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Dynamic augmentation and complexity reduction of set-based constrained control

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Abstract: Computing polytopic controlled invariant sets which are maximal inside a prescribed region often yields sets which have a really complex representation. Since the associated control has a complexity which drastically increases with that of the region, this turns out to be a major problem in the implementation of the theory of controlled invariant regions. In this paper, we consider the problem of reducing the complexity of these regions and/or the complexity of the associated compensator. We propose two heuristic techniques based on spectral properties of some relevant matrices and on vertex-elimination methods. The paper presents several preliminary results which are interesting on their own such as dynamic augmentation and the properties of complex (as opposed to real) polytopic invariant regions.

1. INTRODUCTION

It is well known that the problem of controlling a system in the presence of control and state constraints can be faced via controlled invariant sets. Controlled invariant sets of several types have been addressed in the literature, in particular the ellipsoidal and the polytopic ones. It is also well established that there is a fundamental tradeoff between the complexity of the representation of one of such regions and its size. Indeed regions and compensator of fixed complexity (e.g. ellipsoids) may be much smaller in size than the largest invariant set compatible with the constraints Blanchini (1999). Polytopic sets do not have this problem since they can approximate the largest controlled invariant region with arbitrary precision Gutman and Cwikel (1987); Keerthi and Gilbert (1987). Unfortunately, the price to pay is the resulting complexity of the region and, more important, of the control.

In this paper, we consider the problem of generating controlled-invariant polytopic sets and control laws of reduced (or limited) complexity. As a first result we consider the problem of deriving a dynamic linear control which fulfills the constraints starting from a controlled invariant polytope. It turns out that we can always achieve a controlled-invariant set in the extended (plant+compensator) space which can be associated with a linear control. The projection of this set on the plant state space is equal to the original controlled invariant set. We show how this dynamic augmentation procedure can produce a significant complexity reduction of the compensator with respect to previous nonlinear static compensators proposed in the literature.

The mentioned dynamic augmentation method may simplify the compensator, not the region. Then we face the problem of reducing the region complexity. We introduce an approach based on the pole selection of a relevant matrix appearing in the controlled invariance basic condition. This decomposition is shown to be affected by a restriction on the eigenvalues of this matrix. Such a restriction disappears when complex (instead of real) regions are considered Brayton and Tong (1980); Miani and Savorgnan (2006).

Finally, we propose an heuristic method to reduce the region complexity. This is based on the elimination of vertices which are "close to others" and seem not so significant in the region structure.

2. PRELIMINARIES AND DEFINITIONS

In this paper, we consider discrete-time systems of the form

$$x(t+1) = Ax(t) + Bu(t) \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ are the state and input vectors respectively and continuous-time systems

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{2}$$

We assume that the couple (A, B) is stabilizable and that the system is subject to constraints

$$M \begin{bmatrix} x \\ u \end{bmatrix} \leq \bar{1} \tag{3}$$

where $\bar{1} = [\ 1\ 1\ \dots\ 1\ ]^T$. It is well known Blanchini (1999) how the theory of controlled-invariant sets plays a fundamental role in the constrained control problem, according to the following definition and theorem.

Definition 1. A set $\mathcal{P}$ is controlled invariant and compatible with the constraints (3) if and only if there exists a control $u = \Phi(x)$ such that, for every $x(0) \in \mathcal{P}$ we have that $x(t) \in \mathcal{P}$ and that

$$M \begin{bmatrix} x(t) \\ \Phi(x(t)) \end{bmatrix} \leq \bar{1}$$

Theorem 1. There exists a feedback control for which no constraint violations occur from the initial condition $x(0)$ if and only if the initial state belongs to an invariant set

$\mathcal{P}$ which is controlled invariant and compatible with the constraints.

It is also known that controlled invariance is not enough to enforce convergence to 0. What we need in practice is contractivity. Without entering in the detail (the reader is referred to Blanchini (1999) Blanchini e Miani (2007)), we remind the reader that if we modify the systems as follows

$$x(t+1) = \frac{A}{\lambda}x(t) + \frac{B}{\lambda}u(t)$$

and, respectively,

$$\dot{x}(t) = (I+\beta)Ax(t) + Bu(t)$$

with $0 < \lambda < 1$ (respectively $\beta > 0$), then any controlled invariant set for the modified systems is contractive for the original ones. If a contractive C-set $\mathcal{P}$ [1] is known then the condition

$$\Psi(x(t)) \leq \lambda^t \Psi(x(0))$$

(respectively $\Psi(x(t)) \leq e^{-\beta t}\Psi(x(0))$) is satisfied ($\Psi$ is the Minkowski function associated with $\mathcal{P}$), hence $x(t) \to 0$.

In this paper we will mainly (although not exclusively) consider polytopic controlled invariant sets. A polytopic set can be described in term of its vertex matrix $X = [x_1\ x_2\ \dots\ x_r]$ as

$$\begin{aligned}\mathcal{P} &= \{x = \sum_{k=1}^{r} p_k x_k :\ \ \sum_{k=1}^{r} p_k = 1,\ \ p_k \geq 0\}\\ &= \{x = Xp :\ \ \bar{1}^T p = 1,\ \ p \geq 0\}\end{aligned}$$

In the following, to keep the notation simple, we will limit our attention to 0-symmetric polytopes. These can be represented as

$$\mathcal{P} = \{x = Xp,\ \ \|p\|_1 \leq 1\}$$

For symmetric polytopes, with a slight abuse of notation, we will call X the vertex matrix, though the actual vertex matrix of the set is $[X\ \ -X]$.

Definition 2. A matrix P belongs to the class $\mathcal{H}$ if there exists $\tau > 0$ such that $\|I + \tau P\|_1 < 1$.

As a preliminary statement of the paper we need the following Blanchini e Miani (2007).

Theorem 2. The 0-symmetric polytopic C-set $\mathcal{P}$ with vertex matrix X is contractive for the discrete-time (resp. continuous-time) system if and only if there exists a matrix P with $\|P\|_1 \leq 1$ ($P \in \mathcal{H}$) and a matrix U such that

$$AX + BU = XP \tag{4}$$

(the columns of U represent the input associated to each vertex of the polytope). Furthermore $\mathcal{P}$ is compatible with the constraints if and only if

$$\pm M \begin{bmatrix} x_k \\ u_k \end{bmatrix} \leq \bar{1} \tag{5}$$

for all k, where x_k and u_k represent the kth columns of X and U.

If the conditions of the theorem hold, then P is a stable matrix in both the discrete and continuous-time case. It is known how, by applying algorithms presented in the literature (see Keerthi and Gilbert (1987); Gutman and Cwikel (1987); Blanchini (1994) and based on dynamic programming (see Bertsekas (1972)), it is possible to approximate (to determine, in lucky cases) the largest controlled invariant set by means of a polytope which is controlled invariant.

[1] convex and compact set including 0 in its interior

The major trouble with this kind of sets is their complexity and the complexity of the associated compensator. Indeed in general the compensator that can be associated is nonlinear. This is in contrast with the case of ellipsoidal controlled-invariants sets which can be always associated with a proper linear gain $u = Kx$. Among the nonlinear controllers we can take the piecewise linear controller proposed in Gutman and Cwikel (1986), the on-line-LP-solving considered in Blanchini (1994). For continuous-time systems it is possible to partly reduce the compensator complexity by using a controller based on a smooth approximation of $\mathcal{P}$ (see Blanchini e Miani (1998)). However, the complexity of the set $\mathcal{P}$ is very high and, as a consequence, the compensator complexity is very high. Therefore the main issue in this paper is basically the following.

Problem 1. Given the matrices X and P in equation (4) how can we obtain a new invariant (contractive) set and a compensator of a simpler description?

To deal with this problem we first consider some basic properties of the dynamic augmentation as in the next section.

3. DYNAMIC AUGMENTATION

We now assume that a controlled invariant polytope is given. It is important to stress that the complexity of the compensator may grow even exponentially with the complexity of the set. For instance the Gutman and Cwikel control requires a simplex partition of the polytope. A proper linear gain is applied inside each simplex. Unfortunately the number of simplices is typically much greater than the number of vertices (see Blanchini e Miani (1998) for an example). Therefore a first question is how to achieve a compensator whose complexity is not explosive with respect to such a number.

An interesting property which can be useful to give an answer (and we think interesting on its own) is the following.

Theorem 3. Given a controlled invariant polytopic symmetric C-set $\mathcal{P}$, compatible with constraints, the system state space can always be extended to achieve a new polytopic C-set $\mathcal{S}$ (affine to a diamond) in such a way that

- $\mathcal{P}$ is the projection of $\mathcal{S}$ on the original state space;
- $\mathcal{S}$ is controlled invariant is the extended state space;
- $\mathcal{S}$ can be associated with a linear stabilizing compensator;
- $\mathcal{S}$ is compatible with the constraints.

Proof If $\mathcal{P}$ is a C-set, then X has full row rank. Take an augmentation matrix Z such that

$$\hat{X} \doteq \begin{bmatrix} X \\ Z \end{bmatrix}$$

is square invertible. Then the polytope generated by this matrix $\hat{X}$

$$\mathcal{S} = \left\{ \begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} X \\ Z \end{bmatrix} p, \quad \|p\|_1 \leq 1 \right\}$$

has clearly $\mathcal{P}$ as projection on the original state space.

Define $V = ZP$ and write

$$\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X \\ Z \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} X \\ Z \end{bmatrix} P$$

Consider the linear dynamic controller

$$\begin{bmatrix} u(t) \\ z(t+1) \end{bmatrix} = \begin{bmatrix} K & H \\ G & F \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix}$$

where

$$\begin{bmatrix} K & H \\ G & F \end{bmatrix} := \begin{bmatrix} U \\ V \end{bmatrix} \begin{bmatrix} X \\ Z \end{bmatrix}^{-1} \tag{6}$$

Then the closed-loop system has state matrix A_{cl} which satisfies the equation

$$\begin{bmatrix} A+BK & BH \\ G & F \end{bmatrix} \begin{bmatrix} X \\ Z \end{bmatrix} = A_{cl} \begin{bmatrix} X \\ Z \end{bmatrix} = \begin{bmatrix} X \\ Z \end{bmatrix} P$$

Note that A_{cl} is stable since it is similar to P. This is an invariance condition for $\mathcal{S}$. The compatibility with the constraints comes for the fact that if $[x^T\ z^T]^T \in \mathcal{S}$ then the corresponding input is

$$u = Kx + Hz = [\,K\ \ H\,] \begin{bmatrix} X \\ Z \end{bmatrix} p = Up$$

thus satisfies constraints in view of (5).

Remark 1. An interesting property (see Blanchini e Pellegrino (2003)) is that this approach allows for the determination of a *stable compensator* (strong stabilization). Indeed, instead of fixing X and determining the compensator via (6), we can fix a stable F and solve for Z and G the following linear matrix equation

$$GX + FZ = ZP$$

If a (nontrivial) solution is found and if $[X^T\ Z^T]^T$ is invertible (a condition generically satisfied), then we can find H and K.

The nice property is that the dimension of the state space of the compensator is $N = r - n$, where r is the number of columns of Z. The linear compensator is quite simple to be implemented, via standard techniques. The only problem is its initialization. Indeed, given the initial plant state $x(0) \in \mathcal{P}$ we must initialize $z(0)$ in such a way that

$$\begin{bmatrix} x(0) \\ z(0) \end{bmatrix} \in \mathcal{S}$$

Once we have performed this operation, the dynamic controller assures convergence and constraint satisfaction.

It is interesting to note the following.

Corollary 1. Consider any invariant convex set $\mathcal{S}$ (not necessarily a C-set!) in the extended state space, compatible with the constraints for the extended system whose state matrix is

$$A_{cl} = \begin{bmatrix} A+BK & BH \\ G & F \end{bmatrix}$$

Then its projection $\mathcal{P}$ on the original state space is controlled invariant and compatible with constraints.

Remark 2. The corollary is valid for convex sets in general. For instance consider an invariant ellipsoidal set $\{\hat{x} : \hat{x}^T(Q^{-1})\hat{x} \leq 1\}$ for the system A_{cl} (not necessarily derived by a controlled invariant in the original space) namely such that

$$\begin{bmatrix} A+BK & BH \\ G & F \end{bmatrix} \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{bmatrix} + \begin{bmatrix} Q_1 & Q_{12}^T \\ Q_{12} & Q_2 \end{bmatrix} \begin{bmatrix} A+BK & BH \\ G & F \end{bmatrix}^T$$

$$= -\begin{bmatrix} S_1 & S_{12} \\ S_{12}^T & S_2 \end{bmatrix}$$

with the last matrix S positive definite. If we project this set on the original state space we get the ellipsoid $\{x^T(Q_1^{-1})x \leq 1\}$ and

$$AQ_1 + Q_1A^T + BR + R^TB^T = -S_1$$

with $R = KQ_1 + HQ_{12}$ which implies controlled invariance of $\{x : x^T(Q_1^{-1})x \leq 1\}$. Other types of sets can be considered, e.g. the semi-ellipsoidal ones presented in O'Dell and Misawa (2002); Artstein and Raković (2008).

The mentioned properties have the following meaning.

- Although a controlled invariant polytope does not admit a linear static controller as in the case of controlled invariant ellipsoids, this property becomes in some sense true if we extend the state space;
- We can generate controlled invariant sets and generate linear dynamic controllers or consider linear dynamic controllers, find an invariant set, and derive controlled invariants in the original space via projection.

3.1 A simple example of dynamic augmentation

Consider the simple system $\dot{x} = Ax + Bu$ with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and the constraint $u \in \mathcal{U} = \{u : \ |u| \leq 8\}$. The set $\mathcal{P}$ represented by the matrix X below, together with the control matrix U

$$X = \begin{bmatrix} 3 & 0 & -4 \\ 0 & 4 & 4 \end{bmatrix} \quad U = [\,-8\ \ -8\ \ -4\,]$$

satisfy the conditions in Theorem 2 with

$$P = \frac{1}{3}\begin{bmatrix} -8 & 4 & 0 \\ 0 & -6 & 0 \\ -6 & 0 & -3 \end{bmatrix}$$

The Gutman and Cwikel piecewise linear control for this system is given by (see Blanchini e Miani (2007), Chapter 4)

$$u = K^{(i)}x$$

The whole set of control gains is reported in the next table.

sector number	control gain
1	$[-8/3\ \ -2]$
2	$[-1\ \ -2]$
3	$[-5/3\ \ -8/3]$
4	$[-8/3\ \ -2]$
5	$[-1\ \ -2]$
6	$[-5/3\ \ -8/3]$

As an alternative approach we can "square" the matrix X by adding the matrix Z which can be taken, for instance as

$$Z = [\,0\ \ 0\ \ 1]$$

so that

$$V = ZP = [\,-2\ \ 0\ \ -1\,]$$

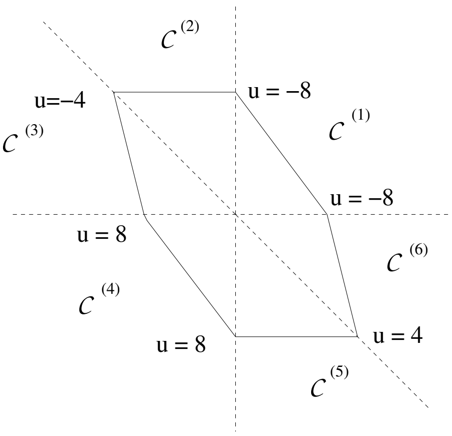

Fig. 1. The sector partition

The resulting dynamic compenstator is

$$\left[\begin{array}{c|c} \text{K} & \text{H} \\ \hline \text{G} & \text{F} \end{array}\right] = \begin{bmatrix} U \\ V \end{bmatrix} \begin{bmatrix} X \\ Z \end{bmatrix}^{-1} = \left[\begin{array}{cc|c} \text{-7/3} & \text{-7/3} & \text{-20/3} \\ \hline \text{-2/3} & 0 & \text{-11/3} \end{array}\right]$$

The closed-loop eigenvalues are those of P and precisely -1, $-8/3$ and -2. Note that, in this case, the compensator is stable and has its pole equal to $-11/3$. This compensator has to be initialized. This can be done as follows. For any $x(0) \in \mathcal{P}$ one can take $z(0)$ in such a way that $[x(0)^T\ z(0)^T]^T \in \mathcal{S}$ namely such that $[x(0)^T\ z(0)^T]^T = [X^T\ Z^T]^T p$ with some $\|p\|_1 \leq 1$. In view of the invertibility of $[X^T\ Z^T]^T$, $z(0)$ must be such that

$$\left\| \begin{bmatrix} X \\ Z \end{bmatrix}^{-1} \begin{bmatrix} x(0) \\ z(0) \end{bmatrix} \right\|_1 \leq 1$$

The so obtained linear compensator is simpler than the piecewise-linear compensator that would require to keep in memory all the information on the sectors of $\mathcal{P}$ and the associated gains.

In general, the complexity reduction becomes more effective as the system dimension increases. For instance, for two dimensional systems, the number of sectors is (up to symmetry) equal to the number r of columns of X. The dimension of the dynamic compensator is $r-2$ and therefore the complexity is basically the same. For the 4 dimensional system proposed in Blanchini e Miani (1998) the matrix X has 30 columns. Thus the dynamic compensator would have $30-4=26$ state variables. Conversely, the simplicial partition would give 296 sectors with 296 linear gains (actually 148 distinct gains by symmetry).

Since in general the compensator turns out to be quite complex, one might think about simplifying the region rather than seeking for a simpler compensator for that region. This is what will be done next.

4. SPECTRAL REDUCTION

We consider for brevity the discrete-time case only and we assume (4) can be rewritten as follows

$$A\ [X_1\ X_2] + B\ [U_1\ U_2] = [X_1\ X_2] \begin{bmatrix} P_1 & P_{12} \\ 0 & P_2 \end{bmatrix} \qquad (7)$$

Note that this kind of decomposition can be achieved by applying a state transformation to (4) as follows

$$AXT + B\ UT = XT\ T^{-1}PT$$

where T is a proper similarity transformation for P. Then we get the reduced equation

$$AX_1 + BU_1 = X_1 P_1 \qquad (8)$$

If $P_1 \in \mathcal{M}$ we have a simplified contractive region represented by X_1. The first trouble is the following.

Problem 2. How can we find a "good" transformation T?

It is intuitive that we can use a spectral decomposition.

$$T = [T_1\ T_2]$$

where T_1 is any basis matrix of a proper eigenspace of P, while T_2 is any complement which makes T invertible. The resulting matrix $X_1 = XT_1$ gives a new invariant polytope.

Unfortunately, some problems arise if we work with real regions since P_1 has part of the eigenvalues of P. If these are real, then we can take, for instance, P_1 diagonal so that $\|P_1\| < 1$. Conversely, if P_1 has complex eigenvalues, a block diagonal matrix must be composed by 2×2 blocks of the form

$$P_1 = \text{block diag} \left\{ \begin{bmatrix} \xi_k & \theta_k \\ -\theta_k & \xi_k \end{bmatrix},\quad k = 1, 2, \ldots \right\}$$

This imposes the restriction, known in the literature (see Vassilaki, Hennet and Bitsoris (1988)), $|\xi_k| + |\theta_k| < 1$ (in the continuous-time case this condition becomes $\xi_k < |\theta_k|$ Castelan and Hennet (1993); Bitsoris (1991)). Unfortunately, this restriction penalizes the procedure if we start from a given P (for instance that associated with the maximal controlled invariant set) since the eigenvalues of P_1 must be selected among those of P. This restriction disappears if we consider regions which are projections of complex polytopes on the real space Miani and Savorgnan (2006). One of such polytopes can be written as

$$\mathcal{P} = \{x = Xp,\quad \|p\|_1 \leq 1\}$$

where X is a matrix of complex vertices and p is a complex vector and $\|p\|_1 = \sum_i |p_i|$. Since P is stable, any diagonal matrix P_1 whose nonzero elements are given by any selection of eigenvalues of P is such that (8) holds (possibly with complex eigenvalues, the basic conditions of controlled invariance for complex regions hold) since any diagonal stable P_1 has 1-norm strictly less than 1 in the complex domain. According to Miani and Savorgnan (2006), the projection of controlled invariant complex regions on the real space produces controlled invariant regions which can be used for the determination of a controller. The following comments are important.

- Unfortunately, due to the transformation T, it is possible that the transformed matrices X_1 and U_1, once projected on the real space, produce regions which do not satisfy the constraints. Thus, the produced region must be scaled by a factor $\bar{\rho}$

$$\bar{\rho} = \arg\max\{\rho > 0 : \rho\ M\,[\,X_1, U_1\,] \leq \bar{1}\}$$

(here we assume that X_1 and U_1 are real, possibly after projection).
- There is no clear way on how to select the poles. "Bad selections" can produce very small regions compared with the maximal.

- The advantage of this procedure is that by selecting the number of eigenvalues, we fix a priori the complexity of the region.
- We can assign the poles by fixing P and by solving for X and U of fixed complexity. This basically extends previous work Benzaouia (1994); Castelan and Hennet (1993) (note that we can consider dynamic compensators rather than purely static ones). The derived region is not maximal, however fixing the poles may be a task of its own interest.

In the next section we propose a completely different criterion to reduce complexity.

5. AN HEURISTIC PROCEDURE

In this section we illustrate an heuristic algorithm to reduce the complexity of a given invariant set. For simplicity only the discrete-time case is considered. The main idea is that of suppressing vertices which are "nearly redundant". The final set is a polytope of fixed complexity whose vertices will be stored in the matrix

$$\tilde{X} = [\, \mu_1 \bar{x}_1 \ \mu_2 \bar{x}_2 \ \dots \mu_r \bar{x}_r \,] \tag{9}$$

where the column vectors $\bar{x}_j$ are a subset of the columns of X and $0 \leq \mu_j \leq 1$. We chose such target region in order to preserve some of the information about the shape of the original region given by X.

We now just outline the steps performed in the algorithm. The details of the single procedures are discussed later in this section.

Algorithm 1. (1) Set the desired number of vertices n_v of the final region.
(2) Select n_v columns from the matrix X as to obtain a full rank matrix $\bar{X}$ (*select* procedure).
(3) Calculate the coefficients $\mu_1, \dots, \mu_r$ in equation (9) in order to maximize the volume of the final polytope and such that there exist $\tilde{U}$ and $\tilde{P} \in \mathcal{M}$ which satisfy the equation

$$A\tilde{X} + B\tilde{U} = \tilde{X}\tilde{P} \tag{10}$$

and the state and input constraints (*shrink–enlarge* procedure).

5.1 The "select" procedure

The selection of the vertices of the original polytope is a crucial point in the algorithm since the goodness of the reduced complexity polytope depends on it. We propose a procedure which is reasonably effective and is based on the following idea: to preserve the information about the shape of the original polytope we discard the vertices which are close to other vertices with respect to the Euclidean norm $\|\cdot\|$. The procedure we use is the following:

(1) Set $\bar{X} = X$ and $\bar{n}_v$ equal to the number of columns of $\bar{X}$.
(2) If $\bar{n}_v \leq n_v$ terminate the procedure, otherwise go to step 3.
(3) Select two column indices i and j such that $i \neq j$ and the quantity $\|x_i - x_j\|$ is minimized.
(4) Select two column indices $\bar{i}$ and $\bar{j}$ such that $\bar{i} \neq i$ and $\bar{j} \neq j$ and the quantities $d_i = \|x_{\bar{i}} - x_i\|$ and $d_j = \|x_{\bar{j}} - x_j\|$ are minimized.
(5) If $d_i < d_j$ eliminate the j-th column from $\bar{X}$, otherwise eliminate the i-th column.
(6) Update the value of $\bar{n}_v$ and go to step 2.

At the end of the procedure we have to check if $\bar{X}$ is full rank (if it is not, it is necessary to change the selection, possibly increasing n_v).

5.2 The "shrink–enlarge" procedure

The first step in this procedure is finding a scalar $\bar{\mu}$ such that the polytope represented by $\bar{\mu}\bar{X}$ is contractive. Equivalently we search a $\bar{\mu}$ for which there exist $\bar{U}$ and $\bar{P}$ such that

$$A\bar{\mu}\bar{X} + B\bar{U} = \bar{\mu}\bar{X}\bar{P} \tag{11}$$

is satisfied. Such scalar can be found by solving the linear programming problem

$$\begin{aligned} \frac{1}{\bar{\mu}} &= \min_{\gamma, U, \bar{P}} \ \gamma \\ & s.t. \ \ \left\|\bar{P}\right\|_1 \leq 1 \\ & \quad M \begin{bmatrix} x_i \\ u_i \end{bmatrix} \leq \gamma \bar{1} \quad \forall i = 1, \dots, n_v \\ & \quad A\bar{X} + BU = \bar{X}\bar{P} \end{aligned} \tag{12}$$

Equation (11) is satisfied with

$$\bar{U} = \bar{\mu} U$$

When algorithm 1 is used to reduced the complexity of a contractive set for which the constraint (3) is active on some of the vertices, the value of γ in (11) will be greater than 1 (in order to relax the constraint). In this case $\bar{\mu} < 1$ and therefore $\bar{\mu}\bar{X}$ is shrinked with respect to $\bar{X}$. The shrinking phase of the procedure is necessary to find an initial contractive polytope for the second part of the procedure where $\bar{\mu}\bar{X}$ is enlarged.

The method proposed to enlarge the polytope increases the value of every μ_i in an iterative manner. To use this procedure we need to set the value of γ which is the minimum rate of improvement under which the iteration is stopped.

(1) Set $\mu_i = \bar{\mu}$ for all $i = 1, \dots, r$. Set $terminated = false$.
(2) If $terminated = true$ stop the procedure, otherwise continue.
(3) Set $i = 1$ and $terminated = true$.
(4) If $i > n_v$ go to step 2, otherwise continue.
(5) By bisection method, find the maximal value of μ_i such that

$$\begin{aligned} & \min_{u,p} \sum_{j=1}^{n_v} p, \quad p_j \geq 0, \\ & \text{s.t.} \ \ M \begin{bmatrix} \mu_i \bar{x}_i \\ u \end{bmatrix} \leq \bar{1} \\ & A\mu_i \bar{x}_i + Bu = \tilde{X}p \end{aligned}$$

is less or smaller than 1 (the values of the lower and upper bounds used initially by the bisection method are the previous value of μ_i and 1, respectively).
(6) If the increment obtained for μ_i in the preceding step is greater γ set $terminated = false$.
(7) Set $i = i + 1$ and go to step 4.

Example 1. Consider a system whose matrices are

$$A = \begin{bmatrix} 1 & 0.1 \\ 0 & 0.98 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0.98 \end{bmatrix}$$

If we consider the constraints

$$M = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}^T$$

we obtain a maximal invariant set which is represented by 28 vertices. By using algorithm 1 and eliminating 20 vertices we achieve the regions depicted in figure 2.

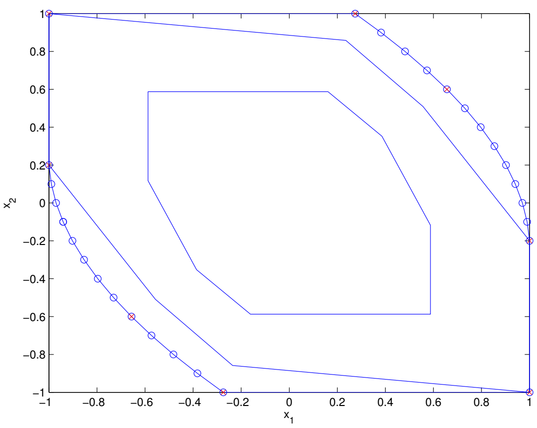

Fig. 2. The largest set represents the maximal invariant set. The other sets starting from the most internal one are the sets $\bar{\mu}\bar{X}$ and $\tilde{X}$ obtained by the shrink–enlarge procedure, respectively.

Remark 3. Algorithm 1 can be easily extended to deal with the uncertain case:

$$x(t+1) = A(w(t))x(t) + B(w(t))u(t)$$

where

$$[A(w)\ B(w)] = \sum_{i=1}^{s} [A_i\ B_i] w_i, \quad w_i \geq 0, \quad \sum_{i=1}^{s} w_i = 1$$

Even in this case an invariant polytopic region can be determined by means of the procedure presented in Blanchini (1994).

6. CONCLUSIONS

A viable approach to reduce complexity in determining a controlled invariant polytopic set and the associated compensator is that based on dynamic augmentation which allows for a linear, hence easily implementable, compensator. We have shown that this can always be done and how we can derive such compensator. Since the stability of the compensator (granted the closed-loop stability) is also important, we have given conditions to achieve strong stabilization (see Remark 1).

We have proposed an approach to reduce the complexity of the region based of an eigenvalue selection procedure. At the moment it is not clear at all how to select the eigenvalues to derive a good approximation of the maximal controlled invariant set compatible with the constraints. The tested examples showed that the procedure finds simplified controlled invariant set which are, unfortunately, quite smaller that the maximal.

Finally we have presented an heuristic procedure, whose main idea is to eliminate some vertices. This approach seems, at list tested on simple examples to provide better results in terms of tradeoff between complexity and volume of the region.

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