j=1}^{M_T}\left(s - \varrho_j(t)\right) \tag{32}$$

with M_T complex roots $\varrho_j(t) \in \mathcal{C}$ $\;1 \le j \le M_T$, and define the vector $\hat{\omega}(t) = [\hat{\omega}_1, \hat{\omega}_2, \ldots, \hat{\omega}_{M_T}]^T \in \Re^{M_T}$,

$$\hat{\omega} \triangleq sort\left[\sqrt{\|\varrho_1(t)\|}, \ldots, \sqrt{\|\varrho_{M_T}(t)\|}\right], \tag{33}$$

obtained by sorting in descending magnitude order the square roots of the euclidean norms of the complex numbers $\varrho_j(t) \in \mathcal{C}$, $1 \le j \le M$. In order to determine if each angular frequency estimate $\hat{\omega}_j(t)$, $1 \le j \le m_T$, is associated to the input reference $u_r(t)$ or to the output disturbance $\delta(t)$, consider the piece-wise continuous time functions $\ell_1(t), \ldots \ell_{M_T}(t)$, $\bar{\ell}_1(t), \ldots \bar{\ell}_{M_T}(t)$, defined as

$$\begin{cases} \ell_h(t) = \beta_h(t) & \text{if } \; \hat{\omega}_h(t) \notin \mathcal{S}_\delta \\ \ell_h(t) = 0 & \text{if } \; \hat{\omega}_h(t) \in \mathcal{S}_\delta \end{cases} \quad 1 \le h \le M_T \tag{34}$$

$$\begin{cases} \bar{\ell}_j(t) = \beta_j(t) & \text{if } \; \hat{\omega}_j(t) \in \mathcal{S}_\delta \\ \bar{\ell}_j(t) = 0 & \text{if } \; \hat{\omega}_j(t) \notin \mathcal{S}_\delta \end{cases} \quad 1 \le j \le M_T \tag{35}$$

where $\mathcal{S}_\delta$ is the known subset of $\Re^+$ to which the angular frequencies $\delta(t)$ belong, while $\beta_1, \beta_2, \ldots, \beta_{M_T}$ are defined in (26). Define iteratively the vectors $V_i \in \Re^{M_T}$, $\bar{V}_i \in \Re^{M_T}$, with $i = M_T, \ldots, 1, 0$ as follows:
$V_{M_T} = \bar{V}_{M_T} = [0, 0, \ldots, 0, 1]^T$;
$V_{i-1}(t) = \left[A_{M_T} + \hat{\omega}_i^2(t)\ell_i(t)I_{M_T}\right]V_i(t), \quad M_T \ge i \ge 1;$
$\bar{V}_{i-1}(t) = \left[A_{M_T} + \hat{\omega}_i^2(t)\bar{\ell}_i(t)I_{M_T}\right]\bar{V}_i(t), \quad M_T \ge i \ge 1.$
It can be shown that the first m entries of $V_0(t)$ tend to θ and the first $\bar{m}$ entries of $\bar{V}_0(t)$ tend to $\bar{\theta}$. Continuous estimates of these constant parameter vectors are given by the filters with states $\hat{\theta} \in \Re^M$, $\hat{\bar{\theta}} \in \Re^{\bar{M}}$ respectively,

$$\dot{\hat{\theta}} = -c_6\left(\hat{\theta} - [I_M, 0]\,V_0(t)\right), \; \dot{\hat{\bar{\theta}}} = -c_6\left(\hat{\bar{\theta}} - [I_{\bar{M}}, 0]\,\bar{V}_0(t)\right), \tag{36}$$

with suitable initial conditions $\hat{\theta}(0) \in \Re^M$, $\hat{\bar{\theta}}(0) \in \Re^{\bar{M}}$, where c_6 is a constant design parameter. The functions $\sum_{h=1}^{M_T}\ell_h(t)$ and $\sum_{h=1}^{M_T}\bar{\ell}_h(t)$ can be shown to define suitable estimates of the positive numbers m and $\bar{m}$. Continuous estimates of these integers are given by the filters with states $\hat{m} \in \Re$ and $\hat{\bar{m}} \in \Re$ defined as

$$\begin{cases} \dot{\hat{m}} = -c_6[\hat{m}(t) - \sum_{h=1}^{M_T}\ell_h(t)] \\ \dot{\hat{\bar{m}}} = -c_6[\hat{\bar{m}}(t) - \sum_{h=1}^{M_T}\bar{\ell}_h(t)] \end{cases} \tag{37}$$

with suitable initial conditions $\hat{m}(0) \in \Re$ and $\hat{\bar{m}}(0) \in \Re$, where c_6 is the positive design parameter in (36). Let h, j be integers such that $h \in [0, M_T]$, $j \in [0, M_T]$ and consider a partition of the vectors $\hat{\theta}(t) \in \Re^M$, $\hat{\bar{\theta}} \in \Re^{\bar{M}}$ into subvectors whose dimension depends on the integers h, j as follows:

$$\begin{cases} \hat{\theta}^{[h]} = [I_h, 0]\,\hat{\theta}, \quad \hat{\theta}^{[h]} \in \Re^h, \quad h \in [0, M] \\ \hat{\bar{\theta}}^{[j]} = [I_j, 0]\,\hat{\bar{\theta}}, \quad \hat{\bar{\theta}}^{[j]} \in \Re^j, \; j \in \left[0, \bar{M}\right]. \end{cases} \tag{38}$$

The vector functions in (36) and (37) defined above can be shown to be bounded, and by construction (see Marino and Tomei (1992)) comply with the following property:

Claim 2. (i): The functions $\hat{m}(t)$, $\hat{\bar{m}}(t)$, $\hat{\theta}(t)$, $\hat{\bar{\theta}}(t)$ in (37) and (38) are bounded. (ii): $\lim_{t\to\infty}\hat{m}(t) = m$, $\lim_{t\to\infty}\hat{\bar{m}}(t) = \bar{m}$, $\lim_{t\to\infty}\hat{\theta}^{[m]}(t) = \theta$ and $\lim_{t\to\infty}\hat{\bar{\theta}}^{[\bar{m}]}(t) = \bar{\theta}$. (iii): the last $M_T - m$ entries of $\hat{\theta}(t)$ and the last $M_T - \bar{m}$ entries of $\hat{\bar{\theta}}(t)$ tend exponentially to zero.

2.3 Compensator design

In this section we describe the control law constructed after the estimation of the state $\bar{\zeta} \in \Re^{v_T}$ of the system (19) and the unknown parameter vector $\vartheta \in \Re^{m_T}$ along with the vectors $\theta \in \Re^m$, $\bar{\theta} \in \Re^{\bar{m}}$ via the estimates $\hat{\theta}(t) \in \Re^M$, $\hat{\bar{\theta}}(t) \in \Re^{\bar{M}}$. The crucial step in the controller design is obtaining estimates of the state $x_o \in \Re^v$ of system (11) and of the reference input u_r given by (21). We follow the adaptive saturation approach in Marino and Santosuosso (2007) to the problem in this note. In particular, let $s_C(z): \Re \rightarrow \Re$ be the continuous and piece-wise linear even scalar function defined as: $s_C(z) = 1$ for $|z| \leq 1/4$ and $s_C(z) = 0$ for $|z| \geq 3/4$; define the $(M+1)$ functions $\kappa_h(t) = s_C(h - \hat{m}(t))$ for $0 \leq h \leq M$ and the $(\bar{M}+1)$ functions $\bar{\kappa}_j(t) = s_C(j - \hat{\bar{m}}(t))$ for $0 \leq j \leq \bar{M}$. Consider the $M + \bar{M} + 2$ scalar filters with states $p_h(t)$, $0 \leq h \leq M$, and $\bar{p}_j(t)$, $0 \leq j \leq \bar{M}$, defined as

$$\begin{aligned} \dot{p}_h &= c_7[-c_8 p_h + |1-\kappa_h| + \tfrac{2}{\pi}\arctan(c_9 \left|\tilde{y} - \widehat{\zeta}_1\right|)], \\ \dot{\bar{p}}_j &= c_7[-c_8 \bar{p}_j + |1-\bar{\kappa}_j| + \tfrac{2}{\pi}\arctan(c_9 \left|\tilde{y} - \widehat{\zeta}_1\right|)], \end{aligned} \quad (39)$$

driven by the inputs $(1-\kappa_h(t))$, $(1-\bar{\kappa}_j(t))$, $0 \leq h \leq M$, $0 \leq j \leq \bar{M}$, along with the estimation error $\left|\tilde{y} - \widehat{\zeta}_1\right|$, where c_7, c_8, and c_9 are positive design parameters. It can be shown that if $p_h(0) > 0$, $\bar{p}_j(0) > 0$ then all $p_h(t)$, $\bar{p}_j(t)$ with $h \neq m$, $j \neq \bar{m}$ are greater than a positive lower bound, while $p_m(t)$ and $\bar{p}_{\bar{m}}(t)$ both tend exponentially to zero as t goes to infinity. The filters in (39) are tools to estimate (x_o, u_r) given by (21) via the adaptive saturation algorithm (see Marino and Santosuosso (2007))

$$\begin{cases} \Upsilon_{hj} = \begin{cases} \dfrac{1}{\det \widehat{\mathcal{O}}_{h,j}} & \text{if } \left|\det \widehat{\mathcal{O}}_{h,j}\right| > (p_h + \bar{p}_j), \\ \dfrac{\det \widehat{\mathcal{O}}_{h,j}}{(p_h + \bar{p}_j)^2} & \text{if } \left|\det \widehat{\mathcal{O}}_{h,j}\right| \leq (p_h + \bar{p}_j), \end{cases} \\ \begin{pmatrix} \hat{x}_o \\ \hat{u}_r \end{pmatrix} = \begin{pmatrix} \bar{x}_o \\ 0 \end{pmatrix} + \sum_{h=0}^{M}\sum_{j=0}^{\bar{M}} \kappa_h \bar{\kappa}_j \left(\Upsilon_{hj} \widehat{z}_{h,j}\right). \end{cases} \quad (40)$$

where $\widehat{z}_{h,j} = z_{h,j}(\hat{\theta}^{[h]}, \hat{\bar{\theta}}^{[j]}, \widehat{\xi}_1^{[h+j]}, \ldots, \widehat{\xi}_{m_T}^{[h+j]}, \widehat{\zeta}^{[h+j]})$ and $\widehat{\mathcal{O}}_{h,j} = \mathcal{O}_{h,j}(\hat{\theta}^{[h]}, \hat{\bar{\theta}}^{[j]})$ are obtained from (17) and (20) with h and j in place of m, $\bar{m}$ respectively, and $\hat{\theta}^{[h]}$, $\hat{\bar{\theta}}^{[j]}$, $\widehat{\xi}_1^{[h+j]}$, $\ldots$, $\widehat{\xi}_{m_T}^{[h+j]}$, $\widehat{\zeta}^{[h+j]}$ are defined via (31), (38), in place of θ, $\bar{\theta}$, ξ_1, $\ldots$,$\xi_{m+\bar{m}}$, ζ. The task of the positive signals $p_h(t)$, $\bar{p}_j(t)$ is to avoid the singularities in which $\det \widehat{\mathcal{O}}_{h,j}(t) = 0$, while the functions κ_h, $\bar{\kappa}_j$ select the correct disturbance estimate of (x_0, u_r) given by (21) with indices $(m, \bar{m})$ among all combinations $\Upsilon_{hj}\widehat{z}_{h,j}$ with $h \in [0, M]$, $j \in [0, \bar{M}]$. Since by hypothesis (H1) the system (1) is stabilizable and by virtue of (H2) the matrix A_u is Hurwitz, then the couple (A_o, b_o) is stabilizable and there exists a suitable row vector $k_c \in \Re^v$ such that the matrix $A_o + b_o k_c$ is Hurwitz. The overall compensating control law is

$$u = k_c \hat{x}_o + \hat{u}_r \quad (41)$$

Previous arguments lead to the following result.

Proposition 2.1. Consider system (1)-(3). If assumptions (H1)-(H5) are satisfied, then Problem 1.1 is solvable any initial condition $x(0) \in \Re^n$, $w_t(0) \in \Re^r$, $w_\delta(0) \in \Re^{r_\delta}$ via the control law (40)-(41) with dynamics (12), (22), (24), (25), (29), (30), (36), (37), (39), for any initial condition of the dynamic compensator such that $\Omega(0) \geq 0$, $\chi_i(0) > 0$, $i \in [1, M+\bar{M}]$, $p_h(0) > 0$, $\bar{p}_j(0) > 0$, $h \in [0, M]$, $j \in [0, \bar{M}]$, for any admissible choice of the design parameters, which are: any positive real numbers f_1, $f_{2,}$, $\ldots f_v$, and $\bar{\alpha}_1$, $\bar{\alpha}_2$, $\ldots \bar{\alpha}_{2(M+\bar{M})+2}$, such that the polynomials $p_f(s)$ in (13) and $p_\alpha(s)$ in (23) respectively, are Hurwitz; any vectors $\bar{d}[i] \in \Re^{v+2i}$, $0 \leq i \leq M+\bar{M}$ such that the polynomials $s^{v+2i} + \left[s^{v+2i-1}, s^{v+2i-2}, \ldots, 1\right] \bar{d}[i]$ are Hurwitz; any class $\mathcal{K}$ functions $\sigma_j(\varrho_j)$ and $\tilde{\sigma}_j(\varrho_{M_T+1})$ with $1 \leq j \leq M+\bar{M}$; any positive real numbers $c_j \in \Re^+$, with $1 \leq j \leq 9$, $\gamma_i \in \Re^+$, $\bar{\gamma}_i \in \Re^+$, $1 \leq i \leq M+\bar{M}$, $\chi_0 \in \Re^+$, $\bar{\lambda} \in \Re^+$, and any row vector $k_c \in \Re^v$ such that the matrix $(A_o + b_o k_c)$ is Hurwitz.

Proof. It is a consequence of the Claims 1, 2 along with the arguments in Marino and Santosuosso (2007), Marino, Santosuosso and Tomei (2008). ■

3. AN EXAMPLE

Consider the system

$$\begin{cases} \dot{x} = \begin{bmatrix} -3 & 1 \\ 1 & 2 \end{bmatrix} x + \begin{bmatrix} -1 \\ 2 \end{bmatrix} (u - u_r) \\ e = \begin{bmatrix} 1 & 0 \end{bmatrix} x; \qquad y = e + \delta, \end{cases} \quad (42)$$

with state $x = [x_1, x_2]^T \in \Re^2$, control input $u \in \Re$. System (42) has non minimum phase and unstable unforced dynamics, that are stabilized via the state feedback control law $u = k_c x$ with $k_c = [-1, 0]$. The output $y = x_1 + \delta$ is available for measurement, where $u_r(t)$ is an input reference and $\delta(t)$ is a disturbance. The regulation task is to drive the state x_1 to zero. We consider two operating conditions, in particular for $0 < t \leq 100$ we set $u_r(t) = -1$, $\delta(t) = \sin(1.5t)$ and for $100 < t \leq 200$ we set $u_r(t) = \sin(0.5t)$, $\delta(t) = \frac{1}{2}(\sin(t) + \sin(2t))$. We assume that $\mathcal{S}_\delta$ is the set of all angular frequencies $\omega \geq 0.7$. We construct a controller to regulate the output trajectory $x_1(t)$ to zero via the algorithm proposed in this note, by setting $M = 1$, $\bar{M} = 2$, so that the extended system (1)-(3) has order 9. We simulate the algorithm for $0 < t \leq 200$ choosing the numerical values of the constant design parameters as follows: in system (12) we choose $f = [5, 6]$, in system (22) we set the parameters $\bar{\alpha}_1$, $\bar{\alpha}_2$, $\ldots$, $\bar{\alpha}_8$, so that $p_\alpha(s) = s^8 + \bar{\alpha}_1 s^5 + ... + \bar{\alpha}_7 s + \bar{\alpha}_8$ is the polynomial whose roots coincide all with the number $-3/2$. In system (24) we let $c_1 = 1$, $c_2 = 10^4$, in system (25) we set $\sigma_1 = 0.5\arctan(0.5\varrho_1)$, $\sigma_2 = 0.5\arctan(5\varrho_2)$, $\sigma_2 = 0.5\arctan(200\varrho_3)$, $\tilde{\sigma}_1 = \tilde{\sigma}_2 = \tilde{\sigma}_3 = 50\arctan(100\varrho_3)$, $\chi_0 = 10^6$, in expression (26) we set $c_3 = 10^{-4}$, in (28) we set $\bar{d}[0]$, $\bar{d}[1]$, $\bar{d}[2]$, $\bar{d}[3]$ so that the roots of the polynomials $s^{2+2h} + \left[s^{1+2h}, s^{2h}, \ldots, 1\right] \bar{d}[h]$ with $h = 0, 1, 2, 3$ coincide with the number $-3/2$ and in system (29) we set $c_4 = 1$. In system (30) we set $c_5 = 1$, $\bar{\lambda} = 10$, $\gamma_1 = 10^5$; $\gamma_2 = 15 \cdot 10^5$, $\gamma_3 = 2 \cdot 10^5$; $\bar{\gamma}_1 = \bar{\gamma}_2 = \bar{\gamma}_3 = 500$; in system (36) and (37) we set $c_6 = 1$, in system (39) we set $c_7 = 1$, $c_8 = 50$, $c_9 = 10^3$. The compensator dynamics have been simulated starting from zero initial conditions except for $\chi_i(0) = 10^6$, $i \in [1, 2, 3]$ and $p_j(0) = \bar{p}_h(0) = 50$, $j \in [0, 1]$ and $h \in [0, 2]$.

The simulation results are reported in Figures 1-3. In Figure 1 the input $u_r(t)$ along with disturbance $\delta(t)$ are plotted. In Figure 2 the time histories of the estimates $\hat{\theta}_1(t)$, $\hat{\bar{\theta}}_1(t)$, $\hat{\bar{\theta}}_2(t)$ of the parameters θ_1, $\bar{\theta}_1$, $\bar{\theta}_2$, are described. For $0 < t \leq 100$ in the reference exosystem (6) there aren't unknown parameters while in the disturbance exosystem (7) we have $\bar{\theta}_1 = 2.25$, so that $\hat{\theta}_1(t) \rightarrow 0$, $\hat{\bar{\theta}}_1(t) \rightarrow 2.25$ and $\hat{\bar{\theta}}_2(t) \rightarrow 0$. For $100 < t \leq 200$ the exosystem (6) has the

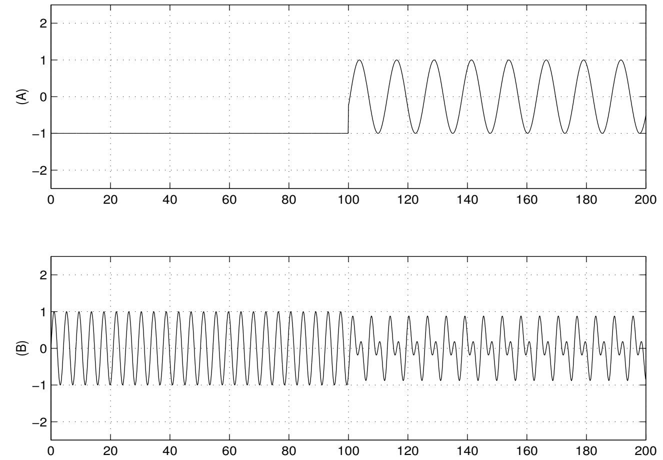

Fig. 1. In (A): the reference input $u_r(t)$; in (B): the disturbance $\delta(t)$.

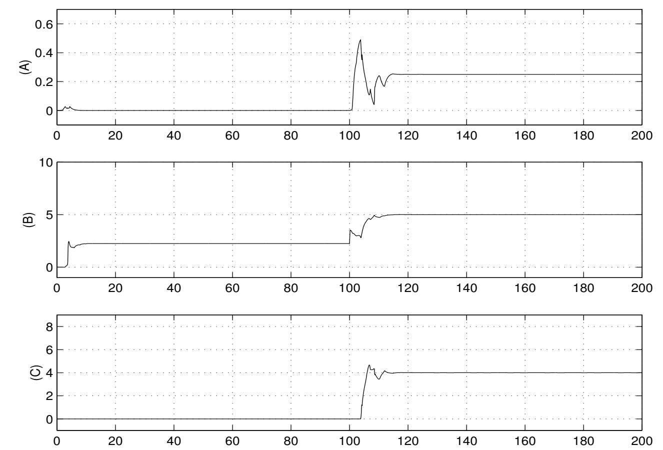

Fig. 2. In (A): the estimate $\hat{\theta}_1(t)$; in (B): the estimate $\hat{\bar{\theta}}_1(t)$, in (C) the estimate $\hat{\bar{\theta}}_2(t)$.

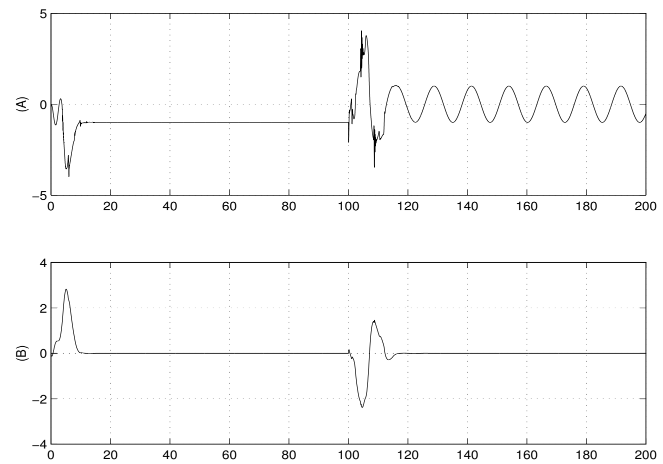

Fig. 3. In (A): the control input $u(t)$; in (B): the system undisturbed output $e(t) = x_1(t)$.

unknown parameter $\theta_1 = 0.25$, while in the exosystem (7) there are the parameters $\bar{\theta}_1 = 5$, $\bar{\theta}_2 = 4$; as a consequence $\hat{\theta}_1(t) \to 0.25$, $\hat{\bar{\theta}}_1(t) \to 5$, $\hat{\bar{\theta}}_2(t) \to 4$. Figure 3 reports the control input $u(t)$ that drives exponentially to zero the regulation error $e(t)$ in all three operating conditions.

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