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Stabilization of Multidimensional Wave Equations under Non-Collocated Controls and Observations ⋆

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Abstract: The objective of this paper is to deal with the stabilization of multi-dimensional wave equations under non-collocated control and observation with the following cases: a) internal distributed control and boundary observation; b) boundary control and internal distributed observation; c) locally internal distributed control and boundary observation.

Keywords: Distributed parameter system; Exponentially stable; Observers; Non-collocated control and observation; Wave equations.

1. INTRODUCTION

Collocated method is commonly used in most of the control designs for the systems described by partial differential equations. This control design method makes actuators and sensors in the same positions. The idea is so natural in the sense that a collocated control system is always passive. That is to say, the increase of the energy stored in the system does not exceed the energy that enters from the external world.

However, it has been found by engineers for a long time that the performance of the collocated control design in engineering practice is not always good enough, see Chodavarapu et al (1996). Although the non-collocated control method has been widely used in the engineering systems control, see Lacarbonara et al (2004), Liu and Yuan (2003), Queiroz et al (2002), Ryu et al (2003), Spector et al (1990) ,Udwadia (1991) and Wu (2001), the theoretical studies from the mathematical control point of view for these systems are quite few. The first difficulty for the non-collocated control is that the open-loop forms are usually not minimum-phase, which makes the closed-loop systems unstable with a small increment of feedback controller gain. The second difficulty arises from the non-dissipativity for the closed-loop forms, which gives rise to difficulty in applying the traditional Lyapunov methods or the energy multiplier methods to analyze the stability. Compared with the huge works on the stabilization of collocated PDEs in literature, the study of the non-collocated PDEs is fairly scarce.

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In order to overcome these difficulties arising in non-collocated control systems, the observer-based feedback was used recently to stabilize a one-dimensional wave equation with boundary control and non-collocated observation in Guo and Xu (2007). The same idea was applied in Guo et al (2007) to stabilize an Euler-Bernoulli beam equation with the boundary control and non-collocated observation. In Deguenon et al (2006), the abstract observers for a class of well-posed regular infinite-dimensional systems were designed but the stabilization was not addressed.

In this paper, we consider the stabilization of multi-dimensional wave equations under non-collocated control and observation with the following cases: a) internal distributed control and boundary observation; b) boundary control and internal distributed observation; c) locally internal distributed control and boundary observation. To our knowledge, this is the first attempt to deal with the stabilization of the multi-dimensional PDEs' system with the non-collocated control and observation.

We proceed as follows. In section 2, we introduce an abstract second order system with some basic assumptions. Section 3 is devoted to the design and the well-posedness of the observer. In Section 4, we design the observed state feedback control and prove the exponential stability of the corresponding closed-loop system. The stabilizability of three multi-dimensional wave equations with non-collocated controls and observations is proved in Section 5 as the application of the abstract result and the method.

2. ABSTRACT SETTING

We consider the following infinite-dimensional abstract second order linear system in three Hilbert spaces, the state space X, control space U and observation space Y

$$\begin{cases} w_{tt} + \mathscr{A} w = \eta \mathscr{B} u, \\ w(0) = w_1, \ w_t(0) = w_2, \\ y(t) = \varepsilon \mathscr{C} w_t, \end{cases} \tag{1}$$

where $\mathscr{A} : \mathcal{D}(\mathscr{A})(\subset X) \to X$ is a positive and self-adjoint operator in X. η, ε are positive constants. For the two operators $\mathscr{B}$ and $\mathscr{C}$, we make assumptions (H1) and (H2) that correspond to the different problems respectively:

(H1). $\mathscr{B} \in \mathcal{L}(U; [\mathcal{D}(\mathscr{A}^{\frac{1}{2}})]')$; that is, $\mathscr{A}^{-\frac{1}{2}}\mathscr{B} \in \mathcal{L}(U; X)$ or $\mathscr{B}^*\mathscr{A}^{-\frac{1}{2}} \in \mathcal{L}(X; U)$, where $\mathscr{B}^* \in \mathcal{L}(\mathcal{D}(\mathscr{A}^{\frac{1}{2}}); U)$ is defined by

$$\langle \mathscr{B}u, f \rangle_{[\mathcal{D}(\mathscr{A}^{\frac{1}{2}})]' \times \mathcal{D}(\mathscr{A}^{\frac{1}{2}})} = \langle u, \mathscr{B}^* f \rangle_U, \forall f \in \mathcal{D}(\mathscr{A}^{\frac{1}{2}}), u \in U.$$

And $\mathscr{C} \in \mathcal{L}(X; Y)$.

(H2). $\mathscr{C} \in \mathcal{L}(\mathcal{D}(\mathscr{A}^{\frac{1}{2}}); Y)$; that is, $\mathscr{C}\mathscr{A}^{-\frac{1}{2}} \in \mathcal{L}(X; Y)$ or $\mathscr{A}^{-\frac{1}{2}}\mathscr{C}^* \in \mathcal{L}(Y; X)$, where $\mathscr{C}^* \in \mathcal{L}(Y; [\mathcal{D}(\mathscr{A}^{\frac{1}{2}})]')$ is defined by

$$\langle \mathscr{C}^* y, f \rangle_{[\mathcal{D}(\mathscr{A}^{\frac{1}{2}})]' \times \mathcal{D}(\mathscr{A}^{\frac{1}{2}})} = \langle y, \mathscr{C} f \rangle_Y, \forall f \in \mathcal{D}(\mathscr{A}^{\frac{1}{2}}), y \in Y.$$

And $\mathscr{B} \in \mathcal{L}(U; X)$.

The system (1) can be written as the first order system in the Hilbert state space $H = \mathcal{D}(\mathscr{A}^{\frac{1}{2}}) \times X$:

$$\begin{cases} z_t = Az + \eta B u, \\ z(0) = (w_1, w_2)^\top, \\ y(t) = \varepsilon C z, \end{cases} \tag{2}$$

where

$$A = \begin{pmatrix} 0 & I \\ -\mathscr{A} & 0 \end{pmatrix} = -A^* : \mathcal{D}(A)(\subset H) \to H, \quad \mathcal{D}(A) = \mathcal{D}(A^*) = \mathcal{D}(\mathscr{A}) \times \mathcal{D}(\mathscr{A}^{\frac{1}{2}}); \tag{3}$$

$$B = \begin{pmatrix} 0 \\ \mathscr{B} \end{pmatrix}, \quad A^{-1}B = \begin{pmatrix} -\mathscr{A}^{-1}\mathscr{B} \\ 0 \end{pmatrix} \in \mathcal{L}(U; H), \tag{4}$$

$$B^* = \begin{pmatrix} 0 & \mathscr{B}^* \end{pmatrix}; \tag{5}$$

$$C = \begin{pmatrix} 0 & \mathscr{C} \end{pmatrix}, \quad A^{-1}C^* = \begin{pmatrix} -\mathscr{A}^{-1}\mathscr{C}^* \\ 0 \end{pmatrix} \in \mathcal{L}(Y; H), \tag{6}$$

$$C^* = \begin{pmatrix} 0 \\ \mathscr{C}^* \end{pmatrix}. \tag{7}$$

We give the following additional assumptions.

(H3). The C_0-semigroup $e^{A_\eta t}$ generated by A_η is exponentially stable on H: there exist constants $M \geq 1$ and $\delta > 0$ such that

$$\|e^{A_\eta t}\|_{\mathcal{L}(H)} \leq M e^{-\delta t}, \ t \geq 0, \tag{8}$$

where

$$\begin{cases} A_\eta := \begin{pmatrix} 0 & I \\ -\mathscr{A} & -\eta\mathscr{B}\mathscr{B}^* \end{pmatrix} = A - \eta B B^* \\ \quad : \mathcal{D}(A_\eta)(\subset H) \to H, \\ \mathcal{D}(A_\eta) = \{(z_1, z_2)^\top |\ z_1, z_2 \in \mathcal{D}(\mathscr{A}^{\frac{1}{2}}), \\ \qquad \mathscr{A}^{\frac{1}{2}} z_1 + \eta \mathscr{A}^{-\frac{1}{2}} \mathscr{B}\mathscr{B}^* z_2 \in \mathcal{D}(\mathscr{A}^{\frac{1}{2}})\}. \end{cases} \tag{9}$$

(H4). The C_0-semigroup $e^{A_\varepsilon t}$ generated by A_ε is exponentially stable on H: there exist constants $L \geq 1$ and $\gamma > 0$ such that

$$\|e^{A_\varepsilon t}\|_{\mathcal{L}(H)} \leq L e^{-\gamma t}, \ t \geq 0. \tag{10}$$

where

$$\begin{cases} A_\varepsilon := \begin{pmatrix} 0 & I \\ -\mathscr{A} & -\varepsilon\mathscr{C}^*\mathscr{C} \end{pmatrix} = A - \varepsilon C^* C \\ \quad : \mathcal{D}(A_\varepsilon)(\subset H) \to H, \\ \mathcal{D}(A_\varepsilon) = \{(z_1, z_2)^\top |\ z_1, z_2 \in \mathcal{D}(\mathscr{A}^{\frac{1}{2}}) \subset \mathcal{D}(\mathscr{C}), \\ \qquad \mathscr{A}^{\frac{1}{2}} z_1 + \varepsilon \mathscr{A}^{-\frac{1}{2}} \mathscr{C}^*\mathscr{C} z_2 \in \mathcal{D}(\mathscr{A}^{\frac{1}{2}})\}. \end{cases} \tag{11}$$

(H5). The system (1) or (2) is well-posed in the sense of D. Salamon (see e.g., Curtain (1997)), i.e., there exists a $T > 0$ such that for any $u \in L^2(0, T; U)$ and any initial datum $z(0) \in H$, there exists a unique solution $z(t) \in C(0, T; H)$ to the equation (2) such that

$$\|z(T)\|_H^2 + \int_0^T \|y(\tau)\|_Y^2 d\tau \leq C_T \left(\|z(0)\|_H^2 + \int_0^T \|u(\tau)\|_U^2 d\tau \right), \tag{12}$$

where $C_T > 0$ is a constant independent of u and $z(0)$.

3. OBSERVER DESIGN

In this section, we first design the infinite-dimensional version of Luenberger type observer for the system (1) as following

$$\begin{cases} \hat{w}_{tt} + \mathscr{A}\hat{w} + (\eta\mathscr{B}\mathscr{B}^*\hat{w}_t + \varepsilon\mathscr{C}^*\mathscr{C}\hat{w}_t) = \mathscr{C}^* y, \\ \hat{w}(0) = \hat{w}_0, \hat{w}_t(0) = \hat{w}_1, \end{cases} \tag{13}$$

which, in the first order form in H, is

$$\begin{cases} \hat{z}_t = \mathbb{A}\hat{z} + C^* y, \\ \hat{z}(0) = \hat{z}_0, \end{cases} \tag{14}$$

where $\hat{z} = (\hat{w}, \hat{w}_t)^\top$, $\hat{z}_0 = (\hat{w}_0, \hat{w}_1)^\top$ and

$$\mathbb{A} = \begin{pmatrix} 0 & I \\ -\mathscr{A} & -(\eta\mathscr{B}\mathscr{B}^* + \varepsilon\mathscr{C}^*\mathscr{C}) \end{pmatrix} : \mathcal{D}(\mathbb{A})(\subset H) \to H; \tag{15}$$

$$\begin{aligned} \mathcal{D}(\mathbb{A}) = \ & \{(z_1, z_2)^\top |\ z_1, z_2 \in \mathcal{D}(\mathscr{A}^{\frac{1}{2}}) \subset \mathcal{D}(\mathscr{B}^*) \cap \mathcal{D}(\mathscr{C}), \\ & \mathscr{A}^{\frac{1}{2}} z_1 + \mathscr{A}^{-\frac{1}{2}}(\eta\mathscr{B}^*\mathscr{B} + \varepsilon\mathscr{C}^*\mathscr{C}) z_2 \in \mathcal{D}(\mathscr{A}^{\frac{1}{2}})\}. \end{aligned} \tag{16}$$

The following Theorem 1 assures the well-posedness of the observer in the common sense.

Theorem 1. The following assertions are valid.

a) Suppose assumptions (H1), (H3) and (H5). Then the observer (13) is well-posed, that is, there exists a unique solution $\hat{z}$ to (13) such that for any $T > 0$, there exists a constant $C_T > 0$ such that

$$\begin{aligned} \|\hat{z}(T)\|_H \leq \ & M e^{(-\delta + M\|-\varepsilon C^* C\|_{\mathcal{L}(H)})T} \|\hat{z}_0\|_H \\ & + C_T \|y(\cdot)\|_{L^2(0,T;Y)}, \end{aligned} \tag{17}$$

where M and δ are the constants defined in (8).

b) Suppose assumptions (H2), (H4) and (H5). Then the observer (13) is well-posed, that is, there exists a unique solution $\hat{z}$ to (14) such that for any $T > 0$, there exists a constant $C_T > 0$ such that

$$\|\hat{z}(T)\|_H \leq C_T \left(\|\hat{z}_0\|_H + \|y(\cdot)\|_{L^2(0,T;Y)} \right). \tag{18}$$

4. CONTROL DESIGN AND STABILITY OF CLOSED LOOP SYSTEM

With the solvability of the observer, we can naturally design the feedback based on the estimated state

$$u = -\mathscr{B}^* \hat{w}_t. \tag{19}$$

Then the corresponding closed-loop system becomes

$$\begin{cases} w_{tt} + \mathscr{A} w + \eta \mathscr{B}\mathscr{B}^* \hat{w}_t = 0, \\ w(0) = w_1, w_t(0) = w_2, \\ \hat{w}_{tt} + \mathscr{A} \hat{w} + (\eta \mathscr{B}\mathscr{B}^* + \varepsilon \mathscr{C}^*\mathscr{C})\hat{w}_t = \mathscr{C}^* y, \\ \hat{w}(0) = \hat{w}_0, \hat{w}_t(0) = \hat{w}_1. \end{cases} \tag{20}$$

Set the error $e = w - \hat{w}$. Then e satisfies

$$\begin{cases} e_{tt} + \mathscr{A} e + \varepsilon \mathscr{C}^*\mathscr{C} e_t = 0, \\ e(0) = e_0, e_t(0) = e_1. \end{cases} \tag{21}$$

By assumption (H4), the solution $d = (e, e_t)^\top$ to the system (21) tends to zero exponentially as t goes to infinity, i.e.,

$$\|d(t)\|_H \le L e^{-\gamma t} \|(e_0, e_1)^\top\|_H. \tag{22}$$

Theorem 2. Let $\mathbf{A}$ be the infinitesimal generator of a C_0-semigroup $e^{\mathbf{A}t}$ in a Hilbert space $\mathbf{H}$ satisfying

$$\|e^{\mathbf{A}t}\|_{\mathcal{L}(\mathbf{H})} \le K e^{-\lambda t}$$

for some constants $K > 0, \lambda > 0$. Let f be a function such that $f \in C(0, \infty; \mathbf{H})$. If

$$\|f(t)\|_{\mathbf{H}} \le N e^{-\gamma t} \tag{23}$$

for some constants $N, \gamma > 0$, then, $z(t)$, the mild solution of the non-homogeneous equation of the following

$$z_t = \mathbf{A} z + f, \quad z(0) = z_0 \in \mathbf{H}, \tag{24}$$

satisfies

$$\|z(t)\|_{\mathbf{H}} \le K e^{-\lambda t} \|z_0\|_{\mathbf{H}} + M_0 e^{-\omega t}, \tag{25}$$

where M_0 and ω are positive constants.

Theorem 3. Under either assumptions (H1),(H3), (H4), and (H5) or assumptions (H2), (H3), (H4), and (H5), the closed loop system (20) is exponentially stable.

5. APPLICATIONS

Example 1. We consider the wave equation with the distributed control and the boundary observation

$$\begin{cases} w_{tt}(x,t) - \Delta w(x,t) + \eta\, u(x,t) = 0 & \text{in } \Omega \times (0,\infty), \\ w(x,0) = w_1(x),\ w_t(x,0) = w_2(x) & \text{in } \Omega, \\ w(x,t) = 0 & \text{on } \partial\Omega \times (0,\infty), \\ y(x,t) = -\varepsilon \dfrac{\partial(\mathcal{A}^{-1} w_t)}{\partial \nu}(x,t) & \text{on } \partial\Omega \times (0,\infty), \end{cases} \tag{26}$$

where Ω is a bounded open set in $\mathbb{R}^n$ ($n \ge 2$) with C^2-boundary $\partial\Omega$. Here and in the rest of the paper ν always denotes the unit normal vector field on $\partial\Omega$ pointing towards the exterior of Ω, u and y are the control and the observation, respectively, η and ε are positive constants, the operator $\mathcal{A}$ is given by

$$\mathcal{A} w = -\Delta w,\ \mathcal{D}(\mathcal{A}) = H^2(\Omega) \cap H_0^1(\Omega),\ \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = H_0^1(\Omega).$$

We consider the system (26) in the state space $H = L^2(\Omega) \times H^{-1}(\Omega)$, control space $U = X = H^{-1}(\Omega)$ and observation space $Y = L^2(\partial\Omega)$. Design the observer for the system (26)

$$\begin{cases} \hat{w}_{tt} - \Delta \hat{w} + \eta\, \hat{w}_t = 0 & \text{in } \Omega \times (0,\infty), \\ \hat{w}(x,0) = \hat{w}_0, \hat{w}_t(x,0) = \hat{w}_1 & \text{in } \Omega, \\ \hat{w}(x,t) = \varepsilon \dfrac{\partial(\mathcal{A}^{-1}\hat{w}_t)}{\partial\nu} + y & \text{on } \partial\Omega \times (0,\infty). \end{cases} \tag{27}$$

Theorem 4. The closed-loop system (26) and (27) under the observed state feedback $u = \hat{w}_t$ is exponentially stable in $H \times H$.

Proof. We cast system (26) into the abstract setting (1) or (2) in Section 2. To do this, we introduce the following operators and spaces

$X = [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]' = H^{-1}(\Omega)$, where
the dual $'$ is with respect to the pivot space $L^2(\Omega)$;
$\mathscr{A}$ is defined by

$$\langle \mathscr{A} f, g\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} = \int_\Omega \nabla f(x) \overline{\nabla g(x)} dx,\ \forall f, g \in H_0^1(\Omega);$$

$\mathcal{D}(\mathscr{A}) = H_0^1(\Omega)$, $\mathcal{D}(\mathscr{A}^{\frac{1}{2}}) = L^2(\Omega)$, and
$\mathscr{A}$ is the extension of $\mathcal{A}$ to $H_0^1(\Omega)$;
$\mathscr{B} = -I \in \mathcal{L}(H^{-1}(\Omega))$;

$$\mathscr{C} v = -\frac{\partial(\mathcal{A}^{-1} v)}{\partial \nu}\Big|_{\partial\Omega},\ \forall\, v \in \mathcal{D}(\mathscr{A}^{\frac{1}{2}}) = L^2(\Omega).$$

In this way, assumption (H2) holds for the system (26). Assumption (H3) is equivalent to the exponential stability of the following system in H

$$\begin{cases} v_{tt} - \Delta v + \eta\, v_t = 0 & \text{in } \Omega \times (0,\infty), \\ v(x,0) = v_0,\ v_t(x,0) = v_1 & \text{in } \Omega \\ v(x,t) = 0 & \text{on } \partial\Omega \times (0,\infty). \end{cases} \tag{28}$$

Set $\tilde{v} = \mathcal{A}^{-\frac{1}{2}} v$. We have

$$\begin{aligned} \|\tilde{v}\|_{H_0^1(\Omega)} &= \|v\|_{L^2(\Omega)},\ \forall v \in L^2(\Omega), \\ \|\tilde{v}\|_{L^2(\Omega)} &= \|v\|_{H^{-1}(\Omega)},\ \forall v \in H^{-1}(\Omega). \end{aligned} \tag{29}$$

Then $\tilde{v}$ satisfies

$$\begin{cases} \tilde{v}_{tt} - \Delta \tilde{v} + \eta\, \tilde{v}_t = 0 & \text{in } \Omega \times (0,\infty), \\ \tilde{v}(x,0) = \tilde{v}_0 = \mathcal{A}^{-\frac{1}{2}} v_0 \in H_0^1(\Omega), & \\ \tilde{v}_t(x,0) = \tilde{v}_1 = \mathcal{A}^{-\frac{1}{2}} v_1 \in L^2(\Omega) & \text{in } \Omega, \\ \tilde{v}(x,t) = 0 & \text{on } \partial\Omega \times (0,\infty). \end{cases} \tag{30}$$

By Theorems 4.4 and 2.3 in Liu (1997), it follows that the system (30) is exponentially stable in $H_0^1(\Omega) \times L^2(\Omega)$, which shows in turn by (29) that so is the system (28) in H. That is, the assumption (H3) is satisfied for system (26).

The assumption (H4) is equivalent to the exponential stability of the following system in H

$$\begin{cases} w_{tt}(x,t) - \Delta w(x,t) = 0, & \text{in } \Omega \times (0,\infty) \\ w(x,0) = w_1(x),\ w_t(x,0) = w_2(x) & \text{in } \Omega, \\ w(x,t) = \varepsilon \dfrac{\partial(\mathcal{A}^{-1} w_t)}{\partial\nu}(x,t) & \text{on } \partial\Omega \times (0,\infty). \end{cases} \tag{31}$$

However, this is just Theorem 1.1 in Lasiecka and Triggiani (1992).

Finally, we verify assumption (H5) for the system (26). To do this, it suffices to prove the boundedness of the input-output map with zero initial data since the control operator is bounded, and by Proposition 2.2 in Ammari

(2002), the associated observation operator C is admissible for the semigroup e^{At} generated by the associated operator A.

Set $w_1 = w_2 = 0$ in (26) and $T > 0$. We introduce the following transformation

$$\begin{array}{l} z(t) = \mathcal{A}^{-1} w_t \in C(0,T; H_0^1(\Omega)) \\ \text{continuous w.r.t. } u \in L^2(0,T; H^{-1}(\Omega)). \end{array} \tag{32}$$

Then we have

$$\begin{array}{l} z_t = \mathcal{A}^{-1} w_{tt} = -w + \eta\, \mathcal{A}^{-1} u \in L^2(0,T; L^2(\Omega)) \\ \text{continuous w.r.t. } u \in L^2(0,T; H^{-1}(\Omega)). \end{array} \tag{33}$$

The new variable $z(t)$ satisfies the following equation

$$\begin{cases} z_{tt} - \Delta z + \eta\, \mathcal{A}^{-1} u_t = 0 & \text{in } \Omega \times (0,T) \triangleq Q, \\ z(x,0) = 0,\ z_t(x,t) = z_1 & \text{in } \Omega, \\ z(x,t) = 0 & \text{on } \partial\Omega \times (0,T) \triangleq \Sigma, \\ y(x,t) = -\varepsilon \dfrac{\partial z}{\partial \nu}(x,t) & \text{on } \Sigma, \end{cases} \tag{34}$$

where the Dirichlet boundary condition is obtained by the transformation (32). By (33), we can take $z_1 = 0$ for u in the class (36) defined below.

Multiplying the first equation of (34) by $(T-t)h \cdot \nabla z$ with h a C^2-vector field on $\overline{\Omega}$, $h|_{\partial\Omega} = \nu$, (see Lemma 2.1 in Komornik (1994) on p. 18), and integrating by parts, we obtain the following identity (see also equation (2.27) of Lasiecka et al (1986))

$$\begin{aligned} &\frac{1}{2}\int_\Sigma (T-t)\left|\frac{\partial z}{\partial \nu}\right|^2 d\Sigma = \int_Q (T-t) Dh(\nabla z, \nabla z)\, dQ \\ &+\frac{1}{2}\int_Q (T-t)(|z_t|^2 - |\nabla z|^2)\, \mathrm{div}(h)\, dQ \\ &+\int_Q z_t\, h \cdot \nabla z\, dQ + \eta \int_Q (T-t)(\mathcal{A}^{-1} u_t)\, h \cdot \nabla z\, dQ, \end{aligned} \tag{35}$$

where Dh is the Jacobian matrix of the vector field h.

By (32) and (33) we see that the first three integral terms on the right hand side of (35) are well defined and are continuous w.r.t. $u \in L^2(0,T; H^{-1}(\Omega))$.

Now we treat the last term $\eta \int_Q (T-t)(\mathcal{A}^{-1} u_t)\, h \cdot \nabla z\, dQ$ in (35). Suppose that u is in the following class

$$u \in C([0,T]; H^{-1}(\Omega)), \quad u(0) = u(T) = 0, \tag{36}$$

that is dense in $L^2(0,T; H^{-1}(\Omega))$. Using (36) and integrating by parts in t, we obtain

$$\begin{aligned} &\eta \int_Q (T-t)(\mathcal{A}^{-1} u_t)\, h \cdot \nabla z\, dQ \\ &= \eta \int_Q (\mathcal{A}^{-1} u)\, h \cdot \nabla z dQ + \eta \int_Q (t-T)(\mathcal{A}^{-1} u) h \cdot \nabla z_t dQ \\ &= \eta \int_Q (\mathcal{A}^{-1} u) h \cdot \nabla z dQ + \eta \int_\Sigma (t-T) z_t (\mathcal{A}^{-1} u) h \cdot \nu d\Sigma \\ &\quad -\eta \int_Q (t-T) z_t h \cdot \nabla(\mathcal{A}^{-1} u) dQ \\ &\quad -\eta \int_Q (t-T)\mathrm{div}(h) z_t (\mathcal{A}^{-1} u) dQ \text{ (noticing that } z|_\Sigma = 0) \\ &= \eta \int_Q (\mathcal{A}^{-1} u) h \cdot \nabla z dQ - \eta \int_Q (t-T) z_t h \cdot \nabla(\mathcal{A}^{-1} u) dQ \\ &\quad -\eta \int_Q (t-T)\mathrm{div}(h) z_t (\mathcal{A}^{-1} u) dQ. \end{aligned}$$

Since $\mathcal{A}^{-1} u \in L^2(0,T; H_0^1(\Omega))$ is continuous w.r.t. $u \in L^2(0,T; H^{-1}(\Omega))$, it follows from (32) and (33) that the terms on the right hand side of the above equality are continuous in u in the class of (36), then so are for $u \in L^2(0,T; H^{-1}(\Omega))$ by the density argument. Assumption (H5) is therefore verified for (26).

By the abstract results in previous sections, the observer for the system (26) should be (27).

The desired result then follows from Theorem 3. The proof is complete.

Example 2. We consider the wave equation with the Dirichlet boundary control and the distributed observation

$$\begin{cases} w_{tt} - \Delta w = 0 & \text{in } \Omega \times (0,\infty), \\ w(x,0) = w_1,\ w_t(x,0) = w_2 & \text{in } \Omega, \\ w(x,t) = \eta\, u(x,t) & \text{on } \partial\Omega \times (0,\infty), \\ y(x,t) = \varepsilon\, w_t(x,t) & \text{in } \Omega \times (0,\infty), \end{cases} \tag{37}$$

where the domain Ω and ε, η are the same as in system (26), u is the control and y is the observation.

Again we consider the system (37) in the state space $H = L^2(\Omega) \times H^{-1}(\Omega)$, the control space $U = L^2(\partial\Omega)$ and the observation space $Y = X = H^{-1}(\Omega)$. Design the observer for the system (37) as

$$\begin{cases} \hat{w}_{tt} - \Delta \hat{w} + \varepsilon \hat{w}_t - y = 0 & \text{in } \Omega \times (0,\infty), \\ \hat{w}(x,0) = \hat{w}_t(x,0) = 0 & \text{in } \Omega, \\ \hat{w}(x,t) = \eta \dfrac{\partial(\mathcal{A}^{-1}\hat{w}_t)}{\partial \nu}(x,t) & \text{on } \partial\Omega \times (0,\infty). \end{cases} \tag{38}$$

Theorem 5. The closed-loop system (37) and (38) under the observed state feedback $u = \frac{\partial(\mathcal{A}^{-1}\hat{w}_t)}{\partial \nu}\big|_{\partial\Omega}$ is exponentially stable in $H \times H$.

Proof. Again we introduce the associated operators and spaces to cast the system (37) into the abstract setting in Section 2.

$\mathcal{A}w = -\Delta w,\ \mathcal{D}(\mathcal{A}) = H^2(\Omega) \cap H_0^1(\Omega),\ \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = H_0^1(\Omega);$
$X = [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]' = H^{-1}(\Omega),$ where
the dual $'$ is with respect to the pivot space $L^2(\Omega)$;
$\mathscr{A}$ is defined by

$$\langle \mathscr{A} f, g\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} = \int_\Omega \nabla f(x)\overline{\nabla g(x)}dx,\ \forall f, g \in H_0^1(\Omega);$$

$\mathcal{D}(\mathscr{A}) = H_0^1(\Omega),\ \mathcal{D}(\mathscr{A}^{\frac{1}{2}}) = L^2(\Omega),$
$\mathscr{A}$ is the extension of $\mathcal{A}$ to $H_0^1(\Omega)$;
$f = Dg \Leftrightarrow \{\Delta f = 0 \text{ in } \Omega \text{ and } f|_{\partial\Omega} = g\},$
$D \in \mathcal{L}(L^2(\partial\Omega), H^{\frac{1}{2}}(\Omega));$
$\mathscr{B} = -\mathcal{A}D \in \mathcal{L}(U; [\mathcal{D}(\mathscr{A}^{\frac{1}{2}})]'),\ \mathscr{B}^* = -\left.\frac{\partial \mathcal{A}^{-1}}{\partial \nu}\right|_{\partial\Omega}$
(see (2.16) and (2.18) in Guo and Zhang (2005));
$\mathscr{C} = I \in \mathcal{L}(H^{-1}(\Omega)).$

Assumption (H1) holds true for system (37), and assumption (H3) is also satisfied since the system (31) is exponentially stable in H. Similar to Example 1, assumption (H4) also holds since the solution of equation (28) tends to zero exponentially in H. Finally, the well-posedness of the system (37) in the sense of D.Salamon was actually proved by Proposition 2.2 in Ammari (2002) since the observation operator $\mathscr{C}$ is bounded. Hence assumption (H5) also holds true.

Now the abstract results in previous sections give the observer (38) for the system (37). The result follows again from Theorem 3.

To end this paper, we give an example that does not exactly fit the abstract setting in Section 2, but the stabilizability still can be proved in the same spirit.

Example 3. Consider the following wave equation with the locally distributed control and the Neumann boundary observation

$$\begin{cases} w_{tt} - \Delta w + \eta\,\chi_G u = 0, & \text{in } \Omega \times (0,\infty), \\ w(x,0) = w_1(x),\ w_t(x,0) = w_2(x) & \text{in } \Omega, \\ w(x,t) = 0 & \text{on } \Sigma, \\ y(x,t) = \dfrac{\partial w}{\partial \nu}(x,t) & \text{on } \Sigma_1, \end{cases} \tag{39}$$

where Ω is an open bounded domain in $\mathbb{R}^n$ $(n \geq 2)$, with C^2-boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$, $\Gamma_0 \neq \emptyset$, $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset$ satisfying the geometrical condition: there exists a vector field $h(x) \in C^2(\bar{\Omega}; \mathbb{R}^n)$ such that

$$h \cdot \nu \leq 0 \text{ on } \Gamma_0,$$

and for some constant $\rho > 0$, and all vectors $v(x) \in (L^2(\Omega))^n$,

$$\int_\Omega Dh(v,v)\,dx \geq \rho \int_\Omega |v|^2\,dx,$$

where Dh is the Jacobian matrix of h. $\Sigma = \partial\Omega \times (0,\infty)$, $\Sigma_0 = \Gamma_0 \times (0,\infty)$, $\Sigma_1 = \Gamma_1 \times (0,\infty)$, G is a Lebesgue measurable subset of Ω and satisfies the following geometrical condition

$(g;G)$: there exist open sets $\Omega_j \subset \Omega$ with Lipschitz boundary $\partial\Omega_j$ and points $x_0^j \in \mathbb{R}^n$, $j = 1, \cdots, J$ such that $\Omega_i \cap \Omega_j = \emptyset$ for any $1 \leq i < j \leq J$ and

$$G \supset \Omega \cap \mathcal{N}_\alpha \left[\left(\bigcup_{j=1}^J \Gamma_j\right) \bigcup \left(\Omega \setminus \bigcup_{j=1}^J \Omega_j\right)\right]$$

for some $\alpha > 0$ where

$$\mathcal{N}_\alpha[S] := \bigcup_{x\in S}\{y \in \mathbb{R}^n : |y - x| < \alpha\} \text{ for } S \in \mathbb{R}^n,$$

$$\Gamma_j = \{x \in \partial\Omega_j : (x - x_0^j)\cdot \nu^j(x) > 0\}$$

with $\nu^j(x)$, the unit normal vector of $\partial\Omega_j$ at x pointing towards the exterior of $\partial\Omega_j$, being defined almost everywhere on $\partial\Omega$ and belonging to $L^\infty(\partial\Omega_j; \mathbb{R}^n)$.

$\chi_G(\cdot)$ is the characteristic function of G, η is a positive parameter, u is the control and y is the observation.

We consider the system (39) in the state Hilbert space $H = H_0^1(\Omega) \times L^2(\Omega)$, the control space $U = L^2(G)$ and the observation space $Y = L^2(\Gamma_1)$. Design the observer for the system (39) as

$$\begin{cases} \hat{w}_{tt} - \Delta\hat{w} + \eta\,\chi_G \hat{w}_t = 0 & \text{in } \Omega \times (0,\infty), \\ \hat{w}(x,0) = \hat{w}_0,\ \hat{w}_t(x,0) = \hat{w}_1 & \text{in } \Omega, \\ \hat{w}(x,t) = 0 & \text{on } \Sigma_0, \\ \dfrac{\partial \hat{w}}{\partial \nu}(x,t) = -\varepsilon \hat{w}_t(x,t) + y(x,t) & \text{on } \Sigma_1, \end{cases} \tag{40}$$

where ε is a positive constant. The system (40) is considered in a different Hilbert state space $\mathcal{H} = H_{\Gamma_0}^1(\Omega) \times L^2(\Omega)$ with $H_{\Gamma_0}^1(\Omega) = \{f \in H^1(\Omega)|\ f|_{\Gamma_0} = 0\}$ which is larger than H.

Theorem 6. The observer system (40) is well-posed in $\mathcal{H}$, and the system (39) can be exponentially stabilized by the observed state feedback $u = \hat{w}_t$.

Proof. Supposing that (40) is well-posed in the common sense, we first show the stability of the system (39) under the feedback $u = \hat{w}_t$.

By Theorem 2.1 of Lasiecka et al (1986), we know that the system (39) is well-posed in the sense of D. Salamon, i.e., for each $u \in L_{loc}^2(0,\infty; L^2(G))$ and initial data $w_1 \in H_0^1(\Omega)$, $w_2 \in L^2(\Omega)$, there exists a unique solution $(w, w_t) \in C(0,\infty; H)$ to equation (39), and for each $T > 0$, there exist some $C_T > 0$ independent of u and (w_1, w_2) such that

$$\|(w(T), w_t(T))\|_H^2 + \int_0^T \|y(s)\|_{L^2(\Gamma_1)}^2\,ds \leq C_T\left(\|(w_1, w_2)\|_H^2 + \int_0^T \|u(s)\|_{L^2(G)}^2\,ds\right).$$

Design the observed state feedback $u = \hat{w}_t$ in (39), and let the error $e = w - \hat{w}$. Then formally we obtain the following system satisfied by e:

$$\begin{cases} e_{tt}(x,t) - \Delta e(x,t) = 0 & \text{in } \Omega \times (0,\infty), \\ e(x,0) = e_1(x),\ e_t(x,0) = e_2(x) & \text{in } \Omega, \\ e(x,t) = 0 & \text{on } \Sigma_0, \\ \dfrac{\partial e}{\partial \nu}(x,t) = -\varepsilon e_t(x,t) & \text{on } \Sigma_1. \end{cases} \tag{41}$$

By Theorem 1.2 of Lasiecka and Triggiani (1992), we know that the solution to (41) tends to zero exponentially in $\mathcal{H}$, i.e. there exist constants $\gamma > 0$ and $L = L_\gamma > 0$ such that

$$\|(e(t), e_t(t))\|_{\mathcal{H}} \leq Le^{-\gamma t}\|(e_1, e_2)\|_{\mathcal{H}},\ \forall t \geq 0. \tag{42}$$

Now the system (39) with $u = \hat{w}_t$ can be represented as

$$\begin{cases} w_{tt} - \Delta w + \eta\chi_G w_t - \eta\chi_G e_t = 0 & \text{in } \Omega \times (0,\infty), \\ w(x,0) = w_1(x),\ w_t(x,0) = w_2(x) & \text{in } \Omega, \\ w(x,t) = 0 & \text{on } \Sigma. \end{cases} \tag{43}$$

By Theorem 2.3 and Theorem 4.2 in Liu (1997), the solution $(w(t), w_t(t))$ to the following system in $H = H_0^1(\Omega) \times L^2(\Omega)$

$$\begin{cases} w_{tt} - \Delta w + \eta\,\chi_G w_t = 0 & \text{in } \Omega \times (0,\infty), \\ w(x,0) = w_0(x),\ w_t(x,0) = w_1(x) & \text{in } \Omega, \\ w(x,t) = 0 & \text{on } \partial\Omega \times (0,\infty), \end{cases} \tag{44}$$

tends to zero exponentially as t tends to infinity.

Noting this fact and (42), it follows from Theorem 2 that the solution to the system (43) is exponentially stable in H. In other words, the system (39) can be exponentially stabilized by $u = \hat{w}_t$.

The remaining is to validate the solvability of the observer system (40) in $\mathcal{H}$. To this end, we cast the system (40) into an abstract setting and then make use of Theorem 1.

Introduce the following operators

$$\mathscr{A} f = -\Delta f,$$

$$\mathcal{D}(\mathscr{A}) = \left\{ f \in H^2(\Omega) \Big|\ f|_{\Gamma_0} = 0,\ \frac{\partial f}{\partial \nu}|_{\Gamma_1} = 0 \right\},$$

$$\mathcal{D}(\mathscr{A}^{\frac{1}{2}}) = H^1_{\Gamma_0}(\Omega);$$

$$f = Ng \Leftrightarrow \left\{ \Delta u = 0 \text{ in } \Omega,\ f|_{\Gamma_0} = 0,\ \frac{\partial f}{\partial \nu}|_{\Gamma_1} \right\},$$

$$N \in \mathcal{L}(L^2(\Gamma_1); H^{\frac{3}{2}}(\Omega));$$

$$\mathscr{C}^* = -\mathscr{A} N,\ \mathscr{C} f = -N^* \mathscr{A} f = f|_{\partial\Omega},\ \forall\, f \in \mathcal{D}(\mathscr{A})$$

(see Eqn.(3.3.1.12) of Lasiecka et al (2000) on p. 196);

$$\mathscr{B} = -\tilde{u}(\cdot),\ \forall u \in U = L^2(G),$$

$\tilde{u} \in L^2(\Omega)$ is the zero extension of u to Ω, $\mathscr{B} \in \mathcal{L}(U, X)$.

With these operators at hand and (14), we can write the observer system (40) into a first order abstract system in $\mathcal{H} = H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$

$$\hat{z}_t = A_\varepsilon \hat{z} - \eta B B^* \hat{z} + C^* y, \tag{45}$$

where the operator A_ε is defined by (10), the operators B, B^*, C and C^* are defined by (4), (5), (6), and (7) respectively.

Since the exponential stability of (42) is equivalent to the assumption (H4), it follows from Theorem 7.6.2.2 of Lasiecka et al (2000) on p. 665 and the fact $y \in L^2(0,T;Y)$ that

$$(\hat{w}, \hat{w}_t) \in L^2(0,T;\mathcal{H}) \cap C(0,T;\mathcal{H}),$$

that is, the observer (40) is solvable in $\mathcal{H}$. The proof is complete.

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