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Output Variance–Constrained LQG Control of Discrete-Time Systems

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Abstract: The constrained infinite-horizon LQG control problem can be solved via semidefinite programming if the state-control constraints are given by variance-bounds on linear functions of the state and control input. Given a nonzero initial state covariance matrix, each suboptimal dynamic output feedback controller is initially time-varying but reaches time-invariance after a finite number of time steps. This number of time steps can be determined iteratively via repetitive execution of semidefinite programs.

1. INTRODUCTION

Recently, Lee and Khargonekar [2007] considered the constrained infinite-horizon LQR problem, where instead of the usual polyhedral state-control constraints, norm-bounds on linear functions of state and control variables are taken to the constraints. They established a semidefinite programming–based procedure for obtaining an optimal state-feedback solution. In this paper, we extend these ideas to constrained infinite-horizon LQG control; state-control constraints are given by bounds on the "instantaneous" variance of linear functions of state and control variables under zeromean white Gaussian disturbance. Thus, the problem formulation is somewhat similar in spirit to that of constrained receding-horizon, or model predictive, control [Keerthi and Gilbert, 1988, García et al., 1989, Mayne et al., 2000], but is different from that of the usual variance-constrained LQG control procedures [Mäkilä et al., 1984, Zhu et al., 1997], where the steady-state variance of output variables are constrained.

The control problem addressed in this paper concerns a linear time-invariant system with multiple outputs. Each suboptimal control problem is posed as a multi-objective control problem where closed-loop stability is to be maintained while the steady-state and/or instantaneous variance of each of the outputs is driven below a given level. It turns out that, for each given initial state covariance matrix, an output feedback solution to any suboptimal problem, if exists, is initially time-varying but becomes time-invariant after a certain number of time steps. The number of time steps until the controller reaches time-invariance is iteratively determined over an increasing family of systems of linear matrix inequalities until a feasible system is found. This result is certainly consistent with that of constrained LQR [Lee and Khargonekar, 2007]. However, while the constrained LQR problem requires one to bound the "zero-input" response of the system, one needs to bound both the "zero-input" and "zero-state" responses of the system simultaneously in the case of our constrained LQG problem.

Our results are most closely related to the multiple objective $\mathcal{H}_2$ control results [Mäkilä, 1989, Khargonekar and Rotea, 1991a] because the same LQG type of norms are used for all the outputs. However, while the multiple objective $\mathcal{H}_2$ problems are concerned with steady-state performance only, the control objective for our constrained LQG problem requires us to consider instantaneous performance under nonzero initial state distribution as well. On the other hand, our synthesis condition for dynamic output feedback is from the linear matrix inequality–based change-of-variable techniques developed by Scherer et al. [1997] and Masubuchi et al. [1998]. These techniques are applicable to a wide range of mixed objective control problems such as the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem [Bernstein and Haddad, 1989, Khargonekar and Rotea, 1991b, Kaminer et al., 1993] and the aforementioned multi-objective $\mathcal{H}_2$ control problem. Moreover, they "linearize" output-feedback control requirements without any assumptions on system coefficients such as the standard orthogonality and rank conditions [Doyle et al., 1989]. However, since these techniques are originally developed for steady-state performance objectives, we adopt the approach taken in Lee and Khargonekar [2008] and use a modification of the change-of-variable formula of Scherer et al. [1997] to cope with the time-variance of closed-loop system coefficients.

In general, constrained optimal control problems lead to nonlinear feedback laws (see, e.g., Keerthi and Gilbert [1987], Bemporad et al. [2002]). However, to solve the constrained LQG control problem for each given initial state covariance matrix, we focus on linear dynamic output feedback controllers only. The underlying assumption here is that each nonlinear feedback control law has a linear time-varying representation for each given initial state covariance; in particular, this assumption holds true in the case of memoryless nonlinear state feedback laws [Lee and Khargonekar, 2007].

Notation. The set of real numbers and the set of nonnegative integers are denoted by $\mathbb{R}$ and $\mathbb{N}_0$, respectively. If

$x \in \mathbb{R}^n$, then $\|x\|$ is the Euclidean norm of x. If $\mathbf{X} \in \mathbb{R}^{n\times n}$, then $\operatorname{tr}\mathbf{X}$ is the trace of $\mathbf{X}$. If $\mathbf{X}$, $\mathbf{Y} \in \mathbb{R}^{n\times n}$ are symmetric, then we write $\mathbf{X} > \mathbf{Y}$ (resp. $\mathbf{X} \geq \mathbf{Y}$) to mean that $\mathbf{X} - \mathbf{Y}$ is positive definite (resp. nonnegative definite).

2. ANALYSIS RESULT

Let $\mathbf{A}(t) \in \mathbb{R}^{n\times n}$, $\mathbf{B}(t) \in \mathbb{R}^{n\times m}$, $\mathbf{C}_0(t) \in \mathbb{R}^{l_0\times n}$, ..., $\mathbf{C}_N(t) \in \mathbb{R}^{l_N\times n}$, and $\mathbf{D}_0(t) \in \mathbb{R}^{l_0\times m}$, ..., $\mathbf{D}_N(t) \in \mathbb{R}^{l_N\times m}$ for $t \in \mathbb{N}_0$, where n, m, l_0, ..., l_N are positive integers. Then the indexed family

$$\boldsymbol{\mathcal{G}} = \{(\mathbf{A}(t), \mathbf{B}(t), \mathbf{C}_0(t), \ldots, \mathbf{C}_N(t), \mathbf{D}_0(t), \ldots, \mathbf{D}_N(t)) \colon t \in \mathbb{N}_0\} \quad (1)$$

defines the linear time-varying system with a state equation

$$x(t+1) = \mathbf{A}(t)x(t) + \mathbf{B}(t)w(t)$$

and $N+1$ output equations

$$z_i(t) = \mathbf{C}_i(t)x(t) + \mathbf{D}_i(t)w(t), \quad i = 0,\ 1,\ \ldots,\ N,$$

for all $t \in \mathbb{N}_0$, where $x(t)$, $w(t)$, and $z_i(t)$ denote the state, the disturbance input, and the i-th output, respectively, at time t. Following the terminology used by Farhood and Dullerud [2002], the system $\boldsymbol{\mathcal{G}}$ is said to be *k-eventually time-invariant* if there exists a time instant k such that

$$\mathbf{A}(t) = \mathbf{A}(k), \quad \mathbf{B}(t) = \mathbf{B}(k),$$
$$\mathbf{C}_i(t) = \mathbf{C}_i(k), \quad \mathbf{D}_i(t) = \mathbf{D}_i(k)$$

for all $t = k$, $k+1$, ... and for all $i = 1$, ..., N.

We assume that $x(0)$, $w(t)$, $t = 0$, 1, ... are Gaussian random vectors satisfying

$$\mathrm{E}[x(0)] = 0, \quad \mathrm{E}[w(t)] = 0, \quad (2a)$$
$$\mathrm{E}[x(0)x(0)^{\mathrm{T}}] = \mathbf{P}, \quad \mathrm{E}[x(0)w(t)^{\mathrm{T}}] = \mathbf{0}, \quad (2b)$$
$$\mathrm{E}[w(t_1)w(t_2)^{\mathrm{T}}] = \begin{cases} \mathbf{I}, & t_1 = t_2; \\ \mathbf{0}, & \text{otherwise} \end{cases} \quad (2c)$$

for all t, t_1 and t_2, where $\mathrm{E}[\cdot]$ indicates the expectation. Under this assumption, it is readily seen that

$$\mathrm{E}[x(t+1)x(t+1)^{\mathrm{T}}] = \mathbf{A}(t)\,\mathrm{E}[x(t)x(t)^{\mathrm{T}}]\mathbf{A}(t)^{\mathrm{T}} + \mathbf{B}(t)\mathbf{B}(t)^{\mathrm{T}}$$

and

$$\mathrm{E}\,\|z_i(t)\|^2 = \operatorname{tr}\left[\mathbf{C}_i(t)\,\mathrm{E}[x(t)x(t)^{\mathrm{T}}]\mathbf{C}_i(t)^{\mathrm{T}} + \mathbf{D}_i(t)\mathbf{D}_i(t)^{\mathrm{T}}\right]$$

for all $t \in \mathbb{N}_0$ and for all $i = 0$, 1, ..., N.

Definition 1. The system $\boldsymbol{\mathcal{G}}$ is said to be *(uniformly exponentially) stable* if there are $c > 1$ and $\lambda \in (0,1)$ such that

$$\|x(t)\| \leq c\lambda^{t-s}\|x(s)\|$$

for all s, $t \in \mathbb{N}_0$ with $t \geq s$, and for all $x(s) \in \mathbb{R}^n$.

Definition 2. Let γ_0, ..., $\gamma_N > 0$. The pair $(\boldsymbol{\mathcal{G}}, \mathbf{P})$ is said to deliver *(output regulation) performance* γ_0 subject to *(output) constraint* $(\gamma_1, \ldots, \gamma_N)$ if there exist $\eta_i \in [0, \gamma_i)$, $i = 1$, ..., N, such that

$$\limsup_{T\to\infty} \frac{1}{T} \sum_{s=0}^{T-1} \mathrm{E}\,\|z_0(s)\|^2 < \gamma_0^2,$$
$$\mathrm{E}\,\|z_1(t)\|^2 \leq \eta_1, \quad \ldots, \quad \mathrm{E}\,\|z_N(t)\|^2 \leq \eta_N$$

for all $t \in \mathbb{N}_0$.

Theorem 3. Let $\boldsymbol{\mathcal{G}}$ be as in (1) and k-eventually time-invariant for some $k \in \mathbb{N}_0$; let γ_0, ..., $\gamma_N > 0$. Suppose (2) holds. The system $\boldsymbol{\mathcal{G}}$ is uniformly exponentially stable and the pair $(\boldsymbol{\mathcal{G}}, \mathbf{P})$ delivers output regulation performance γ_0 subject to output constraint $(\gamma_1, \ldots, \gamma_N)$ if and only if there exist an integer $T \geq k$, and matrices $\mathbf{Y}(0)$, ..., $\mathbf{Y}(T) > \mathbf{0}$ such that

$$\mathbf{A}(T)\mathbf{Y}(T)\mathbf{A}(T)^{\mathrm{T}} + \mathbf{B}(T)\mathbf{B}(T)^{\mathrm{T}} < \mathbf{Y}(T) \quad (3a)$$

with

$$\mathbf{P} < \mathbf{Y}(0), \quad (3b)$$
$$\mathbf{A}(t)\mathbf{Y}(t)\mathbf{A}(t)^{\mathrm{T}} + \mathbf{B}(t)\mathbf{B}(t)^{\mathrm{T}} < \mathbf{Y}(t+1) \quad (3c)$$

for all $t = 0$, ..., $T-1$, and such that

$$\operatorname{tr}\left[\mathbf{C}_0(T)\mathbf{Y}(T)\mathbf{C}_0(T)^{\mathrm{T}} + \mathbf{D}_0(T)\mathbf{D}_0(T)^{\mathrm{T}}\right] < \gamma_0^2 \quad (3d)$$

with

$$\operatorname{tr}\left[\mathbf{C}_i(t)\mathbf{Y}(t)\mathbf{C}_i(t)^{\mathrm{T}} + \mathbf{D}_i(t)\mathbf{D}_i(t)^{\mathrm{T}}\right] < \gamma_i^2 \quad (3e)$$

for all $t = 0$, ..., T and for all $i = 1$, ..., N.

Proof. To show necessity, suppose that $\boldsymbol{\mathcal{G}}$ is stable and k-eventually time-invariant, and that the pair $(\boldsymbol{\mathcal{G}}, \mathbf{P})$ delivers performance γ_0 subject to constraint $(\gamma_1, \ldots, \gamma_N)$. For each $\delta \geq 0$, put

$$\widehat{\mathbf{Y}}(\delta, 0) = \mathbf{P} + \delta\mathbf{I}, \quad (4a)$$
$$\widehat{\mathbf{Y}}(\delta, t+1) = \mathbf{A}(t)\widehat{\mathbf{Y}}(\delta, t)\mathbf{A}(t)^{\mathrm{T}} + \mathbf{B}(t)\mathbf{B}(t)^{\mathrm{T}} + \delta\mathbf{I} \quad (4b)$$

for all $t \in \mathbb{N}_0$. Since $\boldsymbol{\mathcal{G}}$ is stable and k-eventually time-invariant, for any integer $T \geq k$ and number $\varepsilon \geq 0$ there exists a unique symmetric nonnegative definite matrix $\widetilde{\mathbf{Y}}(\varepsilon)$ satisfying

$$\widetilde{\mathbf{Y}}(\varepsilon) = \mathbf{A}(T)\widetilde{\mathbf{Y}}(\varepsilon)\mathbf{A}(T)^{\mathrm{T}} + \mathbf{B}(T)\mathbf{B}(T)^{\mathrm{T}} + \varepsilon\mathbf{I}. \quad (5)$$

It is readily seen that

$$\widehat{\mathbf{Y}}(0, t) = \mathrm{E}[x(t)x(t)^{\mathrm{T}}],$$
$$\widetilde{\mathbf{Y}}(0) = \lim_{S\to\infty} \frac{1}{S-1} \sum_{s=0}^{S-1} \mathrm{E}[x(s)x(s)^{\mathrm{T}}] = \lim_{S\to\infty} \frac{1}{S-T} \sum_{s=T}^{S-1} \mathrm{E}[x(s)x(s)^{\mathrm{T}}].$$

Since $(\boldsymbol{\mathcal{G}}, \mathbf{P})$ delivers performance γ_0 subject to constraints $(\gamma_1, \ldots, \gamma_N)$, there exist $\eta_i < \gamma_i$, $i = 0, 1, \ldots, N$, such that

$$\limsup_{S\to\infty} \frac{1}{S} \sum_{s=0}^{S-1} \mathrm{E}\,\|z_0(s)\|^2 = \operatorname{tr}\left[\mathbf{C}_0(T)\widetilde{\mathbf{Y}}_0(0)\mathbf{C}_0(T)^{\mathrm{T}} + \mathbf{D}_0(T)\mathbf{D}_0(T)^{\mathrm{T}}\right] \leq \eta_0^2$$

for all $T \geq k$, and

$$\mathrm{E}\,\|z_i(t)\|^2 = \operatorname{tr}\left[\mathbf{C}_i(t)\widehat{\mathbf{Y}}(0,t)\mathbf{C}_i(t)^{\mathrm{T}} + \mathbf{D}_i(t)\mathbf{D}_i(t)^{\mathrm{T}}\right] \leq \eta_i^2$$

for all $t \in \mathbb{N}_0$ and for all $i = 1$, ..., N. Let $\mu > 0$ be sufficiently small so that there exist $\tilde{\eta}_i \in (\eta_i, \gamma_i)$, $i = 0$, 1, ..., N, such that

$$\operatorname{tr}\left[\mu\mathbf{C}_i(T)\mathbf{C}_i(T)^{\mathrm{T}}\right] + \eta_i^2 \leq \tilde{\eta}_i^2 \quad (6)$$

for all $i = 0, 1, \ldots, N$ as long as $T \geq k$. Since $\widetilde{\mathbf{Y}}(\varepsilon) \to \widetilde{\mathbf{Y}}(0)$ as $\varepsilon \to 0$ and since $\widetilde{\mathbf{Y}}(\varepsilon) \geq \widetilde{\mathbf{Y}}(0) + \varepsilon\mathbf{I}$, there exists an $\varepsilon > 0$ such that

$$\varepsilon\mathbf{I} \leq \widetilde{\mathbf{Y}}(\varepsilon) - \widetilde{\mathbf{Y}}(0) \leq (\mu/2)\mathbf{I}. \quad (7)$$

Also, since we have $\widehat{\mathbf{Y}}(0,t) \to \widetilde{\mathbf{Y}}(0)$ as $t \to \infty$, there exists an integer $T \geq k$ such that

$$-\min\{\mu/2, \varepsilon/2\}\mathbf{I} \leq \widehat{\mathbf{Y}}(0,t) - \widetilde{\mathbf{Y}}(0) \leq \min\{\mu/2, \varepsilon/2\}\mathbf{I}$$

for all $t \geq T$. Such $\varepsilon > 0$ and $T \geq k$ lead to

$$(\varepsilon/2)\mathbf{I} \leq \widetilde{\mathbf{Y}}(\varepsilon) - \widehat{\mathbf{Y}}(0,T) \leq \mu\mathbf{I}. \quad (8)$$

With these $\varepsilon > 0$ and $T \geq k$, and with $\tilde{\eta}_i \in (\eta_i, \gamma_i)$, $i = 1$, ..., N, as in (6), let $\hat{\mu} > 0$ be such that

$$\operatorname{tr}\left[\hat{\mu}\mathbf{C}_i(t)\mathbf{C}_i(t)^{\mathrm{T}}\right] + \eta_i^2 \leq \tilde{\eta}_i^2 \quad (9)$$

for all $t = 0, \ldots, T-1$ and for all $i = 1, \ldots, N$, and let $\delta > 0$ be such that

$$\widehat{\mathbf{Y}}(\delta, T) - \widehat{\mathbf{Y}}(0, T) \le \min\{\hat{\mu}, \varepsilon/2\}\mathbf{I}. \quad (10)$$

Then, (8) and (10) lead to

$$\widehat{\mathbf{Y}}(\delta, T) \le \widetilde{\mathbf{Y}}(\varepsilon), \quad (11a)$$

inequalities (6) and (7) to

$$\operatorname{tr}\left[\mathbf{C}_0(T)\widetilde{\mathbf{Y}}(\varepsilon)\mathbf{C}_0(T)^{\mathrm{T}} + \mathbf{D}_0(T)\mathbf{D}_0(T)^{\mathrm{T}}\right] < \gamma_0^2, \quad (11b)$$

inequalities (6) and (8) to

$$\operatorname{tr}\left[\mathbf{C}_i(T)\widetilde{\mathbf{Y}}(\varepsilon)\mathbf{C}_i(T)^{\mathrm{T}} + \mathbf{D}_i(T)\mathbf{D}_i(T)^{\mathrm{T}}\right] < \gamma_i^2 \quad (11c)$$

for $i = 1, \ldots, N$, and (9) and (10) to

$$\operatorname{tr}\left[\mathbf{C}_i(t)\widehat{\mathbf{Y}}(\delta, t)\mathbf{C}_i(t)^{\mathrm{T}} + \mathbf{D}_i(t)\mathbf{D}_i(t)^{\mathrm{T}}\right] < \gamma_i^2 \quad (11d)$$

for $t = 0, \ldots, T-1$ and for $i = 1, \ldots, N$. Now, if we let $\mathbf{Y}(t) = \widehat{\mathbf{Y}}(\delta, t)$ for $t = 0, \ldots, T-1$ and let $\mathbf{Y}(T) = \widetilde{\mathbf{Y}}(\varepsilon)$, then (4), (5), and (11) lead to (3). This proves necessity.

To show sufficiency, suppose there are $T \ge k$, and $\mathbf{Y}(0), \ldots, \mathbf{Y}(T) > \mathbf{0}$ such that (3a) holds with (3b)–(3c) for $t = 0, \ldots, T-1$ and such that (3d) holds with (3e) for $t = 0, 1, \ldots, T$ and for $i = 1, \ldots, N$. Then the system $\boldsymbol{\mathcal{G}}$ is stable due to (3a). It follows from (3a)–(3c) that $\mathbf{Y}(\min\{t, T\}) \ge \mathrm{E}[x(t)x(t)^{\mathrm{T}}]$ for all $t \in \mathbb{N}_0$, so (3d) and (3e) imply the system $\boldsymbol{\mathcal{G}}$ delivers performance γ_0 subject to constraint $(\gamma_1, \ldots, \gamma_N)$. This proves the sufficiency part, and hence completes the proof. □

3. SYNTHESIS RESULT

Let n, m_1, m_2, $l_{10}, \ldots, l_{1N}$, and l_2 be positive integers; let $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B}_1 \in \mathbb{R}^{n\times m_1}$, $\mathbf{B}_2 \in \mathbb{R}^{n\times m_2}$, $\mathbf{C}_{10} \in \mathbb{R}^{l_{10}\times n}, \ldots, \mathbf{C}_{1N} \in \mathbb{R}^{l_{1N}\times n}$, $\mathbf{D}_{110} \in \mathbb{R}^{l_{10}\times m_1}, \ldots, \mathbf{D}_{11N} \in \mathbb{R}^{l_{1N}\times m_1}$, $\mathbf{D}_{120} \in \mathbb{R}^{l_{10}\times m_2}, \ldots, \mathbf{D}_{12N} \in \mathbb{R}^{l_{1N}\times m_2}$, $\mathbf{C}_2 \in \mathbb{R}^{l_2\times n}$, and $\mathbf{D}_{21} \in \mathbb{R}^{l_2\times m_1}$ be given matrices. Then the matrix tuple

$$\boldsymbol{\mathcal{T}} = (\mathbf{A}, \mathbf{B}, \mathbf{C}_{10}, \ldots, \mathbf{C}_{1N}, \mathbf{C}_2, \mathbf{D}_{110}, \ldots, \mathbf{D}_{11N}, \mathbf{D}_{120}, \ldots, \mathbf{D}_{12N}, \mathbf{D}_{21}) \quad (12)$$

defines the controlled linear time-invariant system with a state equation

$$x(t+1) = \mathbf{A}x(t) + \mathbf{B}_1 w(t) + \mathbf{B}_2 u(t), \quad (13a)$$

a set of $N+1$ controlled output equations

$$z_i(t) = \mathbf{C}_{1i}x(t) + \mathbf{D}_{11i}w(t) + \mathbf{D}_{12i}u(t), \quad i = 0, 1, \ldots, N, \quad (13b)$$

and a measured output equation

$$y(t) = \mathbf{C}_2 x(t) + \mathbf{D}_{21}w(t). \quad (13c)$$

As in the previous section, we assume that $x(0)$, $w(t)$, $t = 0, 1, \ldots$ are Gaussian random vectors satisfying (2) for all t, t_1, $t_2 \in \mathbb{N}_0$.

Let n_K be a positive integer, and let $\mathbf{A}_K(t) \in \mathbb{R}^{n_K\times n_K}$, $\mathbf{B}_K(t) \in \mathbb{R}^{n_K\times l_2}$, $\mathbf{C}_K(t) \in \mathbb{R}^{m_2\times n_K}$, and $\mathbf{D}_K(t) \in \mathbb{R}^{m_2\times l_2}$ for all $t \in \mathbb{N}_0$. Then the indexed family

$$\boldsymbol{\mathcal{K}} = \{(\mathbf{A}_K(t), \mathbf{B}_K(t), \mathbf{C}_K(t), \mathbf{D}_K(t)) \colon t \in \mathbb{N}_0\}$$

defines the linear dynamic output feedback controller represented by

$$x_K(t+1) = \mathbf{A}_K(t)x_K(t) + \mathbf{B}_K(t)y(t), \quad (14a)$$

$$u(t) = \mathbf{C}_K(t)x_K(t) + \mathbf{D}_K(t)y(t) \quad (14b)$$

over all $t \in \mathbb{N}_0$. The interconnection of the plant (13) and controller (14), with the closed-loop state defined by

$$\tilde{x}(t) = \left[x(t)^{\mathrm{T}}\ x_K(t)^{\mathrm{T}}\right]^{\mathrm{T}} \in \mathbb{R}^{n+n_K},$$

leads to the closed-loop system $\Sigma(\boldsymbol{\mathcal{T}}, \boldsymbol{\mathcal{K}})$ of the form

$$\tilde{x}(t+1) = \widetilde{\mathbf{A}}(t)\tilde{x}(t) + \widetilde{\mathbf{B}}(t)w(t),$$

$$z_i(t) = \widetilde{\mathbf{C}}_i\tilde{x}(t) + \widetilde{\mathbf{D}}_i w(t), \quad i = 0, 1, \ldots, N,$$

where

$$\widetilde{\mathbf{A}}(t) = \widehat{\mathbf{A}} + \widehat{\mathbf{B}}_2\mathbf{K}(t)\widehat{\mathbf{C}}_2,$$
$$\widetilde{\mathbf{B}}(t) = \widehat{\mathbf{B}}_1 + \widehat{\mathbf{B}}_2\mathbf{K}(t)\widehat{\mathbf{D}}_{21},$$
$$\widetilde{\mathbf{C}}_i(t) = \widehat{\mathbf{C}}_{1i} + \widehat{\mathbf{D}}_{12i}\mathbf{K}(t)\widehat{\mathbf{C}}_2,$$
$$\widetilde{\mathbf{D}}_i(t) = \mathbf{D}_{11i} + \widehat{\mathbf{D}}_{12i}\mathbf{K}(t)\widehat{\mathbf{D}}_{21},$$

and

$$\widehat{\mathbf{A}} = \begin{bmatrix}\mathbf{A} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}, \quad \widehat{\mathbf{B}}_1 = \begin{bmatrix}\mathbf{B}_1\\ \mathbf{0}\end{bmatrix}, \quad \widehat{\mathbf{B}}_2 = \begin{bmatrix}\mathbf{0} & \mathbf{B}_2\\ \mathbf{I} & \mathbf{0}\end{bmatrix},$$

$$\widehat{\mathbf{C}}_{1,i} = [\mathbf{C}_{1i}\ \mathbf{0}], \quad \widehat{\mathbf{C}}_2 = \begin{bmatrix}\mathbf{0} & \mathbf{I}\\ \mathbf{C}_2 & \mathbf{0}\end{bmatrix},$$

$$\widehat{\mathbf{D}}_{12i} = [\mathbf{0}\ \mathbf{D}_{12i}], \quad \widehat{\mathbf{D}}_{21} = \begin{bmatrix}\mathbf{0}\\ \mathbf{D}_{21}\end{bmatrix}$$

for $i = 0, 1, \ldots, N$, with

$$\mathbf{K}(t) = \begin{bmatrix}\mathbf{A}_K(t) & \mathbf{B}_K(t)\\ \mathbf{C}_K(t) & \mathbf{D}_K(t)\end{bmatrix}. \quad (15)$$

Assume zero initial controller state (i.e. $x_K(0) = 0$), and write

$$\mathbf{P} \oplus \mathbf{0} = \begin{bmatrix}\mathbf{P} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix} \in \mathbb{R}^{(n+n_K)\times(n+n_K)}.$$

Definition 4. Let $\gamma_0, \ldots, \gamma_N > 0$. If there exists a linear dynamic output feedback controller $\boldsymbol{\mathcal{K}}$ such that the closed-loop system $\Sigma(\boldsymbol{\mathcal{T}}, \boldsymbol{\mathcal{K}})$ is uniformly exponentially stable and such that the pair $(\Sigma(\boldsymbol{\mathcal{T}}, \boldsymbol{\mathcal{K}}), \mathbf{P} \oplus \mathbf{0})$ delivers output regulation performance γ_0 subject to output constraint $(\gamma_1, \ldots, \gamma_N)$, then the controller $\boldsymbol{\mathcal{K}}$ is said to be $(\gamma_0, \ldots, \gamma_N)$-*admissible* for the pair $(\boldsymbol{\mathcal{T}}, \mathbf{P})$.

Application of Theorem 3 to the closed-loop system $\Sigma(\boldsymbol{\mathcal{T}}, \boldsymbol{\mathcal{K}})$, followed by a Schur complement argument, yields that the pair $(\Sigma(\boldsymbol{\mathcal{T}}, \boldsymbol{\mathcal{K}}), \mathbf{P} \oplus \mathbf{0})$ delivers performance γ_0 subject to constraint $(\gamma_1, \ldots, \gamma_N)$ if and only if there are $\mathbf{Y}(t)$, $\mathbf{Z}_i(t) > \mathbf{0}$, $t = 0, \ldots, T$, $i = 1, \ldots, N$ such that

$$\begin{bmatrix}-\mathbf{Y}(T)^{-1} & \widetilde{\mathbf{A}}(T)^{\mathrm{T}} & \mathbf{0}\\ \widetilde{\mathbf{A}}(T) & -\mathbf{Y}(T) & \widetilde{\mathbf{B}}(T)\\ \mathbf{0} & \widetilde{\mathbf{B}}(T)^{\mathrm{T}} & -\mathbf{I}\end{bmatrix} < \mathbf{0} \quad (16a)$$

with

$$\mathbf{P} \oplus \mathbf{0} < \mathbf{Y}(0), \quad (16b)$$

$$\begin{bmatrix}-\mathbf{Y}(t)^{-1} & \widetilde{\mathbf{A}}(t)^{\mathrm{T}} & \mathbf{0}\\ \widetilde{\mathbf{A}}(t) & -\mathbf{Y}(t+1) & \widetilde{\mathbf{B}}(t)\\ \mathbf{0} & \widetilde{\mathbf{B}}(t)^{\mathrm{T}} & -\mathbf{I}\end{bmatrix} < \mathbf{0} \quad (16c)$$

for all $t = 0, \ldots, T-1$, and such that

$$\begin{bmatrix}-\mathbf{Y}(T)^{-1} & \widetilde{\mathbf{C}}_0(T)^{\mathrm{T}} & \mathbf{0}\\ \widetilde{\mathbf{C}}_0(T) & -\mathbf{Z}_0(T) & \widetilde{\mathbf{D}}_0(T)\\ \mathbf{0} & \widetilde{\mathbf{D}}_0(T)^{\mathrm{T}} & -\mathbf{I}\end{bmatrix} < \mathbf{0}, \quad (16d)$$

$$\operatorname{tr}\mathbf{Z}_0(T) < \gamma_0^2 \quad (16e)$$

with

$$\begin{bmatrix}-\mathbf{Y}(t)^{-1} & \widetilde{\mathbf{C}}_i(t)^{\mathrm{T}} & \mathbf{0}\\ \widetilde{\mathbf{C}}_i(t) & -\mathbf{Z}_i(t) & \widetilde{\mathbf{D}}_i(t)\\ \mathbf{0} & \widetilde{\mathbf{D}}_i(t)^{\mathrm{T}} & -\mathbf{I}\end{bmatrix} < \mathbf{0}, \quad (16f)$$

$$\operatorname{tr}\mathbf{Z}_i(t) < \gamma_i^2 \quad (16g)$$

for all $t = 0, \ldots, T$ and for all $i = 1, \ldots, N$. For all t, partition $\mathbf{Y}(t)$ and $\mathbf{Y}(t)^{-1}$ as

$$\mathbf{Y}(t)^{-1} = \begin{bmatrix} \mathbf{S}(t) & \mathbf{U}(t) \\ \mathbf{U}(t)^{\mathrm{T}} & * \end{bmatrix}, \quad \mathbf{Y}(t) = \begin{bmatrix} \mathbf{R}(t) & \mathbf{T}(t) \\ \mathbf{T}(t)^{\mathrm{T}} & * \end{bmatrix}$$

where $\mathbf{S}(t)$, $\mathbf{R}(t) \in \mathbb{R}^{n\times n}$ and $\mathbf{U}(t)$, $\mathbf{T}(t) \in \mathbb{R}^{n\times n_K}$. Then, the change of variable formula developed by Scherer et al. [1997] suggests

$$\mathbf{W}(t) = \begin{bmatrix} \mathbf{S}(t+1)\mathbf{A}\mathbf{R}(t) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{U}(t+1) & \mathbf{S}(t+1)\mathbf{B}_2 \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \mathbf{K}(t) \begin{bmatrix} \mathbf{T}(t)^{\mathrm{T}} & \mathbf{0} \\ \mathbf{C}_2\mathbf{R}(t) & \mathbf{I} \end{bmatrix} \quad (17a)$$

for $t = 0, \ldots, T-1$, and

$$\mathbf{W}(T) = \begin{bmatrix} \mathbf{S}(T)\mathbf{A}\mathbf{R}(T) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{U}(T) & \mathbf{S}(T)\mathbf{B}_2 \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \mathbf{K}(T) \begin{bmatrix} \mathbf{T}(T)^{\mathrm{T}} & \mathbf{0} \\ \mathbf{C}_2\mathbf{R}(T) & \mathbf{I} \end{bmatrix} \quad (17b)$$

for $t = T$.

Theorem 5. Let $\boldsymbol{\mathcal{T}}$ be as in (12); let $\gamma_0, \ldots, \gamma_N > 0$. Suppose (2) holds. There exists a $(\gamma_0, \ldots, \gamma_N)$-admissible linear dynamic output feedback controller $\boldsymbol{\mathcal{K}}$ for the pair $(\boldsymbol{\mathcal{T}}, \mathbf{P})$ if and only if there exist an integer $T \in \mathbb{N}_0$ and matrices $\mathbf{R}(t)$, $\mathbf{S}(t)$, $\mathbf{W}(t)$, $\mathbf{Z}_i(t)$, for $t = 0, \ldots, T$ and $i = 1, \ldots, N$, and $\mathbf{Z}_0(T)$ such that

$$\mathbf{H}(T,T) + \mathbf{F}^{\mathrm{T}}\mathbf{W}(T)\mathbf{G} + \mathbf{G}^{\mathrm{T}}\mathbf{W}(T)^{\mathrm{T}}\mathbf{F} < \mathbf{0} \quad (18a)$$

with

$$\begin{bmatrix} \mathbf{P} - \mathbf{R}(0) & \mathbf{P}\mathbf{S}(0) - \mathbf{I} \\ \mathbf{S}(0)\mathbf{P} - \mathbf{I} & -\mathbf{S}(0) \end{bmatrix} < \mathbf{0}, \quad (18b)$$

$$\mathbf{H}(t,t+1) + \mathbf{F}^{\mathrm{T}}\mathbf{W}(t)\mathbf{G} + \mathbf{G}^{\mathrm{T}}\mathbf{W}(t)^{\mathrm{T}}\mathbf{F} < \mathbf{0} \quad (18c)$$

for all $t = 0, \ldots, T-1$, and such that

$$\widehat{\mathbf{H}}_0(T) + \widehat{\mathbf{F}}_0^{\mathrm{T}}\mathbf{W}(T)\widehat{\mathbf{G}}_0 + \widehat{\mathbf{G}}_0^{\mathrm{T}}\mathbf{W}(T)^{\mathrm{T}}\widehat{\mathbf{F}}_0 < \mathbf{0}, \quad (18d)$$

$$\operatorname{tr}\mathbf{Z}_0(T) < \gamma_0^2 \quad (18e)$$

with

$$\widehat{\mathbf{H}}_i(t) + \widehat{\mathbf{F}}_i^{\mathrm{T}}\mathbf{W}(t)\widehat{\mathbf{G}}_i + \widehat{\mathbf{G}}_i^{\mathrm{T}}\mathbf{W}(t)^{\mathrm{T}}\widehat{\mathbf{F}}_i < \mathbf{0}, \quad (18f)$$

$$\operatorname{tr}\mathbf{Z}_i(t) < \gamma_i^2 \quad (18g)$$

for all $t = 0, \ldots, T$ and for all $i = 1, \ldots, N$, where

$$\mathbf{H}(t,s) = \begin{bmatrix} -\mathbf{S}(t) & -\mathbf{I} & \mathbf{A}^{\mathrm{T}} & \mathbf{A}^{\mathrm{T}}\mathbf{S}(s) & \mathbf{0} \\ -\mathbf{I} & -\mathbf{R}(t) & \mathbf{R}(t)\mathbf{A}^{\mathrm{T}} & \mathbf{0} & \mathbf{0} \\ \mathbf{A} & \mathbf{A}\mathbf{R}(t) & -\mathbf{R}(s) & -\mathbf{I} & \mathbf{B}_1 \\ \mathbf{S}(s)\mathbf{A} & \mathbf{0} & -\mathbf{I} & -\mathbf{S}(s) & \mathbf{S}(s)\mathbf{B}_1 \\ \mathbf{0} & \mathbf{0} & \mathbf{B}_1^{\mathrm{T}} & \mathbf{B}_1^{\mathrm{T}}\mathbf{S}(s) & -\mathbf{I} \end{bmatrix}$$

for t, $s = 0, \ldots, T-1$,

$$\widehat{\mathbf{H}}_i(t) = \begin{bmatrix} -\mathbf{S}(t) & -\mathbf{I} & \mathbf{C}_{1i}^{\mathrm{T}} & \mathbf{0} \\ -\mathbf{I} & -\mathbf{R}(t) & \mathbf{R}(t)\mathbf{C}_{1i}^{\mathrm{T}} & \mathbf{0} \\ \mathbf{C}_{1i} & \mathbf{C}_{1i}\mathbf{R}(t) & -\mathbf{Z}_i(t) & \mathbf{D}_{11i} \\ \mathbf{0} & \mathbf{0} & \mathbf{D}_{11i}^{\mathrm{T}} & -\mathbf{I} \end{bmatrix}$$

for $t = 0, \ldots, T$ and $i = 1, \ldots, N$, and

$$\mathbf{F} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{B}_2^{\mathrm{T}} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{C}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{D}_{21} \end{bmatrix},$$

$$\widehat{\mathbf{F}}_i = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{D}_{12i}^{\mathrm{T}} & \mathbf{0} \end{bmatrix}, \quad \widehat{\mathbf{G}}_i = \begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{C}_2 & \mathbf{0} & \mathbf{0} & \mathbf{D}_{21} \end{bmatrix}$$

for $i = 1, \ldots, N$. Moreover, if this condition is satisfied, then given any nonsingular $\mathbf{T}(t)$, $\mathbf{U}(t) \in \mathbb{R}^{n\times n}$ such that

$$\mathbf{T}(t)\mathbf{U}(t)^{\mathrm{T}} = \mathbf{I} - \mathbf{R}(t)\mathbf{S}(t) \quad (19)$$

for all $t = 0, \ldots, T$, a $(\gamma_0, \ldots, \gamma_N)$-admissible controller of order $n_K = n$ is obtained by solving (17) for $\mathbf{K}(t)$, $t = 0, \ldots, T$, and letting $\mathbf{K}(t) = \mathbf{K}(T)$ for all $t \geq T$.

Proof. The result is obtained by applying appropriate congruence transformations on inequalities (16); see Scherer et al. [1997, Section IV-B]. □

4. ILLUSTRATIVE EXAMPLES

Let us first consider the case of one output constraint (i.e., $N = 1$) where the plant $\boldsymbol{\mathcal{T}}$ has

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, \quad \mathbf{B}_1 = \begin{bmatrix} 0.25 \\ 0 \end{bmatrix}, \quad \mathbf{B}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$\mathbf{C}_{10} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{D}_{110} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathbf{D}_{120} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$\mathbf{C}_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{D}_{111} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{D}_{121} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$\mathbf{C}_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{D}_{21} = \begin{bmatrix} 0.25 \\ 0 \end{bmatrix}.$$

Let

$$\gamma_1 = 13.18 \quad \text{and} \quad \mathbf{P} = \begin{bmatrix} 100 & -50 \\ -50 & 25 \end{bmatrix}.$$

We are to minimize γ_0 over all linear dynamic output feedback controllers $\boldsymbol{\mathcal{K}}$ that are (γ_0, γ_1)-admissible for the pair $(\boldsymbol{\mathcal{K}}, \mathbf{P})$. That is, given that the initial state $x(0)$ is a zeromean Gaussian random vector with variance $\mathbf{P}$ and that the disturbance input $(w(0), w(1), \ldots)$ is a zeromean white Gaussian sequence independent of $x(0)$, the control objective is to minimize the average output variance per unit time given by

$$\limsup_{S\to\infty} \frac{1}{S} \sum_{s=0}^{S-1} \mathrm{E}\left\{ x(s)^{\mathrm{T}} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x(s) + u(s)^2 \right\}$$

subject to the state-control constraint

$$\mathrm{E}\left\{ \|x(t)\|^2 + u(t)^2 \right\} < \gamma_1, \quad t = 0, 1, \ldots,$$

and closed-loop stability. Due to Theorem 5, this optimization problem can be solved by executing the semidefinite program of minizming γ_0^2 subject to linear matrix inequalities (18) incrementally over $T = 0, 1, \ldots$ until a saturation in the minimum achievable value of γ_0 occurs within a reasonable tolerance level. It turns out that the smallest attainable performance level (up to a few significant digits) is $\gamma_0 = 13.18$ if $T = 0$, $\gamma_0 = 7.291$ if $T = 1$, and $\gamma_0 = 0.3864$ if $2 \leq T \leq 50$. Thus, we determine that the near-optimal performance level $\gamma_0^* = 0.3864$ is achieved with $T = 2$; solving (18) and (19) with $T = 2$ and $\gamma_0 = \gamma_0^*$, and plugging into (17) the resulting matrices $\mathbf{R}(t)$, $\mathbf{S}(t)$, $\mathbf{W}(t)$, $\mathbf{U}(t) = (\mathbf{S}(t) - \mathbf{R}(t)^{-1})^{1/2}$, and $\mathbf{T}(t) = -\mathbf{R}(t)\mathbf{U}(t)$ for $t = 0, 1, 2$, we obtain the corresponding controller coefficients $\mathbf{K}(t)$, $t \in \mathbb{N}_0$, partitioned as (15), where

$$\mathbf{A}_K(t) = \begin{cases} \begin{bmatrix} 0.8986 & 1.8713 \\ -0.5315 & -1.150 \end{bmatrix} & \text{if } t = 0; \\ \begin{bmatrix} -0.5872 & 0.3473 \\ 0.2953 & -0.1747 \end{bmatrix} & \text{if } t = 1; \\ \begin{bmatrix} -0.5113 & 0.3726 \\ 0.1984 & -0.1000 \end{bmatrix} & \text{if } t \geq 2; \end{cases}$$

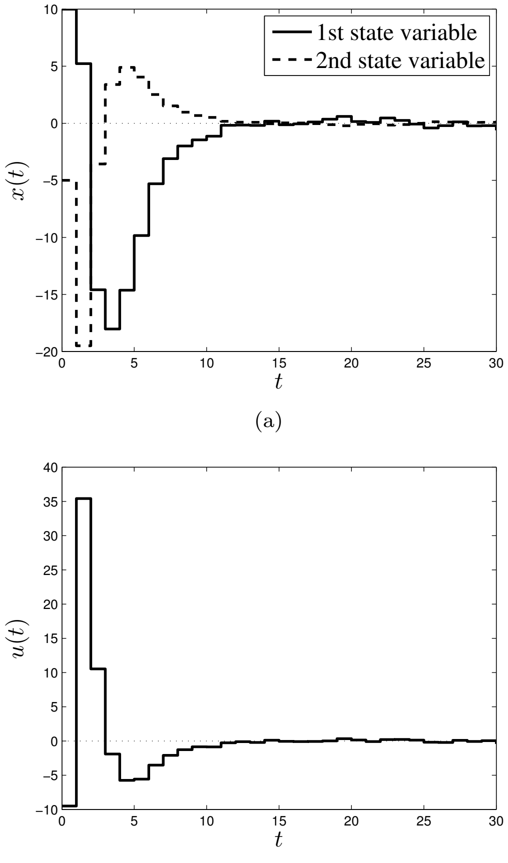


Fig. 1. First example: closed-loop state-control trajectories under the unconstrained optimal controller. (a) Unconstrained state trajectory. (b) Unconstrained control trajectory.

$$\mathbf{B}_K(t) = \begin{cases} \begin{bmatrix} -22500 & 22390 \\ 13790 & -13660 \end{bmatrix} & \text{if } t = 0; \\ \begin{bmatrix} -18180 & -11770 \\ 9147 & 5922 \end{bmatrix} & \text{if } t = 1; \\ \begin{bmatrix} -17960 & -10090 \\ 8536 & 3714 \end{bmatrix} & \text{if } t \geq 2; \end{cases}$$

$$\mathbf{C}_K(t) = \begin{cases} [0.0001179 \; 0.0002473] & \text{if } t = 0; \\ [-0.00007521 \; 0.00004799] & \text{if } t = 1; \\ [-0.00006114 \; 0.00004927] & \text{if } t \geq 2; \end{cases}$$

$$\mathbf{D}_K(t) = \begin{cases} [-0.9728 \; 2.956] & \text{if } t = 0; \\ [-0.3776 \; -1.531] & \text{if } t = 1; \\ [-0.3288 \; -1.227] & \text{if } t \geq 2. \end{cases}$$

Typical state-control trajectories under the "unconstrained" near-optimal time-invariant controller would look like those in Fig. 1. On the other hand, the constrained controller, which is T-eventually time-invariant with $T = 2$, results in state-control trajectories shown in Fig. 2; compared to those in Fig. 1, these trajectories clearly illustrate the striking improvement in transient responses that constrained controllers can make (while maintaining closed-loop stability and essentially optimal steady-state performance after reaching time-invariance).

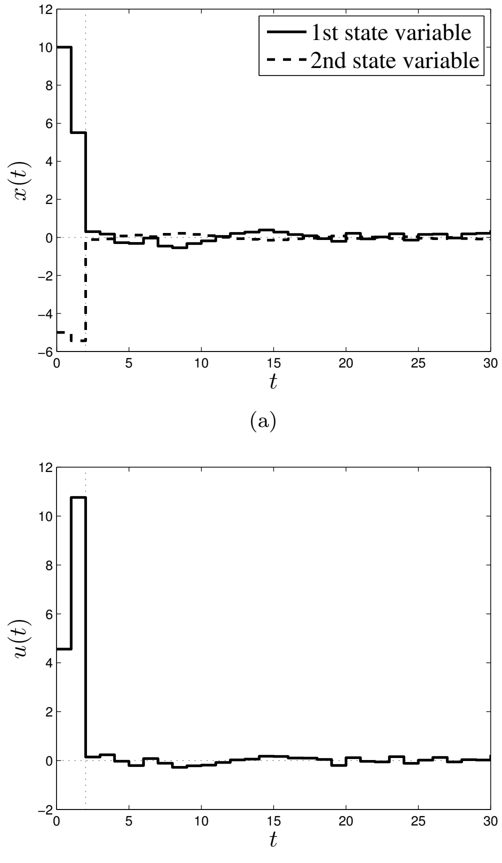


Fig. 2. First example: closed-loop state-control trajectories under the T-eventually time-invariant optimal controller, where $T = 2$. (a) Constrained state trajectory. (b) Constrained control trajectory.

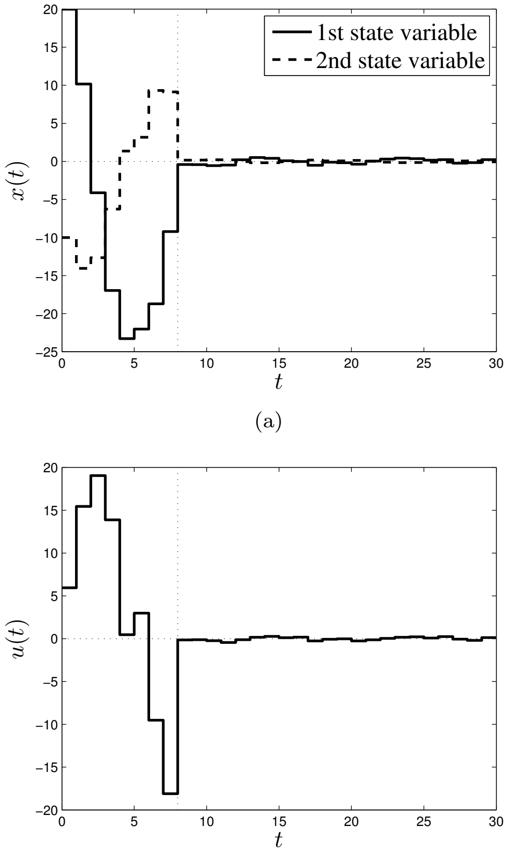
Now, change γ_1 and $\mathbf{P}$ to

$$\gamma_1 = 23.14 \quad \text{and} \quad \mathbf{P} = \begin{bmatrix} 400 & -200 \\ -200 & 100 \end{bmatrix},$$

but keep everything else the same as before. Then the linear matrix inequalities (18) are infeasible for $0 \leq T \leq 3$. However, for $T > 3$, the best achievable performance is given by $\gamma_0 = 23.09$ if $T = 4$, $\gamma_0 = 21.89$ if $T = 5$, $\gamma_0 = 18.05$ if $T = 6$, $\gamma_0 = 11.19$ if $T = 7$, and $\gamma_0 = 0.3864$ if $8 \leq T \leq 50$. Thus, we determine a near-optimal controller is T-eventually time-invariant with $T = 8$. As expected, this controller exhibits the same performance γ_0^* as in the previous case; however, the number of time steps until the controller reaches time-invariance is larger in this case. This shows that, similarly to the case of constrained LQR problem where larger initial state leads to higher computational complexity [Scokaert and Rawlings, 1998, Bemporad et al., 2002, Lee and Khargonekar, 2007], the computational demand for constrained LQG control becomes higher as the initial state variance increases. Typical state-input trajectories under this controller are depicted in Fig. 3.

5. CONCLUSION

A novel approach of using semidefinite programming to solve the constrained LQG control problem was proposed.



(a)

(b)

Fig. 3. Second example: closed-loop state-control trajectories under the T-eventually time-invariant optimal controller, where $T = 8$. (a) Constrained state trajectory. (b) Constrained control trajectory.

Although the approach inherits the computational efficiency of semidefinite programs [Nesterov and Nemirovsky, 1994, Vandenberghe and Boyd, 1996], it also shares computational issues common to constrained optimal control problems. This is because the synthesis of eventually time-invariant feedback controllers involves iteratively determining the number of steps to reach time-invariance, which is analogous to iteratively determining the control horizon in model predictive control. Immediate extensions of this work include to consider mixed constraints that bound both steady-state and instantaneous output variances and/or covariances, and to compute Pareto optimal solutions to these bounds.

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