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Observability analysis for networked control systems: a graph theoretic approach

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Abstract: This paper deals with the state and input observability analysis for Networked Control Systems which are composed of interconnected subsystems that exchange data through communication networks. The proposed method is based on a graph-theoretic approach and assumes only the knowledge of the system's structure. More precisely, for the so-called distributed decentralized and distributed autonomous observation schemes, we express, in simple graphic terms, necessary and sufficient conditions to check whether or not a considered subsystem is strongly observable. These conditions, which allows also to characterize all the strongly observable state and input components of each subsystem, are easy to check because they are based on comparison of integers and on finding paths in a digraph. This makes our approach suited to study large scale distributed systems.

Keywords: Networked Control Systems, input and state observability, graph theory.

1. INTRODUCTION

Networked Control systems (NCS) are in general composed of a large number of interconnected devices or subsystems that exchange data through communication networks. Examples include industrial automation, building supervision, automotive control, ... NCSs provide many advantages such as modular and flexible system design, fast implementation, distribution. However, some disadvantages such as loss of information, time delays in the data transmission may also have an effect on the general performances of the global system. The observability of the internal state or of the input components of each subsystem is one of the main properties which is strongly linked to the configuration of the distributed subsystems and to the data they exchange. Indeed, even if the global system is observable, when we subdivide it into several subsystems, the latter may be not structurally observable. Thus, an analysis of the observability of the distributed system, for different configurations and in function of the informations exchanged on the network, is important for the observer design and so in the general conception of the system.

The issue of this paper is to analyse the observability of each subsystem using the knowledge of its own local measurements and eventually the measurements arriving trough the network. Such problem is very close to the analysis of the strong observability of a given part of the state and the input of a linear system, which is a very significant question in the general observation theory. Indeed, the problem of reconstructing any desired part of the state and/or the unknown input is of a great interest mainly in control law synthesis, fault detection and isolation, fault tolerant control, supervision and so on. In this respect, many works are focused on the design of full or reduced state observers for linear systems with unknown inputs. Among the most important works dealing with the state and input reconstructibility, we can cite the approach developed in (Basile and Marro [1969], Hautus [1983]) where the author gives the definitions of strong detectability and strong observability and the conditions for existence of observers that estimate a functional of the state and unknown inputs.

On the other hand, many studies deal with the observation of decentralized systems even if it is not in the context of Networked Control Systems. In this way, in the early 70's, Sanders et al. [1974] propose, under some decoupling assumptions, the design of a filter for interconnected dynamical systems in which the information pattern is decentralized. More recently, in (Saif and Guan [1992]) the authors propose a method for the design of decentralized reduced state estimator for large scale systems composed by interconnected systems using unknown input observers under some "matching condition". Also on the basis of unknown input observers, in (Hou and Müller [1994]), decentralized state function observer are designed for large scale interconnected systems. Three kinds of interconnections are considered and the design of the state function local observer is done under the solvability of some matrix algebraic equations.

In the latter papers as in most other, the studies on the state or/and input observability or on decentralized systems deal with algebraic and geometric tools (Basile and Marro [1973], Hou and Patton [1998], Trentelman et al. [2001], Yang and Zhang [1995]). The use of such tools requires the exact knowledge of the state space matrices characterizing the system's model. However, in many modeling problems, only zero entries of these matrices, which are determined by the physical laws, are fixed while the remaining entries are not precisely known. To study the properties of these systems in spite of poor knowledge we have on them, the idea is that we only keep the zero/non-zero entries in the state space matrices. Thus, we consider models where the fixed zeros are conserved while the non-zero entries are replaced by free parameters. The analysis of such systems requires a low computational burden which allows one to deal with large scale systems. Many studies on structured systems are related to the graph-theoretic approach to analyse some system properties such as controllability, observability or the solvability of several classical control problems including disturbance rejection, input-output decoupling, ... These

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works are reviewed in the survey (Dion et al. [2003]) from which it results that the graph-theoretic approach provides simple, efficient and elegant solutions. However, the well-known graphic observability conditions for linear structured systems recalled in cannot be applied to systems with unknown inputs. Moreover, the state and input observability conditions provided in (Boukhobza et al. [2007]) for centralized linear systems with unknown inputs are not adapted to study the observability of only a part of state and input components, which is quite necessary to study the observability of Networked Control Systems.
In this context, the purpose of this paper is to use a graph-theoretic approach for providing necessary and sufficient conditions for the generic observability of structured Networked Control systems in function of their configuration. Note finally that our method is mainly an analysis one and we do not deal with the observer design problem.
The paper is organised as follows: after Section 2, which is devoted to the problem formulation, a digraph representation of Networked Control systems is given in Section 3. The main result is enounced in Section 4. Finally, a conclusion ends the paper.

2. PROBLEM STATEMENT

In this paper, we consider Networked Control Systems having the numerically non-specified following model:

$$\Sigma_\Lambda : \begin{cases} \dot{x}(t) = A^\lambda x(t) + B^\lambda u(t) \\ y(t) = C^\lambda x(t) + D^\lambda u(t) \end{cases} \quad (1)$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^q$ and $y(t) \in \mathbb{R}^p$ are respectively the state vector, the unknown input vector and the output vector. A^λ, B^λ, C^λ and D^λ represent matrices which elements are either fixed to zero or assumed to be nonzero free parameters noted λ_i. These parameters forms a vector $\Lambda = (\lambda_1, \ldots, \lambda_h)^T \in \mathbb{R}^h$. If all parameters λ_i are numerically fixed, we obtain a so-called admissible realization of structured system Σ_Λ. We say that a property is true generically if it is true for almost all the realizations of structured system Σ_Λ. Here, "for almost all the realizations" is to be understood as "for all parameter values ($\Lambda \in \mathbb{R}^h$) except for those in some proper algebraic variety in the parameter space". The proper algebraic variety for which the property is not true is the zero set of some nontrivial polynomial with real coefficients in the h system parameters $\lambda_1, \lambda_2, \ldots, \lambda_h$ or equivalently it is an algebraic variety which has Lebesgue measure zero.
Consider that the structured linear system (Σ_Λ) (1) is a distributed system. It is then constituted of several subsystems (Σ_i^R), $i = 1, \ldots, N$. Each subsystem satisfies to a model of the form:

$$(\Sigma_i^R) \begin{cases} \dot{x}_i(t) = A_i^\lambda x_i(t) + B_i^\lambda u_i(t) + \sum_{j=1, j\neq i}^{N} \left(A_{i,j}^\lambda x_j(t) + B_{i,j}^\lambda u_j(t) \right) \\ y_i(t) = C_i^\lambda x_i(t) + D_i^\lambda u_i(t) + \left(\sum_{j=1, j\neq i}^{N} C_{i,j}^\lambda x_j(t) + D_{i,j}^\lambda u_j(t) \right) \end{cases} \quad (2)$$

where for $i, j = 1, \ldots, N, j \neq i$, $x_i \in \mathbb{R}^{n_i}$ is the state vector of subsystem (Σ_i^R), $y_i \in \mathbb{R}^{m_i}$ is the output vector of subsystem (Σ_i^R) and $u_i \in \mathbb{R}^{p_i}$ is the input vector of subsystem (Σ_i^R). Matrices A_i^λ, B_i^λ, $A_{i,j}^\lambda$, $B_{i,j}^\lambda$, C_i^λ, D_i^λ, $C_{i,j}^\lambda$ and $D_{i,j}^\lambda$ represent matrices of appropriate dimensions whose elements are either fixed to zero or assumed to be free non-zero parameters.
All the subsystems are linked together through a network. The measurements arriving to each subsystem (Σ_i^R) can be modeled in the most general form as:

$$\tilde{y}_i = \tilde{C}_i^\lambda x_i(t) + \tilde{D}_i^\lambda u_i(t) + \sum_{j=1, j\neq i}^{N} \tilde{C}_{i,j}^\lambda x_j(t) + \sum_{j=1, j\neq i}^{N} \tilde{D}_{i,j}^\lambda u_j(t)$$

We assume, without loss without loss of generality, that the measurements arriving through the network to subsystem i are linearly independent from the ones constituting y_i.

In this paper, we study the generic partial state and input observability of structured subsystems constituting (Σ_Λ). This notion is related to the strong observability and the left invertibility (Trentelman et al. [2001]). Let us recall the definition of the generic state and input observability in the case of a structured linear system:

Definition 1. We say that structured system (Σ_Λ) is generically state and input observable if and only if it is generically strongly observable and left invertible. In this case, we say that all the state components $x_{i,k}$, $k = 1, \ldots, n_i$ and $u_{i,j}$, $j = 1, \ldots, q_i$ for $i = 1, \ldots, N$ are strongly observable.

Clearly, we cannot guarantee the existence of a causal observer which allows to give an estimate of any strongly observable component. Nevertheless, the strong observability of a component is obviously a necessary condition to the existence of such observer and it ensures the existence of a generalized observer (which can use the measurement derivatives) which allows to give an estimate of any strongly observable component (Hou and Müller [1999]).
For the present study, we are interested in the generic strong observability of only a part of the state or the input of each subsystem (Σ_i^R), $i = 1, \ldots, N$, and we consider two cases. In the first case, we assume that subsystem (Σ_i^R) is linked to the network and can use the measurement vector $\tilde{y}_i$ to reconstruct its state or input components. We call this case, the distributed decentralized observation scheme. In the second case, we consider that there is no external measurements arriving through the network to (Σ_i^R). So, subsystem (Σ_i^R) can use only its own measurements vector y_i to reconstruct its state or input components. We call this case, the distributed autonomous observation scheme.

We define now the strong observability of an input or a state component, relatively to the considered observation scheme, as follows:

Definition 2. Consider structured system (Σ_Λ). For $i = 1, \ldots, N$, we say that state component $x_{i,k}$, $k \in \{1, \ldots, n_i\}$ (respectively input component $u_{i,j}$, $j \in \{1, \ldots, q_i\}$) is generically strongly observable in a distributed decentralized observation scheme if for all initial state x_0 and for every input function $u(t)$, $y_i(t) = 0$ and $\tilde{y}_i(t) = 0$ for $t \geq 0$ implies $x_{i,k}(t) = 0, \forall t \geq 0$ (respectively $u_{i,j}(t) = 0, \forall t > 0$).
Similarly, we say that state component $x_{i,k}$ (respectively input component $u_{i,j}$) is generically strongly observable in a distributed autonomous observation scheme if for all initial state x_0 and for every input function $u(t)$, $y_i(t) = 0$ for $t \geq 0$ implies $x_{i,k}(t) = 0, \forall t \geq 0$ (respectively $u_{i,j}(t) = 0, \forall t > 0$).

Roughly speaking, the generic strong observability of state component $x_{i,k}$ (respectively input component $u_{i,j}$) means that a change in $x_{i,k}(0)$ (respectively $u_{i,j}(0^+)$) is necessarily reflected in a change of measurements accessible to the studied subsystem in the considered observation scheme.

3. GRAPH REPRESENTATION OF STRUCTURED LINEAR SYSTEMS

To structured system (Σ_Λ) constituted by subsystems (Σ_i^R), $i = 1, \ldots, N$, we associate a digraph noted $\mathcal{G}(\Sigma_\Lambda)$ which is constituted by a vertex set $\mathcal{V}$ and an edge set $\mathcal{E}$. More precisely, $\mathcal{V} = \bigcup_{i=1}^{N} \left(\mathbf{X_i} \cup \mathbf{U_i} \cup \mathbf{Y_i} \cup \widetilde{\mathbf{Y}}_\mathbf{i}\right)$, where $\mathbf{X_i} = \{\mathbf{x_{i,1}}, \ldots, \mathbf{x_{i,n_i}}\}$ is the set of state vertices for subsystem i, $\mathbf{U_i} = \{\mathbf{u_{i,1}}, \ldots, \mathbf{u_{i,q_i}}\}$ is the set of input vertices for subsystem i, $\mathbf{Y_i} = \{\mathbf{y_{i,1}}, \ldots, \mathbf{y_{i,p_i}}\}$ is the set of output vertices for subsystem i, $\widetilde{\mathbf{Y}}_\mathbf{i} = \{\tilde{\mathbf{y}}_\mathbf{i,1}, \ldots, \mathbf{y_{i,\tilde{p}_i}}\}$ is the set of output vertices associated to the measurements arriving through the network to subsystem i. The edge set is

$\mathcal{E} = \bigcup_{i=1}^{N} \Big(A_i\text{-edges} \cup B_i\text{-edges} \cup C_i\text{-edges} \cup D_i\text{-edges} \cup \tilde{C}_i\text{-edges} \cup \tilde{D}_i\text{-edges} \bigcup_{j=1, j\neq i}^{N} (A_{i,j}\text{-edges} \cup B_{i,j}\text{-edges} \cup C_{i,j}\text{-edges} \cup D_{i,j}\text{-edges} \cup \tilde{C}_{i,j}\text{-edges} \cup \tilde{D}_{i,j}\text{-edges})\Big)$, where

A_i-edges $= \{(\mathbf{x_{i,j}}, \mathbf{x_{i,k}}) \mid A_i^\lambda(k, j) \neq 0\}$,
B_i-edges $= \{(\mathbf{u_{i,h}}, \mathbf{x_{i,l}}) \mid B_i^\lambda(l, h) \neq 0\}$,
for $j \neq i$, $A_{i,j}$-*edges* $= \{(\mathbf{x_{j,l}}, \mathbf{x_{i,h}}) \mid A_{i,j}^\lambda(h, l) \neq 0\}$,
$B_{i,j}$-edges $= \{(\mathbf{u_{j,l}}, \mathbf{x_{i,h}}) \mid B_{i,j}^\lambda(h, l) \neq 0\}$,
C_i-edges $= \{(\mathbf{x_{i,h}}, \mathbf{y_{i,l}}) \mid C_i^\lambda(l, h) \neq 0\}$,
D_i-edges $= \{(\mathbf{u_{i,h}}, \mathbf{y_{i,l}}) \mid D_i^\lambda(l, h) \neq 0\}$,
for $j \neq i$, $C_{i,j}$-*edges* $= \{(\mathbf{x_{j,l}}, \mathbf{y_{i,h}}) \mid C_{i,j}^\lambda(h, l) \neq 0\}$,
$D_{i,j}$-edges $= \{(\mathbf{u_{j,l}}, \mathbf{y_{i,h}}) \mid D_{i,j}^\lambda(h, l) \neq 0\}$,
$\tilde{C}_i$-edges $= \left\{(\mathbf{x_{i,h}}, \tilde{\mathbf{y}}_\mathbf{i,l}) \mid \tilde{C}_i^\lambda(l, h) \neq 0\right\}$,
$\tilde{D}_i$-edges $= \left\{(\mathbf{u_{i,h}}, \tilde{\mathbf{y}}_\mathbf{i,l}) \mid \tilde{D}_i^\lambda(l, h) \neq 0\right\}$,
for $j \neq i$, $\tilde{C}_{i,j}$-edges $= \left\{(\mathbf{x_{j,l}}, \tilde{\mathbf{y}}_\mathbf{i,h}) \mid \tilde{C}_{i,j}^\lambda(h, l) \neq 0\right\}$
and $\tilde{D}_{i,j}$-edges $= \left\{(\mathbf{u_{j,l}}, \tilde{\mathbf{y}}_\mathbf{i,h}) \mid \tilde{D}_{i,j}^\lambda(h, l) \neq 0\right\}$.

Here, $M(i, j)$ is the $(i, j)th$ element of matrix M and $(\mathbf{v_1}, \mathbf{v_2})$ denotes a directed edge from vertex $\mathbf{v_1} \in \mathcal{V}$ to vertex $\mathbf{v_2} \in \mathcal{V}$. The following example illustrates the previous settings.

Example 3. Consider the following structured distributed system constituted of three subsystems:

<u>Subsystem 1:</u>

$A_1^\lambda = \begin{pmatrix} 0 & \lambda_1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $A_{1,2}^\lambda = \begin{pmatrix} 0 & 0 & 0 \\ \lambda_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $A_{1,3}^\lambda = 0$, $B_1^\lambda = \begin{pmatrix} 0 \\ \lambda_3 \\ 0 \end{pmatrix}$, $B_{1,2}^\lambda = B_{1,3}^\lambda = 0$, $C_1^\lambda = \begin{pmatrix} \lambda_4 & 0 & \lambda_5 \end{pmatrix}$ and $C_{1,2}^\lambda = C_{1,3}^\lambda = D_1^\lambda = D_{1,2}^\lambda = D_{1,3}^\lambda = 0$.

<u>Subsystem 2:</u>

$A_2^\lambda = \begin{pmatrix} 0 & \lambda_6 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $A_{2,1}^\lambda = A_{2,3} = 0$, $B_2^\lambda = \begin{pmatrix} 0 \\ \lambda_7 \\ \lambda_8 \end{pmatrix}$, $B_{2,1}^\lambda = B_{2,3}^\lambda = 0$, $C_2^\lambda = \begin{pmatrix} \lambda_9 & 0 & 0 \\ 0 & 0 & \lambda_{10} \end{pmatrix}$, $C_{2,1}^\lambda = 0$, $C_{2,3}^\lambda = \begin{pmatrix} 0 & 0 \\ 0 & \lambda_{11} \end{pmatrix}$, $D_2^\lambda = \begin{pmatrix} 0 \\ \lambda_{12} \end{pmatrix}$ and $D_{2,1}^\lambda = D_{2,3}^\lambda = 0$.

<u>Subsystem 3:</u>

$A_3^\lambda = \begin{pmatrix} 0 & \lambda_{13} \\ 0 & 0 \end{pmatrix}$, $A_{3,1}^\lambda = A_{3,2}^\lambda = 0$, $B_3^\lambda = \begin{pmatrix} 0 \\ \lambda_{14} \end{pmatrix}$, $B_{3,1}^\lambda = B_{3,2}^\lambda = 0$, $C_3^\lambda = \begin{pmatrix} \lambda_{15} & 0 \\ \lambda_{16} & 0 \end{pmatrix}$, $C_{3,1}^\lambda = \begin{pmatrix} 0 & 0 & \lambda_{17} \\ 0 & 0 & 0 \end{pmatrix}$ and $C_{3,2}^\lambda = D_3^\lambda = D_{3,1}^\lambda = D_{3,2}^\lambda = 0$.

To such a model, we associate the digraph in figure 1. For a sake of simplicity in this example, the set of output vertices associated to the measurements $\tilde{Y}_i$ arriving through the network to each subsystem are not represented. In many cases, this set is a subset of the whole subsystems measurements set *i.e.* $\tilde{\mathbf{Y}}_\mathbf{i} \subseteq \bigcup_{j=1,\, j\neq i}^{N} \mathbf{Y_j}$.

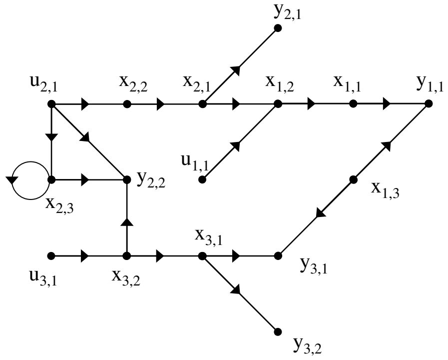

Figure 1. *Digraph associated to Example 3*

Let us now give some useful definitions and notations.

- Two edges $e_1 = (\mathbf{v_1}, \mathbf{v_1'})$ and $e_2 = (\mathbf{v_2}, \mathbf{v_2'})$ are v-disjoint if $\mathbf{v_1} \neq \mathbf{v_2}$ and $\mathbf{v_1'} \neq \mathbf{v_2'}$. Note that e_1 and e_2 can be v-disjoint even if $\mathbf{v_1'} = \mathbf{v_2}$ or $\mathbf{v_1} = \mathbf{v_2'}$. Some edges are v-disjoint if they are mutually v-disjoint.
- Path P containing vertices $\mathbf{v_{r_0}}, \ldots, \mathbf{v_{r_i}}$ is denoted $P = \mathbf{v_{r_0}} \to \mathbf{v_{r_1}} \to \ldots \to \mathbf{v_{r_i}}$, where $(\mathbf{v_{r_j}}, \mathbf{v_{r_{j+1}}}) \in \mathcal{E}$ for $j = 0, 1 \ldots, i-1$. We say that P covers $\mathbf{v_{r_0}}, \mathbf{v_{r_1}}, \ldots, \mathbf{v_{r_i}}$.
- Path P is an $\mathbf{Y}$-*topped* path if its end vertex is an element of $\mathbf{Y}$. An $\mathbf{Y}$-*topped* path family consists of disjoint simple $\mathbf{Y}$-*topped* paths.
- A *cycle* is a path of the form $\mathbf{v_{r_0}} \to \mathbf{v_{r_1}} \to \ldots \to \mathbf{v_{r_i}} \to \mathbf{v_{r_0}}$, where all vertices $\mathbf{v_{r_0}}, \mathbf{v_{r_1}}, \ldots, \mathbf{v_{r_i}}$ are distinct. Some paths are disjoint if they have no common vertex. A path is simple when every vertex occurs only once in this path. A set of disjoint cycles is called a cycle family.
- The union of an $\mathbf{Y}$-*topped* path family, and a cycle family is disjoint if they have no vertices in common. If such union contains a path or a cycle which covers a vertex $\mathbf{v}$, it is said to cover $\mathbf{v}$.

Let $\mathcal{V}_1$ and $\mathcal{V}_2$ denote two subsets of $\mathcal{V}$.

- The cardinality of $\mathcal{V}_1$ is noted $\mathrm{card}\,(\mathcal{V}_1)$.
- A path P is said a $\mathcal{V}_1$-$\mathcal{V}_2$ path if its begin vertex belongs to $\mathcal{V}_1$ and its end vertex belongs to $\mathcal{V}_2$. If the only vertices of P belonging to $\mathcal{V}_1 \cup \mathcal{V}_2$ are its begin and its end vertices, P is said a direct $\mathcal{V}_1$-$\mathcal{V}_2$ path.
- A set of ℓ disjoint $\mathcal{V}_1$-$\mathcal{V}_2$ paths is called a $\mathcal{V}_1$-$\mathcal{V}_2$ linking of size ℓ. The linkings, which consist of a maximal number of disjoint $\mathcal{V}_1$-$\mathcal{V}_2$ paths, are called maximum $\mathcal{V}_1$-$\mathcal{V}_2$ linkings. We define by $\rho\,[\mathcal{V}_1, \mathcal{V}_2]$ the size of these maximum $\mathcal{V}_1$-$\mathcal{V}_2$ linkings.
- $\mu\,[\mathcal{V}_1, \mathcal{V}_2]$ is the minimal number of vertices covered by a maximum $\mathcal{V}_1$-$\mathcal{V}_2$ linking.
- $V_{ess}(\mathcal{V}_1, \mathcal{V}_2) \overset{def}{=} \{\mathbf{v} \in \mathcal{V} \mid \mathbf{v} \text{ is covered by every maximum } \mathcal{V}_1\text{-}\mathcal{V}_2 \text{ linking}\}$.

Roughly speaking, vertex subset $V_{ess}(\mathcal{V}_1, \mathcal{V}_2)$ denotes the set of all essential vertices (van der Woude [2000]), which correspond by definition to vertices present in all the maximum $\mathcal{V}_1$-$\mathcal{V}_2$ linkings.

- $\mathbf{S} \subseteq \mathcal{V}$ is a separator between sets $\mathcal{V}_1$ and $\mathcal{V}_2$ if every path from $\mathcal{V}_1$ to $\mathcal{V}_2$ contains at least one vertex in $\mathbf{S}$. We call minimum separators between $\mathcal{V}_1$ and $\mathcal{V}_2$ any separators having the smallest size. According to Menger's Theorem, the latter is

equal to $\rho\left[\mathcal{V}_1, \mathcal{V}_2\right]$.
• There exist a uniquely determined minimum separator between $\mathcal{V}_1$ and $\mathcal{V}_2$ noted $\mathbf{S^o}(\mathcal{V}_1, \mathcal{V}_2)$ such that:
$\mathbf{S^o}(\mathcal{V}_1, \mathcal{V}_2)$ is the set of begin vertices of all direct $V_{ess}(\mathcal{V}_1, \mathcal{V}_2)$-$\mathcal{V}_2$ paths, where $V_{ess}(\mathcal{V}_1, \mathcal{V}_2) \cap \mathcal{V}_2$ is considered, in the present definition, as input vertices. Vertex subset $\mathbf{S^o}(\mathcal{V}_1, \mathcal{V}_2)$ is called the minimum output separator.
It results, from the previous definitions, that $V_{ess}(\mathcal{V}_1, \mathcal{V}_2) \cap \mathcal{V}_2 \subseteq \mathbf{S^o}(\mathcal{V}_1, \mathcal{V}_2)$.
• $\theta(\mathcal{V}_1, \mathcal{V}_2)$ is the maximal number of v-disjoint edges which begin in $\mathcal{V}_1$ and end in $\mathcal{V}_2$.

Definition 4. For each vertex subsets $\mathbf{V}$ such that $\mathbf{V} \subseteq \mathcal{V}$, we define the following vertex subsets:
• $\bar{\mathbf{X}}(\mathbf{V}) = \mathbf{X} \setminus (\mathbf{V} \cap \mathbf{X})$;
• $\bar{\mathbf{U}}(\mathbf{V}) \subseteq \mathbf{U} \setminus (\mathbf{V} \cap \mathbf{U})$ such that card $(\bar{\mathbf{U}}(\mathbf{V})) = \rho\big[\mathbf{U}, \mathbf{V}\big] = \rho\big[\bar{\mathbf{U}}(\mathbf{V}), \mathbf{V}\big]$ and $\mu\big[\bar{\mathbf{U}}(\mathbf{V}), \mathbf{V}\big] = \mu\big[\mathbf{U}, \mathbf{V}\big]$. Note that $\bar{\mathbf{U}}(\mathbf{V})$ always exists but is not necessarily unique.
• $\mathbf{X_1}(\mathbf{V}) \overset{def}{=} \Big\{\mathbf{x_i} \in \bar{\mathbf{X}}(\mathbf{V}) \,|\, \rho\big[\mathbf{U} \cup \{\mathbf{x_i}\}, \mathbf{V}\big] > \rho\big[\mathbf{U}, \mathbf{V}\big]\Big\}$;
• $\boldsymbol{\Upsilon_0}(\mathbf{V}) \overset{def}{=} \big\{\mathbf{v_i} \in \mathbf{V} \,|\, \rho\big[\mathbf{U}, \mathbf{V}\big] > \rho\big[\mathbf{U}, \mathbf{V} \setminus \{\mathbf{v_i}\}\big]\big\} = \mathbf{V} \cap V_{ess}(\mathbf{U}, \mathbf{V})$;
• $\boldsymbol{\Upsilon_1}(\mathbf{V}) \overset{def}{=} \mathbf{V} \setminus \boldsymbol{\Upsilon_0}(\mathbf{V})$;
• $\mathbf{U_0}(\mathbf{V}) \overset{def}{=} \big\{\mathbf{u_i} \in \bar{\mathbf{U}}(\mathbf{V}) \,|\, \theta\big(\{\mathbf{u_i}\}, \mathbf{X_1}(\mathbf{V}) \cup \boldsymbol{\Upsilon_1}(\mathbf{V})\big) = 0\big\}$;
• $\mathbf{U_1}(\mathbf{V}) \overset{def}{=} \bar{\mathbf{U}}(\mathbf{V}) \setminus \mathbf{U_0}(\mathbf{V})$;
• $\mathbf{S^o}(\mathbf{V}) \overset{def}{=} \mathbf{S^o}(\mathbf{U_0}(\mathbf{V}), \mathbf{V})$;
• $\mathbf{X_s}(\mathbf{V}) \overset{def}{=} \mathbf{S^o}(\mathbf{V}) \cap \bar{\mathbf{X}}(\mathbf{V})$.
• $\mathbf{X_0}(\mathbf{V}) \overset{def}{=} \bar{\mathbf{X}}(\mathbf{V}) \setminus (\mathbf{X_1}(\mathbf{V}) \cup \mathbf{X_s}(\mathbf{V}))$.

• $\beta_1(\mathbf{V})$ is the maximal number of vertices included in $\mathbf{X_1}(\mathbf{V}) \cup \mathbf{X_s}(\mathbf{V}) \cup \mathbf{U_1}(\mathbf{V})$ covered by a disjoint union of

- a $\mathbf{X_s}(\mathbf{V}) \cup \mathbf{U_1}(\mathbf{V})$-$\boldsymbol{\Upsilon_1}(\mathbf{V})$ linking of size $\rho\big[\mathbf{X_s}(\mathbf{V}) \cup \mathbf{U_1}(\mathbf{V}), \boldsymbol{\Upsilon_1}(\mathbf{V})\big]$,
- a $\boldsymbol{\Upsilon_1}(\mathbf{V})$-*topped* path family and
- a cycle family covering only elements of $\mathbf{X_1}(\mathbf{V})$.

• $\beta_0(\mathbf{V}) \overset{def}{=} \mu\Big[\mathbf{U_0}(\mathbf{V}), \mathbf{S^o}(\mathbf{V})\Big] - \rho\Big[\mathbf{U_0}(\mathbf{V}), \mathbf{S^o}(\mathbf{V})\Big]$;
• $\beta(\mathbf{V}) \overset{def}{=} \beta_1(\mathbf{V}) + \beta_0(\mathbf{V}) + \text{card}\,\big(\mathbf{V} \setminus \mathbf{Y}\big)$.

4. MAIN RESULTS

In this section, we enounce the main result of the paper which consists on the exact characterization, in the two considered observation schemes, of the set of all the strongly observable input and state components for each subsystem of NCS (Σ_Λ).
For the sake of simplicity, at first, we do not consider that (Σ_Λ) is a networked Control System. So, let us develop some ideas on a general structured linear system (Σ_Λ) on the form

$$(\Sigma_\Lambda) : \begin{cases} \dot{x}(t) = A^\lambda x(t) + B^\lambda u(t) \\ y(t) = C^\lambda x(t) + D^\lambda u(t) \end{cases} \tag{3}$$

First, let us notice that if some vertices of $\mathbf{X} \cup \mathbf{U}$ are not the begin vertices of an $\mathbf{Y}$-*topped* path, then these state or input components are obviously not observable. So, without loss of generality, we can remove all these vertices from the digraph and the corresponding input or state components from the system, in order to study the strong observability of the other state and input components. Consequently, we will consider in the sequel that all the state and input vertices are the begin vertices of $\mathbf{Y}$-*topped* paths.
Moreover, using the results of (Commault et al. [1997]), we have that input components included in $\mathbf{U} \setminus \bar{\mathbf{U}}(\mathbf{Y})$ can be rendered unobservable using the $\bar{\mathbf{U}}(\mathbf{Y})$ components *i.e.* there exist inputs $\bar{\mathbf{U}}(\mathbf{Y})$ such that output $y(t)$ is not sensitive to the input components associated to $\mathbf{U} \setminus \bar{\mathbf{U}}(\mathbf{Y})$. Hence, the input components associated to vertices $\mathbf{U} \setminus \bar{\mathbf{U}}(\mathbf{Y})$ are not strongly observable and so, for a sake of simplicity, we restrict our study only to the input components associated to $\bar{\mathbf{U}}(\mathbf{Y})$. We denote by $\bar{B}^\lambda$ (resp. $\bar{D}^\lambda$) the submatrix of B^λ (resp. D^λ) associated to $\bar{\mathbf{U}}(\mathbf{Y})$ *i.e.* matrix $\bar{B}^\lambda$ (resp. $\bar{D}^\lambda$) is constituted by the concatenation of columns B^λ_j (resp. D^λ_j) of B^λ (resp. D^λ) where $\mathbf{u_j} \in \bar{\mathbf{U}}(\mathbf{Y})$. Let us denote $\bar{q} = \text{card}\,(\bar{\mathbf{U}}(\mathbf{Y}))$ and the pencil matrix of system $(A^\lambda, \bar{B}^\lambda, C^\lambda, \bar{D}^\lambda)$ by $P^\lambda(s) = \begin{pmatrix} A^\lambda - sI_n & \bar{B}^\lambda \\ C^\lambda & \bar{D}^\lambda \end{pmatrix}$.
For each realization of system (Σ_Λ), we can compute the n-$rank$ of $P^\lambda(s)$. This rank will have the same value for almost all parameter values $\lambda \in \mathbb{R}^h$ (Reinschke [1988], van der Woude [2000]). This so-called generic n-$rank$ of $P^\lambda(s)$ will be denoted by $g\_n\text{-}rank(P^\lambda(s))$. Generic rank of matrix $P^\lambda(s)$, denoted $g\_rank(P^\lambda(s))$, is quite different as it depends on s. Hence, $g\_rank(P^\lambda(s)) = r$, $\forall\, s \in \mathbb{C}$ means that for almost all parameter values $\lambda \in \mathbb{R}^h$, $rank(P^\lambda(s)) = r$, $\forall\, s \in \mathbb{C}$.
On the one hand, applying results of (van der Woude [2000]), we have that system $(A^\lambda, \bar{B}^\lambda, C^\lambda, \bar{D}^\lambda)$ is generically input and state observable iff $g\_rank(P^\lambda(s))$ is equal to n plus the maximal size of a $\bar{\mathbf{U}}(\mathbf{Y})$-$\mathbf{Y}$ linking *i.e.* $g\_rank(P^\lambda(s)) = n + \bar{q}$ or in other words iff $P^\lambda(s)$ generically has full column rank.
On the other hand, as all the state and input components are the begin vertices of $\mathbf{Y}$-*topped* paths and as $\rho\left[\bar{\mathbf{U}}(\mathbf{Y}), \mathbf{Y}\right] = \text{card}\,(\bar{\mathbf{U}}(\mathbf{Y})) = \bar{q}$, from (van der Woude [2000]), we have also that the generic normal rank of $P^\lambda(s)$ is equal to $n + \bar{q}$. According to the generically full column n-$rank$ of $P^\lambda(s)$, this implies that $g\_rank(P^\lambda(s_0)) < n+\bar{q}$ is equivalent Trentelman et al. [2001] to the existence of a nonzero vector $\left(x_0^T, u_0^T\right)^T$ such that the output y resulting from the initial conditions $u(t) = u_0 e^{s_0 t}$ and $x(0) = x_0$ is zero and so that there exists a direction in the extended state and input space which is not strongly observable. Consequently, the generic dimension of the strongly observable subspace in the extended state and input subspace $(x^T, u^T)^T$ is equal to $n + \text{card}\,(\bar{\mathbf{U}})$ minus the generic number of invariant zeros of $P^\lambda(s)$ which are the complex roots of $g\_rank(P^\lambda(s)) < n + \bar{q}$ Trentelman et al. [2001] and are denoted $g\_n_{inv,z}$. The first lemma hereafter, allows us to characterize graphically number $g\_n_{inv,z}$.

Lemma 1. Consider structured system (Σ_Λ) represented by digraph $\mathcal{G}(\Sigma_\Lambda)$. We have that $n + \bar{q} - g\_n_{inv,z} = \beta(\mathbf{Y})$ where $g\_n_{inv,z}$ is the number of invariant zeros of $P^\lambda(s)$.

Proof: Due to the properties of subdivision presented in Definition 4 (Boukhobza et al. [2007]), we have that there is no edge from $\mathbf{X_0}(\mathbf{Y}) \cup \mathbf{U_0}(\mathbf{Y})$ to $\mathbf{X_1}(\mathbf{Y}) \cup \boldsymbol{\Upsilon_1}(\mathbf{Y})$ and $\mathbf{S^o}(\mathbf{U_0}(\mathbf{Y}), \mathbf{Y}) = \mathbf{X_s}(\mathbf{Y}) \cup \boldsymbol{\Upsilon_0}(\mathbf{Y})$. Thus, we can write (Σ_Λ) as:

$$\begin{cases} \dot{X}_0(t) = A^\lambda_{0,0} X_0(t) + A^\lambda_{0,s} X_s(t) + A^\lambda_{0,1} X_1(t) + \\ \qquad B^\lambda_{0,0} U_0(t) + B^\lambda_{0,1} U_1(t) \\ \dot{X}_s(t) = A^\lambda_{s,0} X_0(t) + A^\lambda_{s,s} X_s(t) + A^\lambda_{s,1} X_1(t) + \\ \qquad B^\lambda_{s,0} U_0(t) + B^\lambda_{s,1} U_1(t) \\ \dot{X}_1(t) = A^\lambda_{1,s} X_s(t) + A^\lambda_{1,1} X_1(t) + B^\lambda_{1,1} U_1(t) \\ \Upsilon_0(t) = C^\lambda_{0,0} X_0(t) + C^\lambda_{0,s} X_s(t) + C^\lambda_{0,1} X_1(t) + \\ \qquad D^\lambda_{0,0} U_0(t) + D^\lambda_{0,1} U_1(t) \\ \Upsilon_1(t) = C^\lambda_{1,s} X_s(t) + C^\lambda_{1,1} X_1(t) + D^\lambda_{1,1} U_1(t) \end{cases} \tag{4}$$

where X_0, X_s, U_0, U_1, Υ_0 and Υ_1 represent the variables associated to vertex subsets $\mathbf{X_0}(\mathbf{Y})$, $\mathbf{X_s}(\mathbf{Y})$, $\mathbf{U_0}(\mathbf{Y})$, $\mathbf{U_1}(\mathbf{Y})$, $\boldsymbol{\Upsilon_0}(\mathbf{Y})$ and $\boldsymbol{\Upsilon_1}(\mathbf{Y})$ respectively.

Therefore, with some appropriate permutations on the rows and columns of $P^\lambda(s)$, we can transform $P^\lambda(s)$ into

$$\tilde{P}^\lambda(s) = \begin{pmatrix} A^\lambda_{0,0} - sI_{n_0} & A^\lambda_{0,s} & B^\lambda_{0,0} & A^\lambda_{0,1} & B^\lambda_{0,1} \\ A^\lambda_{s,0} & A^\lambda_{s,s} - sI_{n_s} & B^\lambda_{s,0} & A^\lambda_{s,1} & B^\lambda_{s,1} \\ C^\lambda_{0,0} & C^\lambda_{0,s} & D^\lambda_{0,0} & C^\lambda_{0,1} & D^\lambda_{0,1} \\ 0 & A^\lambda_{1,s} & 0 & A^\lambda_{1,1} - sI_{n_1} & B^\lambda_{1,1} \\ 0 & C^\lambda_{1,s} & 0 & C^\lambda_{1,1} & D^\lambda_{1,1} \end{pmatrix}$$

For a sake of simplicity, let us define $n_0 = \text{card}\,(\mathbf{X_0}(\mathbf{Y}))$, $n_s = \text{card}\,(\mathbf{X_s}(\mathbf{Y}))$, $n_1 = \text{card}\,(\mathbf{X_1}(\mathbf{Y}))$, $q_0 = \text{card}\,(\mathbf{U_0}(\mathbf{Y}))$, $q_1 = \text{card}\,(\mathbf{U_1}(\mathbf{Y}))$, $p_0 = \text{card}\,(\mathbf{\Upsilon_0}(\mathbf{Y}))$ and $p_1 = \text{card}\,(\mathbf{\Upsilon_1}(\mathbf{Y}))$.

Since the edges associated to $A^\lambda_{1,s}$ link $\mathbf{X_s}(\mathbf{Y})$ to $\mathbf{X_1}(\mathbf{Y})$ and the edges associated to $C^\lambda_{1,s}$ link $\mathbf{X_s}(\mathbf{Y})$ to $\mathbf{\Upsilon_1}(\mathbf{Y})$, we have that $g\_rank\begin{pmatrix} A^\lambda_{1,s} \\ C^\lambda_{1,s} \end{pmatrix} = \theta(\mathbf{X_s}(\mathbf{Y}), \mathbf{X_1}(\mathbf{Y}) \cup \mathbf{\Upsilon_1}(\mathbf{Y}))$. According to Statement **St3** of Lemma 6 in (Boukhobza et al. [2007]), this implies that $g\_rank\begin{pmatrix} A^\lambda_{1,s} \\ C^\lambda_{1,s} \end{pmatrix} = n_s$ and so the number of invariant zeros of $P^\lambda(s)$ is equal to the number of invariant zeros of $P^\lambda_e(s)$, where

$$P^\lambda_e(s) = \begin{pmatrix} A^\lambda_{0,0} - sI_{n_0} & A^\lambda_{0,s} & B^\lambda_{0,0} & A^\lambda_{0,1} & B^\lambda_{0,1} & 0 \\ A^\lambda_{s,0} & A^\lambda_{s,s} - sI_{n_s} & B^\lambda_{s,0} & A^\lambda_{s,1} & B^\lambda_{s,1} & 0 \\ C^\lambda_{0,0} & C^\lambda_{0,s} & D^\lambda_{0,0} & C^\lambda_{0,1} & D^\lambda_{0,1} & 0 \\ 0 & I_{n_s} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A^\lambda_{1,1} - sI_{n_1} & B^\lambda_{1,1} & A^\lambda_{1,s} \\ 0 & 0 & 0 & C^\lambda_{1,1} & D^\lambda_{1,1} & C^\lambda_{1,s} \end{pmatrix}$$

Let us denote $P^\lambda_0(s) \overset{def}{=} \begin{pmatrix} A^\lambda_{0,0} - sI_{n_0} & A^\lambda_{0,s} & B^\lambda_{0,0} \\ A^\lambda_{s,0} & A^\lambda_{s,s} - sI_{n_s} & B^\lambda_{s,0} \\ C^\lambda_{0,0} & C^\lambda_{0,s} & D^\lambda_{0,0} \\ 0 & I_{n_s} & 0 \end{pmatrix}$ and $P^\lambda_1(s) \overset{def}{=} \begin{pmatrix} A^\lambda_{1,1} - sI_{n_1} & B^\lambda_{1,1} & A^\lambda_{1,s} \\ C^\lambda_{1,1} & D^\lambda_{1,1} & C^\lambda_{1,s} \end{pmatrix}$.

$P^\lambda_0(s)$ can be seen as the pencil matrix of a square system denoted (Σ_0), defined by input $\mathbf{U_0}(\mathbf{Y})$, state $\mathbf{X_0}(\mathbf{Y}) \cup \mathbf{X_s}(\mathbf{Y})$ and output $\mathbf{Y_s} \cup \mathbf{\Upsilon_0}(\mathbf{Y})$, where $\mathbf{Y_s}$ is a virtual output connected to $\mathbf{X_s}(\mathbf{Y})$ such that $Y_s = X_s$. Matrix $P^\lambda_1(s)$ can be seen as the pencil matrix of a system denoted (Σ_1), defined by input $\mathbf{U_1}(\mathbf{Y}) \cup \mathbf{X_s}(\mathbf{Y})$, state $\mathbf{X_1}(\mathbf{Y})$ and output $\mathbf{\Upsilon_1}(\mathbf{Y})$ and which has generically full column *n-rank* even after the deletion of an arbitrary row (Boukhobza et al. [2007]).

We have that $g\_n\text{-}rank(P^\lambda_0(s))$ is equal to the number of rows of $P^\lambda_0(s)$ and $g\_n\text{-}rank(P^\lambda_1(s))$ is equal to the number of columns of $P^\lambda_1(s)$. Thus, counting the zeros with their multiplicities, it is easy to see that the number of invariant zeros of $P^\lambda_e(s)$ is equal to the sum of the number of invariant zeros of $P^\lambda_0(s)$ and the number of invariant zeros of $P^\lambda_1(s)$. On the one hand, applying Theorem 5.1 of (van der Woude [2000]), we have that the number of invariant zeros of $P^\lambda_0(s)$ is equal to $n_0 + n_s + q_0 - \mu\big[\mathbf{U_0}(\mathbf{Y}), \mathbf{S^o}(\mathbf{U_0}(\mathbf{Y}), \mathbf{Y})\big] + \rho\big[\mathbf{U_0}(\mathbf{Y}), \mathbf{S^o}(\mathbf{U_0}(\mathbf{Y}), \mathbf{Y})\big] - n_s$. Note that the presence of the latter term n_s is due to the fact that the output of system (Σ_0) is Y_s and not X_s. Moreover from Theorem 5.2 of (van der Woude [2000]), the number of invariant zeros of $P^\lambda_1(s)$ is equal to $n_1 + n_s + q_1$ minus the maximal number of vertices of $\mathbf{X_1}(\mathbf{Y}) \cup \mathbf{X_s}(\mathbf{Y}) \cup \mathbf{U_1}(\mathbf{Y})$ covered by a disjoint union of:

- a $\mathbf{X_s}(\mathbf{Y}) \cup \mathbf{U_1}(\mathbf{Y})$-$\mathbf{\Upsilon_1}(\mathbf{Y})$ linking of size $\rho\big[\mathbf{X_s}(\mathbf{Y}) \cup \mathbf{U_1}(\mathbf{Y}), \mathbf{\Upsilon_1}(\mathbf{Y})\big]$,
- a $\mathbf{\Upsilon_1}(\mathbf{Y})$-*topped* path family and
- a cycle family covering only elements of $\mathbf{X_1}(\mathbf{Y})$.

Therefore, using notations of Definition 4, the number of invariant zeros of $P^\lambda_e(s)$ and also of $P^\lambda(s)$ is equal to $n_0 + q_0 + n_1 + n_s + q_1 - \beta_0(\mathbf{Y}) - \beta_1(\mathbf{Y}) = n + \bar{q} - \beta_0(\mathbf{Y}) - \beta_1(\mathbf{Y})$. Thus, the generic dimension of the strongly observable subspace of (Σ_Λ) in the extended state and input subspace is equal to $n + \bar{q} - g\_n_{inv,z} = \beta_1(\mathbf{Y}) + \mu\big[\mathbf{U_0}(\mathbf{Y}), \mathbf{S^o}(\mathbf{U_0}(\mathbf{Y}), \mathbf{Y})\big] - \rho\big[\mathbf{U_0}(\mathbf{Y}), \mathbf{S^o}(\mathbf{U_0}(\mathbf{Y}), \mathbf{Y})\big] = \beta_1(\mathbf{Y}) + \beta_0(\mathbf{Y}) = \beta(\mathbf{Y})$. $\triangle$

The previous lemma allows us to write that the generic dimension of the strongly observable subspace in the extended state and input subspace $(x^T, u^T)^T$ is equal to $\beta(\mathbf{Y})$.

If $\beta(\mathbf{Y}) < n + q$ then (Σ_Λ) is not generically input and state observable and it may be interesting to know which state component x_i (resp. input component u_j) is generically strongly observable. At this aim, we compare $\beta(\mathbf{Y} \cup \{\mathbf{x_i}\})$ or $\beta(\mathbf{Y} \cup \{\mathbf{u_j}\})$ to $\beta(\mathbf{Y})$. Indeed, this amounts to compare the generic dimension of the strongly observable subspace in the extended state and input subspace $(x^T, u^T)^T$ of (Σ_Λ) to the generic dimension of the strongly observable subspace in the extended state and input subspace $(x^T, u^T)^T$ of the same system (Σ_Λ) with an additional sensor which measures the component x_i (resp. u_j). In fact, adding to the system a sensor, which measures the state component x_i (resp. input component u_j) is equivalent to add in the digraph an output vertex $\mathbf{y_{p+1}}$ and an edge $(\mathbf{x_i}, \mathbf{y_{p+1}})$ (resp. $(\mathbf{u_j}, \mathbf{y_{p+1}})$). For the new system obtained by the addition of $\mathbf{y_{p+1}}$, the computation of the generic dimension of the strongly observable subspace in the extended state and input subspace $(x^T, u^T)^T$ can be made by using function $\beta(\mathbf{Y} \cup \{\mathbf{y_{p+1}}\})$. Nevertheless, this requires an effective redraw of the digraph to add effectively an output vertex $\mathbf{y_{p+1}}$ and an edge $(\mathbf{x_i}, \mathbf{y_{p+1}})$ (resp. $(\mathbf{u_j}, \mathbf{y_{p+1}})$). For a sake of simplicity, we have chosen to work on an unique digraph. Thus, we do not add any vertex or edge in the digraph, but we consider vertex $\mathbf{x_i}$ (resp. $\mathbf{u_j}$) as an output. Thus, $\beta(\mathbf{V}) = \beta_1(\mathbf{V}) + \mu\Big[\mathbf{U_0}(\mathbf{V}), \mathbf{S^o}(\mathbf{V})\Big] - \rho\Big[\mathbf{U_0}(\mathbf{V}), \mathbf{S^o}(\mathbf{V})\Big] + \text{card}\,\big(\mathbf{V} \setminus \mathbf{Y}\big)$, for $\mathbf{V} = \mathbf{Y} \cup \{\mathbf{x_i}\}$ (resp. $\mathbf{V} = \mathbf{Y} \cup \{\mathbf{u_j}\}$), represents the generic dimension of the strongly observable subspace in the extended state and input subspace $(x^T, u^T)^T$ for the new system obtained by the addition of $\mathbf{y_{p+1}}$ and an edge $(\mathbf{x_i}, \mathbf{y_{p+1}})$ (resp. $(\mathbf{u_j}, \mathbf{y_{p+1}})$). We take the strong observability of $\mathbf{x_i}$ (resp. $\mathbf{u_j}$) into account by adding term $\text{card}\,\big(\mathbf{V} \setminus \mathbf{Y}\big)$ in the computation of $\beta(\mathbf{V})$.

We can state now the following lemma concerning the strong observability of state or input component:

Lemma 2. Consider structured system (Σ_Λ) represented by digraph $\mathcal{G}(\Sigma_\Lambda)$. Let $\Omega \overset{def}{=} \{\mathbf{v} \in \mathbf{X} \cup \mathbf{U},\ \beta(\mathbf{Y} \cup \{\mathbf{v}\}) = \beta(\mathbf{Y})\}$. A state component x_i (respectively an input component u_j) is strongly observable iff $\mathbf{x_i} \in \Omega$ (resp. $\mathbf{u_j} \in \Omega$)

Proof: Obviously, a state component x_i (resp. input component u_j) is strongly observable iff an additional measure of this state component does not change the generic dimension of the strongly observable subspace. Using notations of Definition 4, this is equivalent to say that state component x_i (resp. input component u_j) is strongly observable iff $\beta(\mathbf{Y}) = \beta(\mathbf{Y} \cup \{\mathbf{x_i}\})$ (resp. $\beta(\mathbf{Y}) = \beta(\mathbf{Y} \cup \{\mathbf{u_j}\})$). $\triangle$

Applying now this result to a NCS we have:

Proposition 1. Consider structured system (Σ_Λ) represented by digraph $\mathcal{G}(\Sigma_\Lambda)$ and constituted by subsystems (Σ^R_i), $i = 1, \ldots, N$. For subsystem i, state component $x_{i,k}$ (resp. input component $u_{i,j}$) is strongly observable in

- a distributed decentralized observation scheme iff $\beta\big(\mathbf{Y_i} \cup \tilde{\mathbf{Y}}_\mathbf{i} \cup \{\mathbf{x_{i,k}}\}\big) = \beta\big(\mathbf{Y_i} \cup \tilde{\mathbf{Y}}_\mathbf{i}\big)$ (resp. $\beta\big(\mathbf{Y_i} \cup \tilde{\mathbf{Y}}_\mathbf{i} \cup \{\mathbf{u_{i,j}}\}\big) = \beta\big(\mathbf{Y_i} \cup \tilde{\mathbf{Y}}_\mathbf{i}\big)$).
- a distributed autonomous observation scheme iff $\beta\big(\mathbf{Y_i} \cup \{\mathbf{x_{i,k}}\}\big) = \beta\big(\mathbf{Y_i}\big)$ (resp. $\beta\big(\mathbf{Y_i} \cup \{\mathbf{u_{i,j}}\}\big) = \beta\big(\mathbf{Y_i}\big)$).

When we study the state strong observability or the whole state and input observability of subsystem (Σ_i^R), we can apply:

Corollary 1. Consider structured system (Σ_Λ) represented by digraph $\mathcal{G}(\Sigma_\Lambda)$ and constituted by subsystems (Σ_i^R), $i = 1, \ldots, N$. Subsystem i is generically input and state observable in

- a distributed decentralized observation scheme iff

$$\beta(\mathbf{U_i} \cup \mathbf{X_i} \cup \mathbf{Y_i} \cup \tilde{\mathbf{Y}}_\mathbf{i}) = \beta(\mathbf{Y_i} \cup \tilde{\mathbf{Y}}_\mathbf{i})$$

- a distributed autonomous observation scheme iff

$$\beta(\mathbf{U_i} \cup \mathbf{X_i} \cup \mathbf{Y_i}) = \beta(\mathbf{Y_i})$$

Subsystem i is generically strongly observable in

- a distributed decentralized observation scheme iff

$$\beta(\mathbf{X_i} \cup \mathbf{Y_i} \cup \tilde{\mathbf{Y}}_\mathbf{i}) = \beta(\mathbf{Y_i} \cup \tilde{\mathbf{Y}}_\mathbf{i})$$

- a distributed autonomous observation scheme iff

$$\beta(\mathbf{X_i} \cup \mathbf{Y_i}) = \beta(\mathbf{Y_i})$$

Let us illustrate the previous results on the simple system presented in Example 3. Consider first the case of decentralized autonomous observation scheme. For Subsystem 1, $\beta(\mathbf{Y_1}) = 3 = \beta(\mathbf{Y_1} \cup \{\mathbf{x_{1,2}}\})$ while $\beta(\mathbf{Y_1} \cup \{\mathbf{u_{1,1}}\}) = 6$ and $\beta(\mathbf{Y_1} \cup \{\mathbf{x_{1,1}}\}) = \beta(\mathbf{Y_1} \cup \{\mathbf{x_{1,3}}\}) = 4$. For Subsystem 2, $\beta(\mathbf{Y_2}) = 5 = \beta(\mathbf{Y_2} \cup \{\mathbf{x_{2,1}}\}) = \beta(\mathbf{Y_2} \cup \{\mathbf{x_{2,2}}\}) = \beta(\mathbf{Y_2} \cup \{\mathbf{u_{2,1}}\})$ while $\beta(\mathbf{Y_2} \cup \{\mathbf{x_{2,3}}\}) = 6$. Finally, for Subsystem 3, $\beta(\mathbf{Y_3}) = 5 = \beta(\mathbf{Y_2} \cup \{\mathbf{x_{3,1}}\}) = \beta(\mathbf{Y_3} \cup \{\mathbf{x_{3,2}}\}) = \beta(\mathbf{Y_3} \cup \{\mathbf{u_{3,1}}\})$.

We can conclude that only Subsystem 3 is input and state observable in an autonomous observation scheme. For Subsystem 1, only state component $x_{1,2}$ and for Subsystem only state components $x_{2,1}$, $x_{2,2}$ and input component $u_{2,1}$ are strongly observable in a distributed autonomous observation scheme.

Note that to make all the state and input components of Subsystem 1 strongly observable in a distributed decentralized observation scheme, it is necessary and sufficient to have $\{\mathbf{y_{2,1}}, \mathbf{y_{3,1}}\} \subseteq \tilde{\mathbf{Y}}_\mathbf{1}$. Similarly, to make all the state and input components of Subsystem 2 strongly observable in a distributed decentralized observation scheme, it is necessary and sufficient to have $\mathbf{y_{3,2}} \in \tilde{\mathbf{Y}}_\mathbf{2}$.

5. CONCLUSION

An important problem that must be considered when dealing with control over network, is the validity of some properties as the observability. For network distributed systems, an alternative to the centralized observation scheme, which can be quite complicated to realize when we deal with a large scale system, is to consider a decentralized distributed observation scheme or a completely autonomous observation scheme. The first scheme corresponds to the case when the subsystem is connected to the network and receive some informations from the other subsystems. The second scheme is related to the case when the subsystem have only its own measurements to reconstruct a part of the state and input components as in a network cut for example.

In this paper, we propose an analysis tool to study the generic observability of any given part of the state and the unknown input for network distributed structured linear systems in both distributed decentralized and distributed autonomous schemes. Using a graphic-theoretic approach, which is well adapted to study structural properties, necessary and sufficient conditions for the strong observability of a state and/or an input component are provided and expressed in graphic terms. The proposed conditions, which need few information about the system, are very easy to check by means of well-known combinatorial techniques and simply by hand for small systems. That makes our approach particularly suited for large scale systems as it is free from numerical difficulties.

Furthermore, starting from the presented results, we can easily deal with the optimisation of the sensor location or of the measurements distribution on the network to achieve the strong observability of the system in different network configurations. Finally, we can highlight another application of the results provided in this paper in the case of systems submitted to faults. For such systems, it is interesting to see whether or not the state is (or remains) observable when the system is faulty. In this case, the failure is considered as unknown input component. This result is quite simple to deduce from Lemma 2.

As it is briefly discussed, the present work can be a point of departure of many studies concerning the generic observability or other structural properties of Networked Controlled Systems.

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