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Output Regulation of Uncertain Nonaffine in Control Systems via Singular Perturbation Technique ⋆

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Abstract: The problem of output regulation for nonlinear control systems with guaranteed transient performances in the presence of uncertainties is discussed, where the nonlinear systems are nonaffine in the control. The fast dynamical controller with the highest output derivative in feedback loop is used, where the controller is proper and can be implemented without ideal differentiation. Two-time-scale motions are induced in the closed-loop system and the method of singular perturbations is used to analyze the closed-loop system properties. Stability conditions imposed on the fast and slow modes and sufficiently large mode separation rate can ensure that the full-order closed-loop system achieves the desired properties in such a way that the output transient performances are desired and insensitive to external disturbances and variations of nonlinear system parameters. The problem of absolute stability analysis of the fast-motion subsystem for nonaffine systems with two-time-scale motions is considered in the presence of a sector-like condition in the control.

1. INTRODUCTION

The importance of output regulation problem for nonlinear time-varying control systems arises from various applications, such as aircraft control, robotics, mechatronics, chemical industry, electrical and electro-mechanical systems. Variations and uncertainties of parameters and disturbances are inherent property of many real-time control plants. Moreover, the control systems must maintain desired transient performances in the presence of uncertainties.

Various methodologies are available for controller design of affine-in-control systems, where the system that is affine in the control is assumed to have the following form

$$\begin{aligned} \dot{X} &= f(X, w) + G(X, w)u, \\ y &= h(X, w), \end{aligned} \tag{1}$$

where $X(t)$ is the state vector of the system; $y(t)$ is the output of the systems; $w(t)$ is the vector of external disturbances or varying parameters; $u(t)$ is the input of the system.

In contrast to the system given by (1), the nonaffine systems in the control may have the following form

$$\begin{aligned} \dot{X} &= f(X, w, u), \\ y &= h(X, w), \end{aligned} \tag{2}$$

where an explicit inversion of the function $z = f(X, w, u)$ with respect to control variable $u(t)$ is impossible for given $z(t), X(t), w(t)$.

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The need of nonaffine-in-control systems investigations rises in various important practical applications, for instance, magnetic servo levitation control system discussed by Gutierrez and Ro (2005), controller design for a nonaffine UAV model reported by Unnikrishnan and Balakrishnan (2006), pendulum control systems presented by Shiriaev *et al.* (1999), Shiriaev and Fradkov (2000) and Young *et al.* (2006), controller design for chemical reactions discussed by Ge *et al.* (1998), etc.

The dynamic inversion control methodology for nonaffine-in-control systems in the presence of uncertainties reported by Lavretsky and Hovakimyan (2005) based on implementation of the radial basis function neural network (RBF NN) approximation of the unknown nonlinearity, state predictor, and the adaptive law for RBF NN weights. Hence, the whole closed-loop system is too complicated one, where the order of the closed-loop system depends on the number of RBF's. The other feature of this design methodology is that the method of singular perturbations is used to analyze the closed-loop system properties. The method of singular perturbations was reported by Tikhonov (1952); Klimushchev and Krasovskii (1962); Kokotović et al. (1976); Kokotović (1984); Kokotović et al. (1999); Naidu (2002) and many others researchers.

In contrast to the control methodology discussed by Lavretsky and Hovakimyan (2005); Hovakimyan *et al.* (2006), the approach reported by Yurkevich (1993, 2004) allows to get the controller of the same order as the relative degree of the uncertain system (2), where the proposed design methodology guarantees the desired output transient performances in presence of unknown external disturbances and variations of parameters of the system.

There are two points of novelty in the paper. First, it has been shown that the design methodology reported by Yurkevich (2004) can be extended for nonaffine in control systems. Second, the fast transients analysis is treated in the paper based on application of absolute stability criteria discussed by Yakubovich (1962, 1963); Kalman (1963); Aizerman and Gantmakher (1964); Popov (1962); Gelig (1965). This paper is the further development of the results discussed by Yurkevich (2008).

2. CONTROL PROBLEM STATEMENT

Consider a SISO nonaffine-in-control continuous-time system

$$x^{(n)} = f(X, w, u), \; X(0) = X^0, \tag{3}$$

where $X = \{x, x^{(1)}, \ldots, x^{(n-1)}\}^T$ is the state vector, it is assumed that components $x^{(1)}, \ldots, x^{(n-1)}$ are not measurable, but the component $x(t)$ is the measurable output (controlled variable) of the system, $x \in \mathbb{R}$; $X(0) = X^0$ is the initial state, $X^0 \in \Omega_X$; Ω_X is a compact set, $\Omega_X \subset \mathbb{R}^n$; w is the vector of external disturbances or varying parameters, which are not measurable, $w \in \Omega_w$, Ω_w is a compact set; $u(t)$ is the input of the system, $u \in \mathbb{R}$. The nonlinear scalar function $f(X, w, u)$ is continuous one for all its arguments $(X, w, u) \in \Omega_{X,w,u} = \Omega_X \times \Omega_w \times \Omega_u$, and $f(X, w, u)$ is unknown function as well. The other conditions imposed on the properties of $f(X, w, u)$ are defined below.

A control system is being designed so that

$$\lim_{t\to\infty} e(t) = 0, \tag{4}$$

where $e(t)$ is an error of the reference input realization, $e(t) \triangleq r(t) - x(t)$, and $r(t)$ is the reference input.

3. INSENSITIVITY CONDITION

Consider the reference model of the desired output behavior for (3) in the following form $x(s) = G^d(s)r(s)$, where

$$G^d(s) = \frac{B^d(s)}{A^d(s)}$$
$$= \frac{T^{-n}[a^d_\rho T^\rho s^\rho + a^d_{\rho-1}T^{\rho-1}s^{\rho-1} + \cdots + a^d_1 Ts + 1]}{s^n + a^d_{n-1}T^{-1}s^{n-1} + \cdots + a^d_1 T^{1-n}s + T^{-n}}. \tag{5}$$

Let the polynomial $A^d(s)$ be stable, the parameters of $A^d(s)$ are selected in accordance with the desired transient performances for $x(t)$, as well as ρ is selected in accordance with the desired system type of the reference model, $\rho < n$. From $G^d(s)$, the reference model of the desired behavior for $x(t)$ in the form of the n-th order stable differential equation

$$x^{(n)} = -\frac{a^d_{n-1}}{T}x^{(n-1)} - \cdots - \frac{a^d_1}{T^{n-1}}x^{(1)} - \frac{1}{T^n}x$$
$$+\frac{a^d_\rho}{T^{n-\rho}}r^{(\rho)} \cdots + \frac{a^d_1}{T^{n-1}}r^{(1)} + \frac{1}{T^n}r \tag{6}$$

results. Let us rewrite the equation (6), for short, as

$$x^{(n)} = F(X, R), \tag{7}$$

where $R = \{r, r^{(1)}, \ldots, r^{(\rho)}\}^T$ and $x(t)$ exponentially converges to r if $r = \text{const}$.

Denote $e^F \triangleq F(X, R) - x^{(n)}$, where e^F is the realization error of the desired behavior assigned by (7) and $x^{(n)}$ is defined by (3). Accordingly, if the condition

$$e^F = 0 \tag{8}$$

holds, then the behavior of $x^{(n)}$ with prescribed dynamics of (7) is fulfilled, that is the same as $x(s) = G^d(s)r(s)$. Accordingly, $e(s) = [1 - G^d(s)]r(s)$, thus $e(t) \to 0$ exponentially as $t \to \infty$ for $r = \text{const}$.

Hence, the output regulation problem given by (4) has been reformulated as the requirement (8). Expression (8) is called as the insensitivity condition for the behavior of the output $x(t)$ with respect to the external disturbances and varying parameters of the system (3). In accordance with (3), the condition (8) can be rewritten as

$$F(X, R) - f(X, w, u) = 0. \tag{9}$$

Assumption 1. Let an isolated root of (9) exists, which can be denoted as the control function

$$u^{id}(t) = f^{-1}(X(t), w(t), F(X(t), r(t))) \tag{10}$$

in some neighbourhood of the point (X, R, w), where $u^{id}(t)$ is not available explicitly as well as $u^{id}(t)$ may be non-unique solution, in general.

The control function $u^{id}(t)$ is called as the nonlinear inverse dynamics solution and one corresponds to the desired output behavior of (3) prescribed by (7).

Remark 1. From the properties of (5) it follows, if the condition (8) holds in the system (3), then we get

(i) robust zero steady-state error of the reference input realization;
(ii) desired output performance specifications such as overshoot, settling time, and system type;
(iii) insensitivity of the output transient behavior with respect to smoothly varying parameters of the system (3) and unknown external disturbances.

4. MAIN RESULTS

4.1 Control law

In order to keep hold of (8), that is (9), under the condition of unknown external disturbances and varying parameters, as well as unknown nonlinear function $f(X, w, u)$ of the system (3), let us consider the feedback control law given by the following differential equation:

$$\mu^q u^{(q)} + d_{q-1}\mu^{q-1}u^{(q-1)} + \cdots + d_1\mu u^{(1)} + d_0 u$$
$$= k_0\{F(X, R) - x^{(n)}\}, \tag{11}$$

where the n-th derivative of $x(t)$ is used in feedback loop, μ is a small positive parameter, $U = \{u, u^{(1)}, \ldots, u^{(q-1)}\}^T$, $U \in \Omega_U \subset \mathbb{R}^q$, and $U(0) \in \Omega^0_U \subset \Omega_U$.

Remark 2. If $q \geq n$ and $n > \rho$, then the controller (11) is proper and one can be rewritten as the system of state space differential equations given by

$$\dot{U} = A_c U + B_c x + E_c r,$$
$$u = C_c U + D_c x. \tag{12}$$

An example of control law given by (11) for $q = n = 2$ and $\rho = 0$ in the form (12) has been shown below in Section 5.

4.2 Two-time-scale motions in the closed-loop system

In accordance with (3) and (11), the closed-loop system is given by

$$x^{(n)} = f(X, w, u), \tag{13a}$$
$$\mu^q u^{(q)} + d_{q-1}\mu^{q-1}u^{(q-1)} + \cdots + d_1\mu u^{(1)} + d_0 u = k_0\{F(X,R) - x^{(n)}\}. \tag{13b}$$

Theorem 1. If $d_0 = 0$ and $\mu \to 0$, then two-time-scale motions are induced in the closed-loop system (13) such that the slow-motion subsystem (SMS) is the same as the reference model equation (7), as well as the fast-motion subsystem (FMS) is given by

$$\begin{aligned} \mu\frac{d}{dt}u_j &= u_{j+1}, \quad j = 1,\ldots,q-1, \\ \mu\frac{d}{dt}u_q &= -k_0 f(X,w,u_1) - d_1u_2 \cdots - d_{q-1}u_q \\ &\quad +k_0F(X,R), \end{aligned} \tag{14}$$

where X and w are treated as the constant values during the transients in (14).

Proof. Substitution of (13a) into (13b) yields

$$x^{(n)} = f(X, w, u), \tag{15a}$$
$$\mu^q u^{(q)} + \cdots + d_1\mu u^{(1)} + d_0 u + k_0 f(X,w,u) = k_0F(X,R). \tag{15b}$$

Let us rewrite the closed-loop system equations (15) as

$$\begin{aligned} \frac{d}{dt}x_i &= x_{i+1}, \qquad i = 1,\ldots,n-1, \\ \frac{d}{dt}x_n &= f(X,w,u_1), \\ \mu\frac{d}{dt}u_j &= u_{j+1}, \qquad j = 1,\ldots,q-1, \\ \mu\frac{d}{dt}u_q &= -d_0u_1 - k_0f(X,w,u_1) \cdots - d_{q-1}u_q \\ &\quad +k_0F(X,R), \end{aligned} \tag{16}$$

where $U_1 = \{u_1, u_2, \ldots, u_q\}^T$ and

$$u_j = \mu^{j-1}u^{(j-1)}, \quad \forall\, j = 1,\ldots,q.$$

Since μ is a small parameter, the above equations are the singularly perturbed differential equations. If $\mu \to 0$, then fast and slow modes are forced in the closed-loop system and the time-scale separation between these modes depends on the parameter μ.

Let us introduce the new fast time scale $t_0 = t/\mu$. Hence, from (16), we get

$$\begin{aligned} \frac{d}{dt_0}x_i &= \mu x_{i+1}, \quad i = 1,\ldots,n-1, \\ \frac{d}{dt_0}x_n &= \mu f(X,w,u_1), \\ \frac{d}{dt_0}u_j &= u_{j+1}, \quad j = 1,\ldots,q-1, \\ \frac{d}{dt_0}u_q &= -d_0u_1 - k_0f(X,w,u_1) - d_1u_2 \cdots - d_{q-1}u_q \\ &\quad +k_0F(X,R), \end{aligned}$$

as the closed-loop system equations in the new time scale t_0. It is easy to see that as $\mu \to 0$, we get the FMS equations in the new time scale t_0, that is

$$\begin{aligned} \frac{d}{dt_0}u_j &= u_{j+1}, \quad j = 1,\ldots,q-1, \\ \frac{d}{dt_0}u_q &= -d_0u_1 - k_0f(X,w,u_1) - d_1u_2 \cdots - d_{q-1}u_q \\ &\quad +k_0F(X,R). \end{aligned}$$

Take $d_0 = 0$ in order to include an integral action into the control loop and, accordingly, provide the robust zero steady-state error. Then, returning to the primary time scale $t = \mu t_0$, we obtain the FMS given by (14).

Next, take $\mu = 0$, then from (14) the condition (9) results. Hence, by (10), $U_1 = U_1^{id}$ is the equilibrium point of the FMS (14), where $U_1^{id} = \{u_1^{id}, 0, \ldots, 0\}^T$ and $u_1^{id} = u^{id}$. Substitution of $d_0 = 0$ and $\mu = 0$ into (16) yields the so-called reduced system (that is the SMS of (16))

$$\begin{aligned} \frac{d}{dt}x_i &= x_{i+1}, \qquad i = 1,\ldots,n-1, \\ \frac{d}{dt}x_n &= F(X,R), \quad X(0) = X^0, \end{aligned} \tag{17}$$

which is the same as the reference model equation (7) that is the exponentially stable linear system.

Assumption 2. Let U_1^{id} be the exponentially stable equilibrium point uniformly in (X, R, w) of the FMS given by the equation (14) and $\Omega_{U_1} = \{U_1 \in \mathbb{R}^q \mid \|U_1 - U_1^{id}\|_2 \le \delta,\ \delta > 0\}$ is a subset of the region of attraction for U_1^{id}.

Remark 3. Assumption 2 validity can be maintained by proper selection of controller parameters as shown below.

Denote $X^{ref} = \{x_1^{ref}, x_2^{ref}, \ldots, x_n^{ref}\}^T$, where $X^{ref}(t)$ is a solution of (17) on $[0,\infty)$ for initial conditions given by $X^{ref}(0) = X(0) \in \Omega_X$. Denote $X^\mu = \{x_1^\mu, x_2^\mu, \ldots, x_n^\mu\}^T$, $U_1^\mu = \{u_1^\mu, u_2^\mu, \ldots, u_q^\mu\}^T$, where $X^\mu(t)$ and $U_1^\mu(t)$ are solutions of (16) on $[0,\infty)$ for initial conditions given by $X^\mu(0) = X(0) \in \Omega_X$, $U_1^\mu(0) = U_1(0) \in \Omega_{U_1}$.

Corollary 1. From the exponential stability of (14) and (17), in accordance with the basic theorem on singular perturbations (see, for instance, Klimushchev and Krasovskii (1962); Hoppensteadt (1966); Khalil (2002)) there exists a positive constant $\mu^\star$ such that for all $t \ge 0$, $X^0 \in \Omega_X$, $U_1 \in \Omega_{U_1}$, and $0 < \mu < \mu^\star$, the unique solution of (16) exists on $[0,\infty)$ and the condition

$$X^\mu - X^{ref} = O(\mu) \tag{18}$$

holds uniformly for $t \in [0,\infty)$.

Remark 4. Assumption 2 implies that the condition (18) holds despite that $f(X,w,u)$ is unknown function. So, if a sufficient time-scale separation between the fast and slow modes in the closed-loop system and exponential convergence of FMS transients to equilibrium are provided, then after the damping of fast transients the desired output behavior prescribed by (7) is fulfilled. Thus, the output transient performance indices are insensitive to parameter variations of the nonlinear system and external disturbances, by that the solution of the discussed control problem (4) is maintained.

4.3 FMS stability analysis via linearization

Let us consider the FMS equations (14), where $X = \text{const}$, $w = \text{const}$, and U_1^{id} is its equilibrium point. Denote

$$\tilde{U}_1 \triangleq U_1 - U_1^{id}, \tag{19}$$

where $\tilde{U}_1 = \{\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_q\}^T$ and $\tilde{u}_1 = u_1 - u_1^{id}$.

Assumption 3. Let $\partial f(X, w, u_1)/\partial u_1$ exists for all $(X, w) \in \Omega_X \times \Omega_w$, and $u_1 \in \Omega_{u_1}$, where $\Omega_{u_1} = \{u_1 \in \mathbb{R} \mid |u_1 - u_1^{id}| \le \delta_u,\ \delta_u > 0\}$.

Due to Assumption 3, the expanding of $f(X, w, u_1)$ into Taylor series around (X, w, u_1^{id}) yields

$$f(X, w, u_1) = f(X, w, u_1^{id}) + g(X, w, u_1^{id})\tilde{u}_1 + O(\tilde{u}_1^2), \tag{20}$$

where

$$g(X, w, u_1^{id}) = \left.\frac{\partial f(X, w, u_1)}{\partial u_1}\right|_{u_1 = u_1^{id}}. \tag{21}$$

Assumption 4. Let the condition

$$0 < g_{min} \le g(X, w, u_1) \le g_{max} < \infty \tag{22}$$

holds for all $(X, w, u_1) \in \Omega_{X,w,u_1} = \Omega_X \times \Omega_w \times \Omega_{u_1}$.

From (14), by neglecting higher-order terms of (20), the linearized FMS equations about the equilibrium U_1^{id}

$$\mu\frac{d}{dt}\tilde{u}_j = \tilde{u}_{j+1}, \quad j = 1, \ldots, q-1, \tag{23}$$
$$\mu\frac{d}{dt}\tilde{u}_q = -k_0 g(X, w, u_1^{id})\tilde{u}_1 - d_1\tilde{u}_2 \cdots - d_{q-1}\tilde{u}_q$$

result, where $g(X, w, u_1^{id})$ is treated as an unknown fixed parameter during the transients in (23) and the condition (22) holds.

The FMS equations (23) may be rewritten as

$$\mu^q\tilde{u}^{(q)} + d_{q-1}\mu^{q-1}\tilde{u}^{(q-1)} + \cdots + d_1\mu\tilde{u}^{(1)} + k_0 g\tilde{u} = 0 \tag{24}$$

where $\tilde{u} = \tilde{u}_1$ and the characteristic polynomial of the FMS is as

$$D_{fms}(\mu s) = \mu^q s^q + d_{q-1}\mu^{q-1}s^{q-1} + \cdots + d_1\mu s + k_0 g. \tag{25}$$

Remark 5. The parameter μ has not an influence on stability of $D_{fms}(\mu s)$, but one affects the rate of FMS transients. Thus μ should be selected to maintain the desired degree of time-scale separation between fast and slow motions in the closed-loop system (16). Usually, the degree of time-scale separation η can be estimated by the ratio of SMS time constant to the maximum value of the FMS time constant, that is $\eta = T_{sms}/T_{fms}$, where $T_{sms} = T$, and $T_{fms} = \mu/\sqrt[q]{k_0 g_{min}}$. Therefore, from the above, $\eta \ge \eta_{min}$ if the following condition is satisfied: $0 < \mu \le \mu_1 = T\sqrt[q]{k_0 g_{min}}/\eta_{min}$.

Theorem 2. Let

(i) Assumptions 3 and 4 hold;
(ii) The parameters $d_1, \cdots, d_{q-1}$ are selected such that $D_{fms}(\mu s)$ is Hurwitz polynomial for all $g \in [g_{min}, g_{max}]$.

Then there exists a positive constant δ such that Assumption 2 holds, where Ω_{U_1} is the subset of the region of attraction for U_1^{id}.

Proof. The proof follows directly from the exponential stability of the origin for the linearized FMS (24).

4.4 Absolute stability analysis of FMS

Assumption 5. Let $f(X, w, u_1)$ be an unknown continuous function of $X(t)$, $w(t)$, and $u_1(t)$, where the following sector-like condition in control variable

$$0 < k_1 \le \frac{f(X, w, u_1) - f(X, w, \hat{u}_1)}{u_1 - \hat{u}_1} \le k_2 < \infty \tag{26}$$

holds for all $(X, w) \in \Omega_{X,w}$, $u_1 \in \Omega_u$, and $\hat{u}_1 \in \Omega_u$, where

$$\Omega_u = \{u_1 \in \mathbb{R} \mid |u_1 - u_1^{id}| \le \delta_{\hat{u}},\ \delta_{\hat{u}} > 0\}.$$

Denote

$$\psi(X, w, u_1^{id}, \tilde{u}_1) \triangleq f(X, w, \tilde{u}_1 + u_1^{id}) - f(X, w, u_1^{id}). \tag{27}$$

From (26), we get the following sector condition

$$0 < k_1 \le \frac{\psi(X, w, u_1^{id}, \tilde{u}_1)}{\tilde{u}_1} \le k_2 < \infty.$$

Hence, from (14), (19), and (27), the FMS equations

$$\mu\frac{d}{dt}\tilde{u}_j = \tilde{u}_{j+1}, \quad j = 1, \ldots, q-1, \tag{28}$$
$$\mu\frac{d}{dt}\tilde{u}_q = -k_0\psi(X, w, u_1^{id}, \tilde{u}_1) - d_1\tilde{u}_2 \cdots - d_{q-1}\tilde{u}_q$$

follow, where $\tilde{U}_1 = 0$ is the equilibrium point of (28).

Then, the FMS equations (28) may be rewritten as

$$\mu^q\tilde{u}^{(q)} + d_{q-1}\mu^{q-1}\tilde{u}^{(q-1)} + \cdots + d_1\mu\tilde{u}^{(1)} + k_0\psi(X, w, u^{id}, \tilde{u}) = 0 \tag{29}$$

where X, w, and u^{id} are treated as unknown constant values during the transients in (29).

Let us rewrite (29) such as

$$\mu^q\tilde{u}^{(q)} + d_{q-1}\mu^{q-1}\tilde{u}^{(q-1)} + \cdots + d_1\mu\tilde{u}^{(1)} + k_0 k_1\tilde{u} + k_0 h(X, w, u^{id}, \tilde{u}) = 0, \tag{30}$$

where

$$h(X, w, u^{id}, \tilde{u}) = \psi(X, w, u^{id}, \tilde{u}) - k_1\tilde{u}$$

and the following sector condition

$$0 < \frac{h(X, w, u^{id}, \tilde{u})}{\tilde{u}} \le k_2 - k_1$$

holds on the specified compact set $\Omega_{X,w,u}$. Denote

$$\tilde{D}_{fms}(\mu s) = \mu^q s^q + d_{q-1}\mu^{q-1}s^{q-1} + \cdots + d_1\mu s + k_0 k_1.$$

Assumption 6. Let the parameters μ, $d_1, \ldots, d_{q-1}$, and k_0 of the polynomial $\tilde{D}_{fms}(\mu s)$ are selected such that this polynomial is stable and the degree of time-scale separation between fast and slow modes is large enough.

Remark 6. In order to provide the requirement $\eta \ge \eta_{min}$ the condition

$$0 < \mu \le \mu_2 = T\sqrt[q]{k_0 k_1}/\eta_{min} \tag{31}$$

can be used for selection of the parameter μ.

Theorem 3. Let

(i) Assumptions 5 and 6 hold;
(ii) The condition

$$\text{Re}\left[\frac{k_0}{\tilde{D}(j\mu\omega)}\right] > -\frac{1}{k_2 - k_1},\ \forall\, \omega \in (-\infty, \infty) \tag{32}$$

is satisfied.

Then, there exists a positive constant $\delta_{\hat{u}}$ such that the origin of (30) is uniformly asymptotically stable for any nonlinearity in the given sector (that is, the FMS (30) is absolutely stable with a finite domain Ω_u).

Proof. The proof follows directly from a straightforward application of the circle criterion (see, for example, Gelig (1965); Cho and Narendra (1968); Khalil (2002)).

Theorem 4. Let

(i) Assumptions 5 and 6 hold;
(ii) There exists a positive number γ such that the condition

$$\mathrm{Re}\left[(1+j\gamma\omega)\frac{k_0}{\tilde{D}(j\mu\omega)}\right]+\frac{1}{k_2-k_1}>0,\ \forall\,\omega\in(-\infty,\infty) \tag{33}$$

is satisfied.

Then, there exists a positive constant $\delta_{\hat{u}}$ such that the origin of (30) is uniformly asymptotically stable for any nonlinearity in the given sector with the finite domain Ω_u.

Proof. The proof follows directly from a straightforward application of Popov's criterion (see, for example, Popov (1962); Aizerman and Gantmakher (1964); Khalil (2002)).

Theorem 5. Let

(i) Assumptions 5 and 6 hold;
(ii) The parameters $d_1,\cdots,d_{q-1}$ are selected such that the polynomial $\tilde{D}_{fms}(\mu s)$ has q repeated left-half plane roots.

Then, there exists a positive constant $\delta_{\hat{u}}$ such that the origin of (30) is uniformly asymptotically stable for any nonlinearity in the given sector with the finite domain Ω_u.

Proof. The proof follows directly from the results reported by Fannin and Rushing (1974).

5. EXAMPLE

The differential equation of a plant model is given by

$$x^{(2)}=x+x|x^{(1)}|+w(t)+u+0.9\sin(u). \tag{34}$$

The reference model for $x(t)$ is chosen as $x^{(2)}=-2x^{(1)}-x+r$. Hence, $F(x^{(1)},x,r)=-2x^{(1)}-x+r$ and s^2+2s+1 is the characteristic polynomial of the reference model with the time constant $T=1$ s. Take $q=n=2$. In accordance with the presented design methodology the control law structure can be chosen as

$$\mu^2u^{(2)}+d_1\mu u^{(1)}+d_0u=k_0[F(x^{(1)},x,r)-x^{(2)}],$$

that is

$$\mu^2u^{(2)}+d_1\mu u^{(1)}+d_0u=k_0[-x^{(2)}-2x^{(1)}-x+r]. \tag{35}$$

From (34), we get that the condition $k_1=0.1\le\partial f/\partial u\le k_2=1.9$ holds. Take, for instance, $k_0k_1=1$, that is $k_0=10$ and $\eta_{min}=20$. From (31), we get $\mu_2=0.05$ s. Take $\mu=0.05$ s, $d_1=2$ and $d_0=0$. Then $\tilde{D}_{fms}(\mu s)=(0.05s+1)^2$. It is clear that the condition (33) holds due to $q=2$.

Control law implementation. The discussed control law (35) can be rewritten in the form given by

$$u^{(2)}+\frac{d_1}{\mu}u^{(1)}+\frac{d_0}{\mu^2}u$$
$$=-\frac{k_0}{\mu^2}x^{(2)}-\frac{k_0a_1^d}{\mu^2T}x^{(1)}-\frac{k_0}{\mu^2T^2}x+\frac{k_0}{\mu^2T^2}r.$$

From the above, we get

$$u^{(2)}+a_1u^{(1)}+a_0u=b_2x^{(2)}+b_1x^{(1)}+b_0x+c_0r \tag{36}$$

where

$$a_1=\frac{d_1}{\mu},\quad a_0=\frac{d_0}{\mu^2},\quad c_0=\frac{k_0}{\mu^2T^2},$$
$$b_2=-\frac{k_0}{\mu^2},\quad b_1=-\frac{k_0a_1^d}{\mu^2T},\quad b_0=-\frac{k_0}{\mu^2T^2}.$$

Then, from (36), we may get the equations of the controller in the state space form (12), that are

$$\begin{aligned}\dot{u}_1&=-a_1u_1+u_2+(b_1-a_1b_2)x,\\ \dot{u}_2&=-a_0u_1+(b_0-a_0b_2)x+c_0r,\\ u&=u_1+b_2x.\end{aligned} \tag{37}$$

The simulation results of the closed-loop system given by (34) and (37) are shown in Figs. 1–2.

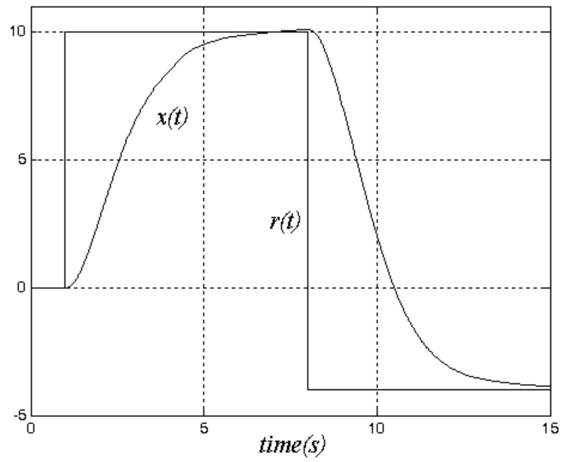

Fig. 1. Plots of $r(t)$ and $x(t)$ in the closed-loop system.

6. CONCLUSION

In accordance with the presented above approach the fast motions occur in the closed-loop system such that after fast ending of the fast-motion transients, the behavior of the overall singularly perturbed closed-loop system approaches that of the SMS, which is the same as the reference model. Hence, the desired output performance specifications are provided, as well as insensitivity of the output transient behavior with respect to unknown external disturbances and varying parameters of the system. The discussed design methodology may be used for a broad class of nonaffine systems. The main advantage of the application of the methods for absolute stability analysis of the fast motions is that the class of nonlinear systems to which the results of two-time-scale motion control design methodology is applicable is significantly enlarged.

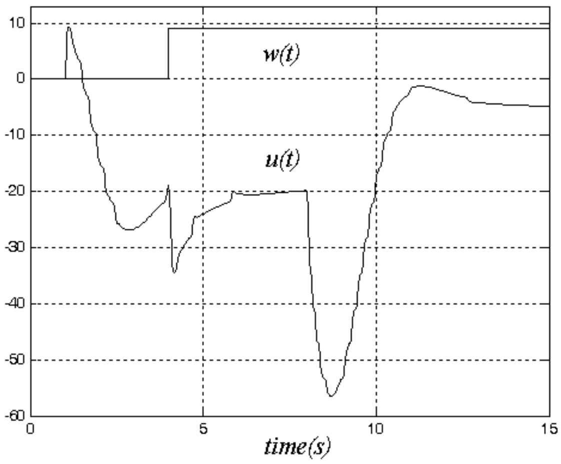

Fig. 2. Plots of $u(t)$ and $w(t)$ in the closed-loop system.

The presented control system design methodology allows to guarantee the desired output transient performances in the presence of plant parameter variations, unknown external disturbances, and uncertain nonlinearities with sector-like condition in control loop. The other advantage, caused by two-time-scale technique for closed-loop system analysis, is that analytical expressions for selection of the controller parameters can be found, where controller parameters depend explicitly on the specifications of the desired output behavior. The presented design methodology may be useful for real-time control system design under uncertainties.

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