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ℓ_p-equivalence of Discretizations of Analog Controllers ⋆

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Abstract: This paper first introduces the fractional-order hold transformation that, together with the generalized bilinear transformation recently proposed in Zhang et al. [2007], contains all commonly used discretization methods as special cases. In light of this, it further shows that at fast sampling, all the digital approximations of an analog controller are *equivalent* in the sense of ℓ_p induced norm for $p \in [1, \infty]$ when the analog controller is stable or in the sense of some gap metric even when it is unstable.

Keywords: sampled-data systems; Generalized bilinear transformation; ℓ_p induced norms; graph metric; fractional-order hold transformation.

1. INTRODUCTION

A digital controller can be designed via a variety of approaches. For instance, it can be designed based on a discrete-time system obtained by lifting the original continuous-time plant. In this way, intersampling behavior can be taken into account. However, unfortunately, it ends up with a controller design in an infinite dimensional space. Alternatively, a digital controller can be designed directly based on the discretization of the original continuous-time plant (the so-called direct design). It can also be designed by discretizing a continuous-time controller designed for the continuous-time plant. Hence the last case can be viewed as approximation of a continuous-time controller. This can be done in roughly three routes. The first is to approximate the differentiator (or equivalently, the integral $1/s$), e.g., the Euler method, the (generalized) bilinear transformation (BT). The second is to derive a discrete-time model via sampling the original continuous-time controller, e.g., the zero-order hold (ZOH) equivalent and first-order hold (FOH) equivalent (Wittenmark et al. [2002]) (see Fig. 1). In Fig. 1, H can be a zero-order hold, first-order hold, or even a fractional-order hold, $K(s)$ is an analog controller followed by an ideal sampler. This transformation, we hereafter call *hold equivalent transformation*, maps an analog controller to a digital one. The third is optimization-based controller redesign studied in Keller and Anderson [1992], Rafee et al. [1997], Hwang et al. [2003], Shieh et al. [1998], etc. In each of the approaches proposed in these papers, given an analog controller, an optimization problem is solved to produce a digital controller which is optimal in a certain sense, e.g., H_2 or H_∞ norm. Contemplating these various approaches in designing a digital controller, one may ask the following question: What are the relations among these approximations at fast sampling? A common belief in doing approximation is that the resulting sampled-data system will perform similarly as the original continuous-time system when the underlying sampling period is sufficiently small. Though this faith is quite intuitive and appealing, its rigorous theoretical foundation is not yet solidly built. This paper is an attempt toward this goal and provide a partial answer for the hold equivalent transformations and integral approximations. More specifically, we focus on the following question: Given a stable controller $K(s)$ in continuous time, denote its discrete-time counterparts by $K_d(z)$, $K_{fd}(z)$ and $K_{bt}(z)$ obtained via ZOH, FOH and BT respectively, and let h be the underlying sampling period, we show that K_d, K_{fd} and K_{bt} converge to each other in ℓ_p-induced norm for all $p \in [1, \infty]$ as h tends to zero. For convenience, we thus call them ℓ_p*-equivalent.*

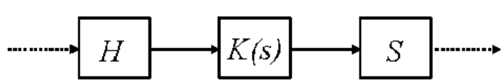

Fig. 1. Hold equivalent transformation

The ℓ_p-equivalence between ZOH and the bilinear transformation is established in Zhang and Chen [2004] under the conditions that $K(s)$ is SISO and its "A" matrix is diagonalizable with all real eigenvalues. These constraints are removed in Zhang et al. [2007]. Actually a stronger result is established in Zhang et al. [2007] for the ℓ_p-equivalence between a so-called generalized bilinear transformation

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(GBT) and ZOH. The generalized bilinear transformation has a free parameter α. When $\alpha = 0$, it is the forward Euler approximation method; when $\alpha = 1/2$, it reduces to the bilinear transformation; when $\alpha = 1$, it is the backward Euler approximation method (Wittenmark et al. [2002]). It is further shown that by choosing a suitable α, GBT may map unstable poles (resp. zeros) to stable poles (resp. zeros). Given that unstable poles or zeros always impose various performance limitations on the pertinent control system, this feature of GBT is hence very appealing. Furthermore, the free parameter α eases the tuning of the controller on-line to trade off various performance specifications. In this paper, instead of merely working on FOH and the bilinear transformation, a new type of fractional-order hold (FROH) is first introduced that contains ZOH and FOH as special cases, then the ℓ_p-equivalence between FROH and GBT is attained, thus partially answering the previously posed question. The discussion in this paper is for linear systems exclusively.

This paper is organized as follows. Section 2 introduces the fractional-order hold (FROH) transformation. After deriving its state-space model, we study its limiting zeros with the aid of those of ZOH equivalents which have been investigated in the literature (Astrom et al. [1984], Hagiwara et al. [1993], Weller et al. [2001]). Section 3 reviews briefly the generalized bilinear transformation. Section 4 investigates the relation between GBT and FROH; we show that the FROH and GBT converge to each other in the ℓ_∞ induced norm as h goes to zero. Section 4 establishes the ℓ_p-equivalence between GBT and the FROH transformation. Section 5 contains some concluding remarks.

The following notation is used in this paper. The norm symbol $\|\cdot\|$ represents the Euclidean norm for a vector or the largest singular value for a matrix; $\|\cdot\|_{\ell_p}$ is the ℓ_p norm if applied to a vector and ℓ_p induced norm if applied to a system. $o(h)$ is a function of h which satisfies $\lim_{h\to 0} o(h)/h \to 0$. Similarly, $O(h)$ satisfies $\lim_{h\to 0} O(h) \to 0$. Here $o(h)$ and $O(h)$ may be either scalar functions or matrix functions. Following the convention, for a discrete-time transfer function $H(z)$ with a state-space realization (A, B, C, D), define

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] := D + C(zI - A)^{-1}B.$$

The discrete-time counterpart is defined similarly.

2. THE FROH TRANSFORMATION

In this section, the fractional-order hold (FROH) transformation is first proposed, then the zeros of the FROH equivalent of an analog controller is investigated. It turns out that they can be estimated via discretization zeros of two ZOH equivalents of the analog controller.

Suppose that an analog controller $K(s)$ has already been designed; now we want to implement it digitally. One obvious way is to precede and follow $K(s)$ with a hold H and an ideal sampler S respectively, as shown in Fig. 1. Because digital controllers are implemented in computers in the forms of algorithms, the physical realization of H is not an issue. Therefore, besides the zero-order and first-order holds (Wittenmark et al. [2002]), other types of holds can also be used. In this paper, H is allowed to be fractional-order holds. For convenience, we call such discretization methods fractional-order hold (FROH) transformations.

More specifically, consider a continuous-time controller $K(s)$ of state-space model:

$$\begin{aligned} \dot{x} &= A_K x + B_K u, \\ y &= C_K x + D_K u, \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, and $y \in \mathbb{R}^p$ is the output. A_K, B_K, C_K, and D_K are all constant matrices of appropriate dimensions. A discrete-time controller can be obtained in the way as shown in Fig. 1, where H is a certain hold and S is an ideal sampler. Clearly, if H is a zero-order hold, then the resulting digital controller is the zero-order hold equivalent of $K(s)$. In this paper we let H be a fractional-order hold, and call the discretization method in Fig. 1 the *fractional-order hold (FROH) transformation*. In what follows we define the (FROH) transformation. At time interval $[kh, (k+1)h)$, where $k \in \mathbb{Z}^+$, if the input $u(\tau)$ is approximated via

$$u(\tau) = u(kh) + \beta \frac{u(kh+h) - u(kh)}{h} (\tau - kh), \tag{2}$$

for $\tau \in [kh, (k+1)h)$, where $\beta \in (-\infty, \infty)$, then the fractional-order hold equivalent of $K(s)$ is defined via

$$\begin{aligned} & x(kh+h) \\ &= e^{A_K h} x(kh) + \int_{kh}^{kh+h} e^{A_K(kh+h-\tau)} B_K \\ &\quad \cdot \left(u(kh) + \beta \frac{u(kh+h) - u(kh)}{h} (\tau - kh) \right) d\tau \\ &= e^{A_K h} x(kh) + (\Gamma - \beta\Gamma_1) u(kh) + \beta \Gamma_1 u(kh+h), \\ y(kh) &= C_K x(kh) + D_K u(kh), \end{aligned} \tag{3}$$

where $\Gamma := \int_0^h e^{A_K\tau} d\tau B$, and $\Gamma_1 := \int_0^h e^{A_K\tau} \frac{h-\tau}{h} d\tau B$. Because $u(kh+h)$ appears on the right-hand side of the equation of the state evolution, define

$$w(kh) := x(kh) - \beta\Gamma_1 u(kh).$$

Then Eq. (3) is converted to

$$\begin{aligned} w(kh+h) &= e^{A_K h} w(kh) + \left(\Gamma + \beta\left(e^{A_K h} - I\right)\Gamma_1\right) u(kh) \\ y(kh) &= C_K w(kh) + (D_K + \beta C_K \Gamma_1) u(kh). \end{aligned} \tag{4}$$

In the sequel, we denote the fractional-order hold equivalent defined in Eq. (4) by $K_{froh}(z)$. Let

$$\begin{aligned} A_{Kfroh} &= e^{A_K h},\ B_{Kfroh} = \Gamma + \beta(A_{Kfroh} - I)\Gamma_1, \\ C_{Kfroh} &= C_K,\ D_{Kfroh} = D_K + \beta C_K \Gamma_1. \end{aligned}$$

Then K_{froh} has a state-space realization

$$(A_{Kfroh}, B_{Kfroh}, C_{Kfroh}, D_{Kfroh}).$$

Remark 1. When $\beta = 0$, the fractional-order hold reduces to the zero-order hold (ZOH); when $\beta = 1$, it becomes the first-order hold (FOH) (pp.17, Wittenmark et al. [2002]). Note that our aim is to get an approximation of a continuous-time controller, and the resulting discrete-time controller will be implemented in a computer, so physical implementation (non-causality) is not an issue.

It is easy to show that the impulse response is

$$
\begin{aligned}
\digamma(t) &= \frac{\beta}{h}(t+h)\left(u_s(t+h) - u_s(t)\right) \\
&\quad + \left(1 - \frac{\beta}{h}t\right)\left(u_s(t) - u_s(t-h)\right) \\
&= \frac{\beta}{h}(t+h)\,u_s(t+h) - \left(\frac{2\beta}{h}t + \beta - 1\right)u_s(t) \\
&\quad - \left(1 - \frac{\beta}{h}t\right)u_s(t-h),
\end{aligned}
$$

where u_s is the unit step function. Applying the Laplace transform to the last equation

$$
\begin{aligned}
\digamma(s) &:= L(f)(s) \\
&= \frac{\beta e^{hs}}{hs^2}\left(1 - e^{-hs}\right)^2 + \frac{\beta}{s}\left(e^{-hs} - 1\right) \\
&\quad + \frac{1}{s}\left(1 - e^{-hs}\right).
\end{aligned}
$$

Denote the continuous-time transfer function of Eq. (1) by $K(s)$. The Z-transform of $\digamma(s)K(s)$, namely $K_{froh}(z)$, is given by

$$
\begin{aligned}
Z\left[\digamma(s)K(s)\right] &= Z\left[\frac{\beta e^{hs}}{hs^2}\left(1 - e^{-hs}\right)^2 K(s)\right] \\
&\quad + Z\left[\frac{1-\beta}{s}K(s)\right](1 - z^{-1}) \\
&= \frac{\beta z\left(1 - z^{-1}\right)}{h} Z\left(\frac{1 - e^{-hs}}{s}\frac{K(s)}{s}\right) \\
&\quad + (1-\beta)\left(1 - z^{-1}\right) Z\left[\frac{K(s)}{s}\right] \qquad (5)
\end{aligned}
$$

In the following, we focus on the limiting zeros of system $Z\left[\digamma(s)K(s)\right]$ as h goes to 0. For simplicity, we here consider a SISO and strictly proper $K(s)$, i.e., $m = p = 1$ and $D = 0$ in Eq. (1). Assume that $K(s)$ is

$$
Q\frac{(s-\gamma_1)\cdots(s-\gamma_{n_m})}{(s-\lambda_1\cdots(s-\lambda_{n_d}))},\ (n_d > n_m), \qquad (6)
$$

where Q is a constant. Moreover, as given in the introduction, assume $K_d(z) = Z\left[\frac{1-e^{-sh}}{s}K(s)\right]$ (namely, the zero-order hold equivalent). Define a differential recurrence relation via

$$
\begin{aligned}
B_1(z) &= 1, \\
B_p(z) &= \left(1 + (p-1)z\right)B_{p-1}(z) \\
&\quad + z(1-z)\frac{dB_{p-1}(z)}{dz}, \quad p = 2, 3, \ldots
\end{aligned}
$$

Moreover, let $\xi_1, \ldots, \xi_{p-1}$ be the roots of $B_p(z)$, $\eta_1, \ldots, \eta_p$ the roots of $B_{p+1}(z)$. Then

$$
\eta_1 < \xi_1 < \eta_2 < \xi_2 < \cdots < \eta_{p-1} < \xi_{p-1} < \eta_p < 0.
$$

Define

$$
C_p(z) := \beta B_{p+1}(z) + (1-\beta)(p+1)B_p(z).
$$

Then we have

Theorem 1. $K_{froh}(z)$ has n_d zeros. Furthermore, as $h \to 0$, $K_{froh}(z)$ approaches

$$
Q\frac{h^{n_d - n_m}}{(n_d - n_m + 1)!}\frac{(z-1)^{n_m}}{(z-1)^{n_d}}C_{n_d - n_m}(z). \qquad (7)
$$

Let $q = n_d - n_m$ and assume that the roots of $C_q(z)$ are $\zeta_1, \ldots, \zeta_q$. Then the following statements hold.

#1. If $\beta \geq 1 + 1/q$, then

$$
\begin{aligned}
&\eta_1 < \zeta_1 < \xi_1 < \eta_2 < \zeta_2 < \xi_2 < \cdots < \eta_{q-1} \\
&\quad < \zeta_{q-1} < \xi_{q-1} < \eta_q < 0 \leq \zeta_q < 1,
\end{aligned}
$$

where "=" holds if and only if $\beta = 1 + 1/q$.

#2. If $1 < \beta < 1 + 1/q$, then

$$
\begin{aligned}
&\eta_1 < \zeta_1 < \xi_1 < \eta_2 < \zeta_2 < \xi_2 < \cdots < \eta_{q-1} \\
&\quad < \zeta_{q-1} < \xi_{q-1} < \eta_q < \zeta_q < 0.
\end{aligned}
$$

#3. If $\beta = 1$, then $\zeta_1, \ldots, \zeta_q$ coincide with $\eta_1, \ldots, \eta_q$.

#4. If $0 < \beta < 1$, then

$$
\begin{aligned}
&\zeta_1 < \eta_1 < \xi_1 < \eta_2 < \xi_2 < \zeta_2 < \cdots < \eta_{q-1} \\
&\quad < \xi_{q-1} < \zeta_{q-1} < \eta_q < 0.
\end{aligned}
$$

#5. If $\beta = 0$, then the roots of $C_q(z)$ are those of $B_q(z)$.

#6. If $\beta < 0$, then

$$
\begin{aligned}
&\eta_1 < \zeta_1 < \xi_1 < \eta_2 < \zeta_2 < \xi_2 < \cdots < \eta_{q-1} \\
&\quad < \zeta_{q-1} < \xi_{q-1} < \eta_q < 0 < \zeta_q.
\end{aligned}
$$

Due to page limitations, its proof is limited.

Remark 2. When $n_d - n_m = 2$, according to Eq. (3) in Sobolev [1977], as $h \to 0$, the FROH equivalent of an analog controller will have two discretization zeros, residing on the opposite side of -1 on the real axis. Therefore it has necessarily one *unstable* discretization zero. If $\beta > 1 + 1/(n_d - n_m) = 3/2$, one of the zeros of $K_{froh}(z)$ is within the interval $(0, 1)$, while the other, no matter whether it is stable or not, is more close to 0 than the discretization zeros of the FOH equivalent. In this sense, the fractional-order hold is superior to the first-order hold.

3. THE GENERALIZED BILINEAR TRANSFORMATION

The generalized bilinear transformation (GBT) is studied in Zhang et al. [2007] and Zhang et al. [2007b]. Given an analog controller $K(s)$, denote the digital controller obtained via GBT by $K_{gbt}(z)$. In terms of state-space data, Let (A_K, B_K, C_K, D_K) be a *minimal* realization of $K(s)$, $K_{gbt}(z)$ has a state-space model $(A_{Kgbt}, B_{Kgbt}, C_{Kgbt}, D_{Kgbt})$, where

$$
\begin{aligned}
A_{Kgbt} &= \left(I - \alpha h A_K\right)^{-1}\left[I + (1-\alpha)hA_K\right], \qquad (8) \\
B_{Kgbt} &= \left(I - \alpha h A_K\right)^{-1} hB_K, \\
C_{Kgbt} &= C_K\left(I - \alpha h A_K\right)^{-1}, \\
D_{Kgbt} &= D_K + \alpha C_K B_{Kgbt},
\end{aligned}
$$

in which $\alpha \in (-\infty, \infty)$.

Remark 3. Interestingly, specific to the discretization of a pure integrator $1/s$, the generalized bilinear transformation and the fractional-order hold transformation are identical provided $\beta = 2\alpha$.

4. ℓ_P EQUIVALENCE OF GBT AND THE FROH TRANSFORMATIONS

In this section, we will study the relation between $K_{gbt}(z)$ and $K_{froh}(z)$, two digital approximations of an analog controller. The main result of this section is: Given a stable analog controller $K(s)$, the ℓ_p induced norm of $K_{gbt}(z) - K_{froh}(z)$ approaches zero for all $p \in [1, \infty]$ as $h \to 0$. In order to establish it, some preliminary results are established first.

Lemma 1. Assume that the pair (A_K, B_K) is stabilizable. Then there exists a single constant matrix F such that both $A_{Kfroh} + B_{Kfroh}F$ and $A_{Kgbt} + B_{Kgbt}F$ are asymptotically stable (in discrete time) for sufficiently small h.

Proof. Upon the hypothesis, there is a matrix F such that $A_K + B_K F$ is stable (in continuous time). Observe that

$$\begin{aligned}
&A_{Kfroh} + B_{Kfroh}F \\
&= e^{A_K h} + \left(\Gamma + \beta\left(e^{A_K h} - I\right)\Gamma_1\right) F \\
&= e^{A_K h} + \int_0^h e^{A_K t} dt B_K F \\
&\quad + \beta\left(e^{A_K h} - I\right)\int_0^h e^{A_K t}\frac{h-t}{h} dt B_K F \\
&= I + h\left(A_K + B_K F\right) + o(h),
\end{aligned}$$

where $o(h)$ satisfies $\lim_{h\to 0}\frac{o(h)}{h} = 0$. Therefore, $A_{Kfroh} + B_{Kfroh}F$ (in discrete time) is stable for sufficiently small h. To prove the stability of $A_{Kgbt} + B_{Kgbt}F$, observe that

$$\begin{aligned}
&A_{Kgbt} + B_{Kgbt}F \\
&= \left(I - \alpha h A_K\right)^{-1}\left[I + h\left((1-\alpha)A_K + B_K F\right)\right] \\
&= \left(I + \alpha h A_K + o(h)\right)\left[I + h\left((1-\alpha)A_K + B_K F\right)\right] \\
&= I + h\left((1-\alpha)A_K + B_K F\right) + \alpha h A_K + o(h) \\
&= I + h\left(A_K + B_K F\right) + o(h).
\end{aligned}$$

Hence, $A_{Kgbt} + B_{Kgbt}F$ is stable (in discrete time) for sufficiently small h. ■

Now we have set up for the following lemma that is crucial for further development. It basically says that K_{froh} and K_{gbt} converge to each other in some graph metric as $h \to 0$.

Lemma 2. Suppose that (A_K, B_K) is stabilizable. Then in the graph metric, $K_{froh}(z) - K_{gbt}(z)$ converges to zero as the sampling period h goes to zero.

Proof. By Lemma 1, there exists a matrix F such that both $A_{Kfroh} + B_{Kfroh}F$ and $A_{Kgbt} + B_{Kgbt}F$ are stable for h sufficiently small. Define

$$\begin{bmatrix} M_{Kfroh}(z) \\ N_{Kfroh}(z)\end{bmatrix} = \left[\begin{array}{c|c} A_{Kfroh} + B_{Kfroh}F & B_{Kfroh} \\ \hline F & I \\ C_{Kfroh} + D_{Kfroh}F & D_{Kfroh}\end{array}\right],$$

$$\begin{bmatrix} M_{Kgbt}(z) \\ N_{Kgbt}(z)\end{bmatrix} = \left[\begin{array}{c|c} A_{Kgbt} + B_{Kgbt}F & B_{Kgbt} \\ \hline F & I \\ C_{Kgbt} + D_{Kgbt}F & D_{Kgbt}\end{array}\right].$$

Then $K_{froh}(z) = N_{Kfroh}(z) M_{Kfroh}^{-1}(z)$ and $K_{gbt} = N_{Kgbt}(z) M_{Kgbt}^{-1}(z)$ are right coprime factorizations (pp. 71-73, Zhou and Doyle [1998]). it is easy to show that

$$\begin{aligned}
&M_{Kfroh}(z) - M_{Kgbt}(z) \\
&= F\left[zI - \left(A_{Kfroh} + B_{Kfroh}F\right)\right]^{-1} B_{Kfroh} \\
&\quad - F\left[zI - \left(A_{Kgbt} + B_{Kgbt}F\right)\right]^{-1} B_{Kgbt} \\
&:= F T(z) B,
\end{aligned}$$

where

$$\begin{aligned}
&T(z) \\
&= \left[zI - \left(A_{Kfroh} + B_{Kfroh}F\right)\right]^{-1}\left(\int_0^h e^{A_K t} dt\right. \\
&\quad \left. + \beta\left(e^{A_K h} - I\right)\int_0^h e^{A_K t}\frac{h-t}{t} dt\right) \\
&\quad - \left[zI - \left(A_{Kgbt} + B_{Kgbt}F\right)\right]^{-1}\left(I - \alpha h A_K\right)^{-1} h.
\end{aligned}$$

Let $z = e^{-j\theta}$, $\theta \in [-\pi, \pi)$. If $\theta = 0$, $z = 1$, then

$$\begin{aligned}
&zI - \left(A_{Kfroh} + B_{Kfroh}F\right) \\
&= I - \left(A_{Kfroh} + B_{Kfroh}F\right) \\
&= -\left(e^{A_K h} - I\right) \\
&\quad \cdot \left[I + \left(A_K^{-1} + \beta\int_0^h e^{A_K t}\frac{h-t}{t} dt\right) B_K F\right],
\end{aligned}$$

therefore

$$\begin{aligned}
&\left[I - \left(A_{Kfroh} + B_{Kfroh}F\right)\right]^{-1} \\
&= -\left[I + \left(A_K^{-1} + \beta\int_0^h e^{A_K t}\frac{h-t}{t} dt\right) B_K F\right]^{-1} \\
&\quad \cdot \left(e^{A_K h} - I\right)^{-1}.
\end{aligned}$$

(The matrix A_K is assumed to be invertible purely for technical simplicity, it can be easily shown that the result also holds when A_K is not invertible.) Define

$$W := A_K^{-1} + \beta\int_0^h e^{A_K t}\frac{h-t}{t} dt.$$

Then

$$\begin{aligned}
W &= A_K^{-1} + \frac{\beta}{h} A_K^{-2}\left(e^{A_K h} - I - A_K h\right) \\
&= (1-\beta) A_K^{-1} + \frac{\beta}{h} A_K^{-2}\left(e^{A_K h} - I\right) \\
&= A_K^{-1} + \frac{\beta}{h} A_K^{-2} o(h).
\end{aligned}$$

Observe that

$$\begin{aligned}
&\int_0^h e^{A_K t} dt + \beta\left(e^{A_K h} - I\right)\int_0^h e^{A_K t}\frac{h-t}{t} dt \\
&= \left(e^{A_K h} - I\right) W,
\end{aligned}$$

and

$$\begin{aligned}
&\left[I - \left(A_{Kfroh} + B_{Kfroh}F\right)\right]^{-1} \\
&\cdot \left(\int_0^h e^{A_K t} dt + \beta\left(e^{A_K h} - I\right)\int_0^h e^{A_K t}\frac{h-t}{t} dt\right) \\
&= -\left[I + W B_K F\right]^{-1}\left(e^{A_K h} - I\right)^{-1}\left(e^{A_K h} - I\right) W \\
&= -\left[I + W B_K F\right]^{-1} W.
\end{aligned}$$

Taking limit with respect to h yields:

$$\begin{aligned}&\left[I-\left(A_{Kfroh}+B_{Kfroh}F\right)\right]^{-1}\\&\cdot\left(\int_0^h e^{A_K t}dt+\beta\left(e^{A_K h}-I\right)\int_0^h e^{A_K t}\frac{h-t}{t}dt\right)\\=&-\left(A_K+B_KF\right)^{-1}.\end{aligned}$$

On the other hand,

$$\begin{aligned}&\left[I-\left(A_{Kgbt}+B_{Kgbt}F\right)\right]^{-1}\\=&-\frac{1}{h}\left(A_K+B_KF\right)^{-1}\left(I-\alpha hA_K\right),\end{aligned}$$

thus

$$\begin{aligned}&\left[I-\left(A_{Kgbt}+B_{Kgbt}F\right)\right]^{-1}\left(I-\alpha hA_K\right)^{-1}h\\=&-\left(A_K+B_KF\right)^{-1}.\end{aligned}$$

Consequently,

$$\lim_{h\to 0}T(1)=0.$$

When $\theta\neq 0$, $T(z)$ is analytic, and it is easy to show that

$$\lim_{h\to 0^+}T(e^{-j\theta})=0,\ \forall\theta\in[-\pi,\pi).$$

Therefore, we conclude that

$$\begin{aligned}&\lim_{h\to 0^+}\left\|M_{Kfroh}(z)-M_{Kgbt}(z)\right\|_{\ell_2}\\=&\lim_{h\to 0^+}\sup_{-\pi\leq\theta<\pi}\left\|M_{Kfroh}(e^{-j\theta})-M_{Kgbt}(e^{-j\theta})\right\|\\=&\,0.\end{aligned}$$

Similarly, we can show that

$$\lim_{h\to 0^+}\left\|N_{Kfroh}(z)-N_{Kgbt}(z)\right\|_{\ell_2}=0.$$

It hence follows that $K_{froh}(z)-K_{gbt}(z)$ converges to zero in the graph metric as $h\to 0$. ■

If K is stable, $K_{froh}(z)$ is stable for any β. Furthermore, according to Eq. (8), it is not hard to show that, for any given $\alpha\in(-\infty,\infty)$, $K_{gbt}(z)$ is stable when h is sufficiently small. In light of this, the graph metric induces the same topology as that induced by the ℓ_2 induced norm. Thus we have

Lemma 3. Suppose $K(s)$ is stable. Then for all $\alpha\in(-\infty,\infty)$ and $\beta\in(-\infty,\infty)$,

$$\lim_{h\to 0^+}\left\|K_{froh}(z)-K_{gbt}(z)\right\|_{\ell_2}=0.$$

With the aid of Lemma 3, we are now in a position to prove the main result of this section.

Theorem 2. Suppose $K(s)$ is stable. Then for any given $\alpha\in(-\infty,\infty)$ and $\beta\in(-\infty,\infty)$,

$$\lim_{h\to 0^+}\left\|K_{froh}(z)-K_{gbt}(z)\right\|_{\ell_p}=0$$

holds for all $1\leq p\leq\infty$.

Proof. Because $K(s)$ is stable, $K_{froh}(z)$ is stable for any β. On the other hand, by Eq. (8), it is easy to show that, for any given $\alpha\in(-\infty,\infty)$, $K_{gbt}(z)$ is stable when h is sufficiently small. Therefore, $\left\|K_{froh}(z)-K_{gbt}(z)\right\|_{\ell_p}$ is well-defined for sufficiently small h. According to Lemma 3,

$$\lim_{h\to 0^+}\left\|K_{froh}(z)-K_{gbt}(z)\right\|_{\ell_2}=0.$$

Let the Hankel singular values of $K_{froh}(z)-K_{gbt}(z)$ be $\sigma_H^h=\left\{\sigma_1^h,\cdots,\sigma_N^h\right\}$, where $\sigma_1^h\geq\cdots\geq\sigma_N^h\geq 0$. According to the discrete-time counterpart of Theorem 7.8 in Zhou and Doyle [1998], we have

$$\begin{aligned}&\left\|K_{froh}(z)-K_{gbt}(z)\right\|_{\ell_\infty}\\\leq&\,2\sum_{k=1}^{N}\sigma_i^h\leq 2N\sigma_1^h\\\leq&\,2N\cdot\lim_{h\to 0^+}\left\|K_{froh}(z)-K_{gbt}(z)\right\|_{\ell_2}.\end{aligned}$$

As a result,

$$\lim_{h\to 0^+}\left\|K_{froh}(z)-K_{gbt}(z)\right\|_{\ell_\infty}=0.$$

Furthermore, based on the discrete-time version of Theorem 9.1.2 in Chen and Francis [1995], it is not hard to show that

$$\lim_{h\to 0^+}\left\|K_{froh}(z)-K_{gbt}(z)\right\|_{\ell_p}=0$$

for all $1\leq p\leq\infty$. (The SISO case follows Theorem 9.1.2 in Chen and Francis [1995] directly; the MIMO case is more tedious, but without essential difficulties.) The result is thus established. ■

5. CONCLUSION

In this paper, a new type of fractional-order hold transformations has been proposed first. Then the property of its limiting zeros of resulting digital controllers has been investigated. Finally, it has been shown that, at fast sampling, the digital approximations of an analog controller obtained via this type of fractional-order hold transformations and the generalized bilinear transformation are convergent to each other in the sense of ℓ_p induced norm for $p\in[1,\infty]$ when the analog controller is stable or in the sense of some gap metric even when it is unstable.

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