Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

State and Parameter Estimation for Systems in Non-canonical Adaptive Observer Form

Ivan Y. Tyukin * **Erik Steur** ** **Henk Nijmeijer** *** **Cees van Leeuwen** ****

* *Department of Mathematics, University of Leicester, Leicester, LE1 7RH, UK (Tel: +44-116-252-5106; e-mail: I.Tyukin@le.ac.uk); Laboratory for Perceptual Dynamics, RIKEN Brain Science Institute, Wako-shi, 351-0198, Japan; Department of Automation and Control Processes, Saint-Petersburg State University of Electrical Engineering, Saint-Petersburg, 197376, Russia*
** *Department of Mechanical Engineering, Eindhoven University of Technology, P.O. Box 513 5600 MB, Eindhoven, The Netherlands, (e-mail: e.steur@student.tue.nl)*
*** *Department of Mechanical Engineering, Eindhoven University of Technology, P.O. Box 513 5600 MB, Eindhoven, The Netherlands, (e-mail: h.nijmeijer@tue.nl)*
**** *Laboratory for Perceptual Dynamics, RIKEN Brain Science Institute, Wako-shi, 351-0198, Japan, (e-mail: ceesvl@brain.riken.jp)*

Abstract: We consider the problem of state and parameter reconstruction for uncertain dynamical systems that cannot be transformed into the canonical adaptive observer form. The uncertainties are allowed to be both linearly and nonlinearly parameterized functions of state and time. We provide a technique that allows successful reconstruction of uncertain state and parameters for a broad range of dynamical systems that belong to this class. In contrast to conventional approaches our technique is based on the concepts of weakly attracting sets, and non-uniform convergence and Poisson stability rather than the notion of Lyapunov stability. Relevance of the proposed approach to the domains of control and system identification is illustrated with examples.

Keywords: Adaptive observers, nonlinear parametrization, weakly attracting sets, unstable attractors, nonlinear systems, Poisson stability

1. INTRODUCTION

Reconstructing internal variables of a system from available input-output data is a frequent requirement in wide areas of science and engineering. Various mathematical formulations of this problem received substantial attention in the past giving rise to the development of powerful techniques for *system identification* (Eykhoff 1975), (Ljung 1987), (Soderstrom & Stoica 1988), (Deistler 1989), (Sastry & Bodson 1989) and *observer design* in both adaptive (Kreisselmeier 1977), (Bastin & Gevers 1988), (Marino 1990) and non-adaptive statements (Isidori 1989), (Nijmeijer & van der Schaft 1990), (O'Reilly 1983).

In the domain of system identification purposeful variations of input signals enabling efficient solution to the problem of identification are often allowed (Gevers & Ljung 1986), (Hildebrand & Gevers 2003). While this strategy is successful in broad areas of engineering applications, its application in control and mathematical modeling of biological and fragile physical systems is restricted. In these areas inputs to the system are given and their shape and amplitude must not be altered by an identification procedure. Hence passive, adaptive observer-based solutions are needed for this class of systems.

Presently available observer-based solutions to the problem of simultaneous state and parameter estimation assume that equations which govern dynamics of the original system can be non-singularly transformed into the *canonical adaptive observer form* (Bastin & Gevers 1988):

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{R}\mathbf{x} + \boldsymbol{\varphi}(y(t),t)\boldsymbol{\theta} + \mathbf{g}(t) \\ \mathbf{R} &= \begin{pmatrix} 0 & \mathbf{k}^T \\ 0 & \mathbf{F} \end{pmatrix}, \ \mathbf{x} = (x_1,\dots,x_n) \\ y(t) &= x_1(t) \end{aligned} \tag{1}$$

In (1) functions $\mathbf{g}: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n$, $\boldsymbol{\varphi}: \mathbb{R}\times\mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n \times \mathbb{R}^d$ are assumed to be known, $\mathbf{k} = (k_1,\dots,k_{n-1})$ is a vector of *known* constants, $\mathbf{F}$ is a *known* $(n-1)\times(n-1)$ matrix (usually diagonal) with eigenvalues in the left half-plane of the complex domain, and $\boldsymbol{\theta} \in \mathbb{R}^d$ is a vector of *unknown* parameters. Algorithms for asymptotic recovery of state variable and parameter vector $\boldsymbol{\theta}$ can be found in (Bastin &

Gevers 1988), (Marino & Tomei 1993*a*), (Marino & Tomei 1995) [1].

There are systems, however, that cannot be transformed into the canonical adaptive observer form specified by equation (1). This is the case, for instance, in the domains of physical and chemical kinetics (Gorban & Karlin 2005), (Bastin & Dochain 1990). Here the equations are of the following class

$$\begin{aligned} \dot{x}_i &= -\lambda_i x_i + \sum_{j=1}^{n} \phi_i(\mathbf{x}, \boldsymbol{\theta}), \ \phi_i : \mathbb{R}^n \times \mathbb{R}^d \rightarrow \mathbb{R}, \\ y(t) &= h(\mathbf{x}(t)), \ h : \mathbb{R}^n \rightarrow \mathbb{R} \end{aligned} \tag{2}$$

and parameters $\lambda_i \in \mathbb{R}_{>0}$ are the reaction rates or relaxations times that can change with time. An example is the following system of equations (Bykov, Volokitin & Treskov 1997)

$$\begin{aligned} \dot{x}_1 &= -\lambda_1 x_1 + \theta_1 (1 - x_2) \exp\left(\frac{x_1}{x_1 + \theta_2 x_1}\right) \\ \dot{x}_2 &= -\lambda_2 x_2 + (1 - x_2) \exp\left(\frac{x_1}{x_1 + \theta_2 x_1}\right) \\ y(t) &= x_1(t) \end{aligned} \tag{3}$$

describing an exothermic reaction in a well-stirred reactor. The measured variable, x_1, corresponds to the temperature, and the hidden variable is the degree of conversion.

Another example of a class of dynamical systems for which representation (1) might not hold are the models of evoked electrical activity in the membranes of neural cells (Koch 2002). Even the simplest of these, e.g. the FitzHugh-Nagumo or Hindmarsh-Rose oscillators (Hindmarsh & Rose 1984):

$$\begin{aligned} \dot{x}_1 &= -x_1^3 + 3x_1^2 + x_2 + u, \ u \in \mathbb{R}, \\ \dot{x}_2 &= 1 - 5x_1^2 - \lambda_2 x_2, \ \lambda_2 \in \mathbb{R}_{>0}, \\ y(t) &= x_1(t) \end{aligned} \tag{4}$$

cannot be reduced to (1) because the value of λ_2, the time constant of slow ionic currents, is hardly available for direct observations.

Last but not least is a class of mechanical oscillators with unknown damping coefficients and friction:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\lambda_2 x_2 - x_1 + f(x_1, x_2, \boldsymbol{\theta}) + u(t), \\ y(t) &= x_1(t), \end{aligned} \tag{5}$$

where $\lambda_2 \in \mathbb{R}_{>0}$ is an unknown damping parameter, function $f : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ models the influence of friction, and $u(t) : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ is an external forcing.

For all these classes of systems conventional adaptive observer-based techniques cannot be directly applied. Hence developing an alternative is needed. In our present work we concentrate on developing a method for parameter and state reconstruction that applies to a broad subclass of systems (2)–(5). We shall allow uncertainties in matrix $\mathbf{F}$ in (1) and nonlinear parametrization of the regressor $\boldsymbol{\varphi}(y, t)\boldsymbol{\theta}$. The method which we propose is based around the concepts of weakly attracting sets and relaxation times (Milnor 1985), (Gorban 1980), (Gorban 2004) rather than global Lyapunov stability of the observer dynamics. This is because, when general nonlinear parametrization is present multiple values of $\boldsymbol{\theta}$ may result in the same input-output properties of the observer. Hence ensuring global Lyapunov stability of the observer dynamics (including the estimates of parameter $\boldsymbol{\theta}$) is generally unlikely.

Mathematical machinery of our method is based on the results of (Tyukin, Steur, Nijmeijer & van Leeuwen 2008*b*) that allow studying asymptotic convergence in nonlinear systems beyond the framework of Lyapunov design and conventional small-gain theorems. With this result we show that, subject to the condition of uniform persistency of excitation (Loria & Panteley 2003) of a linearly parameterized regressor in the observer dynamics, it is possible to reconstruct both linear and nonlinear parameters of the uncertainty model with sufficient accuracy (Theorem 2). The latter automatically implies the possibility for asymptotic recovery of the state variables using the closed form solution of a linear ODE.

The paper is organized as follows. In Section 2 we define notation used throughout the paper. Section 3 describes formal statement of the problem, Section 4 contains main results. In Section 5 we provide a brief discussion, and Section 6 concludes the paper.

2. NOTATION

The following notational conventions are used throughout the paper.

- The symbol $\mathbb{R}$ denotes the real numbers, $\mathbb{R}_{>0} = \{x \in \mathbb{R} \mid x > 0\}$.
- The symbol $\mathbb{Z}$ denotes the set of integers.
- Consider the vector $\mathbf{x} \in \mathbb{R}^n$ that can be partitioned into two vectors $\mathbf{x}_1 \in \mathbb{R}^p$ and $\mathbf{x}_2 \in \mathbb{R}^q$, $p + q = n$, then $\oplus$ denotes their concatenation, i.e. $\mathbf{x}_1 \oplus \mathbf{x}_2 = \mathbf{x}$.
- For the sake of compactness we will use symbol e^s instead of $\exp(s)$ to denote the exponent of s.
- The Euclidian norm of $\mathbf{x} \in \mathbb{R}^n$ is denoted by $\|\mathbf{x}\|$.
- By $L^n_\infty[t_0, T]$, $t_0 \geq 0$, $T \geq t_0$ we denote the space of all functions $\mathbf{f} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n$ such that $\|\mathbf{f}\|_{\infty,[t_0,T]} = \operatorname{ess\,sup}\{\|\mathbf{f}(t)\|, t \in [t_0, T]\} < \infty$; $\|\mathbf{f}\|_{\infty,[t_0,T]}$ stands for the $L^n_\infty[t_0, T]$ norm of $\mathbf{f}(t)$.
- Finally, let $\epsilon \in \mathbb{R}_{>0}$, then $\|\mathbf{x}\|_\epsilon$ stands for the following:

$$\|\mathbf{x}\|_\epsilon = \begin{cases} \|\mathbf{x}\| - \epsilon, & \|\mathbf{x}\| > \epsilon, \\ 0, & \|\mathbf{x}\| \leq \epsilon. \end{cases}$$

3. PROBLEM FORMULATION

We consider the following class of nonlinear systems:

$$\begin{aligned} \dot{x}_0 &= \theta_0^T \boldsymbol{\phi}_0(x_0, t) + \sum_{i=1}^{n} x_i \\ \dot{x}_i &= -\lambda_{2i-1} x_i + \theta_i^T \boldsymbol{\phi}_i(x_0, \lambda_{2i}, t), \\ y &= x_0, \ x_i(t_0) = x_{0,i} \end{aligned} \tag{6}$$

where

$$\boldsymbol{\phi}_i : \mathbb{R} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^{d_i}, \ d_i \in \mathbb{N}/\{0\}, \ i = \{0, \ldots, n\}$$

are continuous functions. Variable x_0 in system (6) is considered to be an *output*, and variables x_i, $i \geq 1$ model internal states of the system. Parameters $\theta_i \in \mathbb{R}^{d_i}$ are the

[1] See also (Marino & Tomei 1993*b*) for the treatment of this problem when regressor $\boldsymbol{\varphi}(y(t), t)\boldsymbol{\theta}$ in (1) is replaced with a nonlinearly parameterized function of output y and parameter vector $\boldsymbol{\theta}$.

linear components of the uncertainties in the right-hand side of (6). Parameters $\lambda_{2i-1} \in \mathbb{R}_{>0}$, $i = \{1, \ldots, n\}$ are the time constants of the internal states, and parameters $\lambda_{2i} \in \mathbb{R}$ constitute the *nonlinearly parameterized* uncertainties.

We consider the case when the system's state $\mathbf{x} = x_0 \oplus x_1 \oplus \cdots \oplus x_n$ cannot measured explicitly, and only the values of output $y(t) = x_0(t)$ (6) are available. Functions $\boldsymbol{\phi}_i(x_0, \lambda_{2i}, t)$ are supposed to be known, and the actual values of parameters $\theta_0, \ldots, \theta_n$, $\lambda_1, \ldots, \lambda_{2n}$, are assumed to be unknown *a-priori*. We assume, however, that domains of admissible values of θ_i, λ_i are known. In particular, we consider the case when $\theta_{i,j} \in [\theta_{i,\min}, \theta_{i,\max}]$, $\lambda_i \in [\lambda_{i,\min}, \lambda_{i,\max}]$, and the values of $\theta_{i,\min}$, $\theta_{i,\max}$, $\lambda_{i,\min}$, $\lambda_{i,\max}$ are available.

For notational convenience we denote

$$\boldsymbol{\theta} = \theta_0 \oplus \theta_1 \oplus \cdots \oplus \theta_n, \ \boldsymbol{\lambda} = \lambda_1 \oplus \cdots \oplus \lambda_{2n},$$

and domains of admissible values for $\boldsymbol{\theta}$, $\boldsymbol{\lambda}$ are denoted by symbols Ω_θ and Ω_λ respectively.

We aim to derive an algorithm which is capable of reconstructing unknown state and parameters of system (6) from the values of $y(t) = x_0(t)$. In our present work we consider this problem within the framework of designing an adaptive observer for (6). In particular, we should find an auxiliary system

$$\begin{aligned} \dot{\mathbf{q}} &= \mathbf{f}(\mathbf{q}, y(t), t) \\ \mathbf{p} &= \mathbf{h}(\mathbf{q}), \end{aligned} \tag{7}$$

such that for some given $\delta \in \mathbb{R}_{>0}$ and all $t_0 \in \mathbb{R}_{\geq 0}$ the following property holds:

$$\exists\, t' \geq t_0 : \|\mathbf{p}(t) - \boldsymbol{\xi}\| \leq \delta, \ \forall\, t \geq t' \tag{8}$$

where $\boldsymbol{\xi} = \boldsymbol{\theta} \oplus \boldsymbol{\lambda}$.

4. MAIN RESULTS

Let us introduce the following function $\boldsymbol{\phi}(x_0, \boldsymbol{\lambda}, t) : \mathbb{R} \times \mathbb{R}^{2n} \times \mathbb{R}_{\geq 0} \to \mathbb{R}^d$, $d = \sum_{i=0}^{n} d_i$:

$$\begin{aligned} &\boldsymbol{\phi}(x_0, \boldsymbol{\lambda}, t) = \\ &\boldsymbol{\phi}_0(x_0, t) \bigoplus_{i=1}^{n} \int_0^t e^{-\lambda_{2i-1}(t-\tau)} \boldsymbol{\phi}_i(x_0(\tau), \lambda_{2i}, \tau) d\tau \end{aligned} \tag{9}$$

The function $\boldsymbol{\phi}(x_0, \boldsymbol{\lambda}, t)$ is a concatenation of $\boldsymbol{\phi}_0(\cdot)$ and integrals

$$\int_0^t e^{-\lambda_{2i-1}(t-\tau)} \boldsymbol{\phi}_i(x_0(\tau), \lambda_{2i}, \tau) d\tau. \tag{10}$$

Given that functions $\boldsymbol{\phi}_i(\cdot)$ are known and that the values of $x_0(\tau)$, $\tau \in [0, t]$ are available, integrals (10) can be explicitly calculated as functions of $\boldsymbol{\lambda}$, t. When $x_0(t)$ is periodic, bounded, and $\boldsymbol{\phi}_i(x_0, \lambda_{2i}, t)$ is locally Lipschitz in x_0 and periodic in t with the same period, function $\boldsymbol{\phi}_i(x_0(t), \lambda_{2i}, t)$ can be expressed in the form of the Fourier series:

$$\begin{aligned} &\boldsymbol{\phi}_i(x_0(t), \lambda_{2i}, t) = \frac{a_{i,0}(\lambda_{2i})}{2} \\ &+ \sum_{j=1}^{\infty} \left(a_{i,j}(\lambda_{2i}) \cos(\omega_j t) + b_{i,j}(\lambda_{2i}) \sin(\omega_j t)\right) \end{aligned} \tag{11}$$

Taking a finite number N of members from the series expansion (11) yields the following computationally effective approximation of (10):

$$\begin{aligned} &\int_0^t e^{-\lambda_{2i-1}(t-\tau)} \boldsymbol{\phi}_i(x_0(\tau), \lambda_{2i}, \tau) d\tau \simeq \frac{a_{0,i}(\lambda_{2i})}{2\lambda_{2i-1}} + \\ &\sum_{j=1}^{N} \frac{a_{i,j}(\lambda_{2i})}{\lambda_{2i-1}^2 + \omega_j^2} \left(\sin(\omega_j t)\omega_j + \lambda_{2i-1} \cos(\omega_j t)\right) \\ &+ \sum_{j=1}^{N} \frac{b_{i,j}(\lambda_{2i})}{\lambda_{2i-1}^2 + \omega_j^2} \left(-\cos(\omega_j t)\omega_j + \lambda_{2i-1} \sin(\omega_j t)\right) + \epsilon(t), \end{aligned} \tag{12}$$

where $\epsilon(t) : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is an exponentially decaying term. In case when functions $\boldsymbol{\phi}_i(x_0(t), \lambda_{2i}, t)$ are not periodic in t integrals (10) can be approximated as follows

$$\begin{aligned} &\int_0^t e^{-\lambda_{2i-1}(t-\tau)} \boldsymbol{\phi}_i(x_0(\tau), \lambda_{2i}, \tau) d\tau \simeq \\ &\int_{t-T}^t e^{-\lambda_{2i-1}(t-\tau)} \boldsymbol{\phi}_i(x_0(\tau), \lambda_{2i}, \tau), \end{aligned} \tag{13}$$

where $T \in \mathbb{R} > 0$ is sufficiently large.

Consider the following system

$$\begin{cases} \dot{\hat{x}}_0 = -\alpha \hat{x}_0 + \hat{\theta}^T \bar{\boldsymbol{\phi}}(x_0, \hat{\boldsymbol{\lambda}}, t) \\ \dot{\hat{\theta}} = -\gamma_\theta (\hat{x}_0 - x_0) \bar{\boldsymbol{\phi}}(x_0, \hat{\boldsymbol{\lambda}}, t), \ \gamma_\theta, \alpha \in \mathbb{R}_{>0} \end{cases} \tag{14}$$

where the function $\bar{\boldsymbol{\phi}}_0(x_0, \hat{\boldsymbol{\lambda}}, t)$ is a computationally realizable approximation of (10):

$$\|\bar{\boldsymbol{\phi}}(x_0, \hat{\boldsymbol{\lambda}}, t) - \boldsymbol{\phi}(x_0, \hat{\boldsymbol{\lambda}}, t)\| \leq \Delta, \ \Delta \in \mathbb{R}_{>0}, \tag{15}$$

and the components of vector $\hat{\boldsymbol{\lambda}} = \mathrm{col}(\hat{\lambda}_1, \ldots, \hat{\lambda}_{2n})$ evolve according to the following equations

$$\begin{aligned} \dot{\hat{x}}_{1,j} &= \gamma_j s(t) \left(\hat{x}_{1,j} - \hat{x}_{2,j} - \hat{x}_{1,j} \left(\hat{x}_{1,j}^2 + \hat{x}_{2,j}^2\right)\right) \\ \dot{\hat{x}}_{2,j} &= \gamma_j s(t) \left(\hat{x}_{1,j} + \hat{x}_{2,j} - \hat{x}_{2,j} \left(\hat{x}_{1,j}^2 + \hat{x}_{2,j}^2\right)\right) \\ \hat{\lambda}_j(\hat{x}_{1,j}) &= \lambda_{j,\min} + \frac{\lambda_{j,\max} - \lambda_{j,\min}}{2} (\hat{x}_{1,j} + 1), \\ s(t) &= \sigma(\|x_0(t) - \hat{x}_0(t)\|_\varepsilon), \ j = \{1, \ldots, 2n\} \end{aligned} \tag{16}$$

$$\hat{x}_{1,j}^2(t_0) + \hat{x}_{2,j}^2(t_0) = 1, \tag{17}$$

where $\sigma(\cdot) : \mathbb{R} \to \mathbb{R}_{\geq 0}$ is a bounded function, i.e. $\sigma(s) \leq S \in \mathbb{R}_{>0}$, and $|\sigma(s)| \leq |s|$ for all $s \in \mathbb{R}$. We set $\gamma_j \in \mathbb{R}_{>0}$ and let γ_j be *rationally-independent*:

$$\sum \gamma_j k_j \neq 0, \ \forall\, k_j \in \mathbb{Z}.$$

In our current contribution we prove that system (14), (16) can serve as the desired observer (7) for the class of systems specified by equations (6). Our result is based on the concept of non-uniform convergence (Milnor 1985), (Gorban 2004), non-uniform small-gain theorems (Tyukin et al. 2008*b*), and the notion of λ-uniform persistency of excitation (Loria & Panteley 2003):

Definition 1. (λ-uniform persistency of excitation). Let $\boldsymbol{\varphi} : \mathbb{R}_{\geq 0} \times \mathcal{D} \to \mathbb{R}^{n \times m}$ be a continuous function. We say that $\boldsymbol{\varphi}(t, \boldsymbol{\lambda})$ is λ-uniformly persistently exciting (λ-uPE) if there exist $\mu, L \in \mathbb{R}_{>0}$ such that for each $\boldsymbol{\lambda} \in \mathcal{D}$

$$\int_t^{t+L} \boldsymbol{\varphi}(t, \boldsymbol{\lambda}) \boldsymbol{\varphi}(t, \boldsymbol{\lambda})^T d\tau \geq \mu I \ \forall t \geq 0.$$

The latter notion, in contrast to conventional definitions of persistency of excitation, allows to deal with parameterized regressors $\boldsymbol{\varphi}(t, \boldsymbol{\lambda})$. This is essential for deriving asymptotic properties of our observer (14), (16). These properties are formulated in the theorem below:

Theorem 2. Let system (6), (14), (16) be given. Assume that function $\bar{\boldsymbol{\phi}}(x_0(t), \boldsymbol{\lambda}, t)$ is λ-uniformly persistently exciting and Lipschitz in $\boldsymbol{\lambda}$:

$$\|\bar{\boldsymbol{\phi}}(x_0(t), \boldsymbol{\lambda}, t) - \bar{\boldsymbol{\phi}}(x_0(t), \boldsymbol{\lambda}', t)\| \leq D\|\boldsymbol{\lambda} - \boldsymbol{\lambda}'\|. \quad (18)$$

Then there exist numbers $\varepsilon > 0$, $\gamma^* > 0$ such that for all $\gamma_i \in (0, \gamma^*]$:

1) trajectories of the closed loop system (14), (16) are bounded;
2) there exists $\boldsymbol{\lambda}^* \in \Omega_\lambda$: $\lim_{t\to\infty} \hat{\boldsymbol{\lambda}}(t) = \boldsymbol{\lambda}^*$;
3) there exist κ, $\delta > 0$ such that the following estimates hold:

$$\begin{aligned}&\limsup_{t\to\infty} \|\hat{\boldsymbol{\theta}}(t) - \boldsymbol{\theta}\| < \kappa(D\delta + 3\Delta),\\ &\lim_{t\to\infty} |\hat{\lambda}_j(t) - \lambda_j| < \delta, \ j = \{1, \dots, 2n\}\\ &\lim_{t\to\infty} \|\hat{x}_0(t) - x_0(t)\|_\varepsilon = 0\end{aligned} \quad (19)$$

Proof of Theorem 2. In order to prove the theorem we will need the following lemma.

Lemma 3. Let system (14),(16) be given, and function $\bar{\boldsymbol{\phi}}(x_0(t), \boldsymbol{\lambda}, t)$ be λ-uniformly persistently exciting. Then there exist numbers $\rho > 0$, $\varepsilon > 0$, $\bar{\gamma} > 0$, $c > 0$ such that for all $\gamma_i \in (0, \bar{\gamma})$ the following holds along the solutions of system (6), (14), (16):

$$\begin{aligned}\|x_0(t) - \hat{x}_0(t)\|_\varepsilon \leq\ & e^{-\rho(t-t_0)}\|x(t_0) - \hat{x}(t_0)\|_\varepsilon\\ &+ c\|\hat{\boldsymbol{\lambda}}(\tau) - \boldsymbol{\lambda}\|_{\infty,[t_0,t]}\end{aligned} \quad (20)$$

Proof of Lemma 3. Consider the following function

$$\boldsymbol{\eta}(\boldsymbol{\lambda}, t) = \bar{\boldsymbol{\phi}}(x_0(t), \boldsymbol{\lambda}, t) - \boldsymbol{\phi}(x_0(t), \boldsymbol{\lambda}, t) \quad (21)$$

According to (15), the function $\boldsymbol{\eta}(\boldsymbol{\lambda}, t)$ is bounded: $\|\boldsymbol{\eta}(\boldsymbol{\lambda}, \tau)\|_{\infty,[0,t]} \leq \Delta$. Let us now introduce the following vector

$$\mathbf{q} = (\hat{x}_0 - x_0) \oplus (\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}). \quad (22)$$

Taking into account (6), (14), its time-derivative can be expressed as follows:

$$\begin{aligned}\dot{q}_1 &= -\alpha\ q_1 + \hat{\boldsymbol{\theta}}^T \bar{\boldsymbol{\phi}}(x_0, \hat{\boldsymbol{\lambda}}, t) - \theta_0^T \boldsymbol{\phi}_0(x_0(t), t)\\ &\quad - \sum_{j=1}^n x_j(t)\\ \dot{q}_i &= -\gamma_\theta\ q_1\ \bar{\phi}_i(x_0(t), \hat{\boldsymbol{\lambda}}, t),\ i = \{1, \dots, n\}\end{aligned} \quad (23)$$

Expressing trajectories $x_i(t)$ in (6) in the closed form

$$\begin{aligned}x_i(t) = &\ e^{-\lambda_{2i-1}(t-t_0)} x_i(t_0) +\\ &\theta_i^T \int_{t_0}^t e^{-\lambda_{2i-1}(t-\tau)} \boldsymbol{\phi}_i(x_0(\tau), \lambda_{2i}, \tau) d\tau,\end{aligned}$$

and taking (9), (21) into account, we can rewrite (23) as

$$\begin{aligned}\dot{q}_1 &= -\alpha\ q_1 + (\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})^T \bar{\boldsymbol{\phi}}(x_0, \hat{\boldsymbol{\lambda}}, t) + u(\hat{\boldsymbol{\lambda}}, \boldsymbol{\lambda}, t)\\ \dot{q}_i &= -\gamma_\theta\ q_1\ \bar{\phi}_i(x_0(t), \hat{\boldsymbol{\lambda}}, t),\ i = \{1, \dots, n\},\end{aligned} \quad (24)$$

where

$$\begin{aligned}u(\hat{\boldsymbol{\lambda}}, \boldsymbol{\lambda}, t) = &\ \boldsymbol{\theta}^T(\bar{\boldsymbol{\phi}}(x_0, \hat{\boldsymbol{\lambda}}, t) - \bar{\boldsymbol{\phi}}(x_0, \boldsymbol{\lambda}, t))\\ &+ \boldsymbol{\theta}^T \boldsymbol{\eta}(t, \boldsymbol{\lambda}) + \epsilon(t),\end{aligned} \quad (25)$$

and $\epsilon(t)$ is a bounded and exponentially decaying term. Rewriting (24) in vector-matrix notation yields:

$$\dot{\mathbf{q}} = \mathbf{A}(\hat{\boldsymbol{\lambda}}(t), t)\mathbf{q} + \mathbf{b}\ u(\boldsymbol{\lambda}, \hat{\boldsymbol{\lambda}}, t), \quad (26)$$

where

$$\mathbf{A}(\hat{\boldsymbol{\lambda}}(t), t) = \begin{pmatrix} -\alpha & \bar{\boldsymbol{\phi}}(x_0(t), \hat{\boldsymbol{\lambda}}(t), t)^T \\ -\gamma_\theta \bar{\boldsymbol{\phi}}(x_0(t), \hat{\boldsymbol{\lambda}}(t), t), & 0 \end{pmatrix},$$

$$\mathbf{b} = (1, 0, \dots, 0)^T.$$

Let $\Phi(t)$ be a solution of

$$\dot{\Phi} = \mathbf{A}(\hat{\boldsymbol{\lambda}}(t), t)\Phi,\ \Phi(t_0) = \mathbf{I} \quad (27)$$

Then the solution of (26) is defined as

$$\mathbf{q}(t) = \Phi(t)\left(\mathbf{q}(t_0) + \int_{t_0}^t \Phi^{-1}(\tau)\mathbf{b}u(\hat{\boldsymbol{\lambda}}(\tau), \boldsymbol{\lambda}, \tau) d\tau\right), \quad t \geq t_0 \quad (28)$$

We are going to show that there exists $\bar{\gamma} \in \mathbb{R}_{>0}$ such that for all $\gamma_j \in [0, \bar{\gamma}]$ solutions of (26), (16) are bounded.

First, we notice that trajectories $\hat{x}_{1,j}(t)$, $\hat{x}_{2,j}(t)$ are globally bounded. Furthermore, the right-hand side of (16) is locally Lipschitz in $\hat{x}_{1,j}$, $\hat{x}_{2,j}$. Hence the following estimate holds:

$$\|\dot{\hat{\boldsymbol{\lambda}}}(t)\| \leq \gamma^* M,\ M \in \mathbb{R}_{>0}, \gamma^* = \max_j\{\gamma_j\}. \quad (29)$$

As follows from assumptions of the lemma, function $\bar{\boldsymbol{\phi}}(x_0(t), \boldsymbol{\lambda}, t)$ is λ-uniform PE. This implies existence of $L, \mu \in \mathbb{R}_{>0}$ such that

$$\begin{aligned}J(\boldsymbol{\lambda}, t) = \int_t^{t+L} \bar{\boldsymbol{\phi}}(x_0(\tau), \boldsymbol{\lambda}, \tau)\bar{\boldsymbol{\phi}}^T(x_0(\tau), \boldsymbol{\lambda}, \tau) d\tau \geq \mu I\\ \forall\ t > 0,\ \boldsymbol{\lambda} \in \Omega_\lambda\end{aligned} \quad (30)$$

Consider the following matrix:

$$\begin{aligned}&J(\hat{\boldsymbol{\lambda}}(t), t) - \int_t^{t+L} \bar{\boldsymbol{\phi}}(x_0(\tau), \hat{\boldsymbol{\lambda}}(\tau), \tau)\bar{\boldsymbol{\phi}}^T(x_0(\tau), \hat{\boldsymbol{\lambda}}(\tau), \tau) d\tau\\ &= \int_t^{t+L} (\bar{\boldsymbol{\phi}}(x_0(\tau), \hat{\boldsymbol{\lambda}}(t), \tau) - \bar{\boldsymbol{\phi}}(x_0(\tau), \hat{\boldsymbol{\lambda}}(\tau), \tau)) \times\\ &\bar{\boldsymbol{\phi}}^T(x_0(\tau), \hat{\boldsymbol{\lambda}}(t), \tau) d\tau + \int_t^{t+L} \bar{\boldsymbol{\phi}}(x_0(\tau), \hat{\boldsymbol{\lambda}}(\tau), \tau) \times\\ &(\bar{\boldsymbol{\phi}}(x_0(\tau), \hat{\boldsymbol{\lambda}}(t), \tau) - \bar{\boldsymbol{\phi}}(x_0(\tau), \hat{\boldsymbol{\lambda}}(\tau), \tau))^T d\tau\\ &= J_1(\hat{\boldsymbol{\lambda}}(t), t) + J_2(\hat{\boldsymbol{\lambda}}(t), t)\end{aligned} \quad (31)$$

Using inequality

$$\|H\mathbf{z}\| \leq \max_{k,l} |h_{k,l}|\|\mathbf{z}\|,\ H \in \mathbb{R}^{m\times n},\ \mathbf{z} \in \mathbb{R}^n$$

and that $\|\bar{\boldsymbol{\phi}}(x_0(t), \boldsymbol{\lambda}, t)\| \leq B$ for all $t \geq 0$, $\boldsymbol{\lambda} \in \Omega_\lambda$ we can conclude that matrix (31) satisfies

$$\begin{aligned}&\left|\mathbf{z}^T(J_1(\boldsymbol{\lambda}(t), t) + J_2(\boldsymbol{\lambda}(t), t))\mathbf{z}\right| \leq\\ &\leq 2BD\|\hat{\boldsymbol{\lambda}}(t) - \hat{\boldsymbol{\lambda}}(\tau)\|_{\infty,[t,t+L]}\|\mathbf{z}\|^2,\ D \in \mathbb{R}_{>0}.\end{aligned}$$

Hence

$$\begin{aligned}&\int_t^{t+L} \bar{\boldsymbol{\phi}}(x_0(\tau), \hat{\boldsymbol{\lambda}}(\tau), \tau)\bar{\boldsymbol{\phi}}^T(x_0(\tau), \hat{\boldsymbol{\lambda}}(\tau), \tau) d\tau\\ &\geq J(\hat{\boldsymbol{\lambda}}(t), t) - 2BD\|\hat{\boldsymbol{\lambda}}(t) - \hat{\boldsymbol{\lambda}}(\tau)\|_{\infty,[t,t+L]} I \geq\\ &(\mu - 2BD\|\hat{\boldsymbol{\lambda}}(t) - \hat{\boldsymbol{\lambda}}(\tau)\|_{\infty,[t,t+L]}) I\end{aligned} \quad (32)$$

Taking (29), (32) into account we can conclude that

$$\begin{aligned}&\int_t^{t+L} \bar{\boldsymbol{\phi}}(x_0(\tau), \hat{\boldsymbol{\lambda}}(\tau), \tau)\bar{\boldsymbol{\phi}}^T(x_0(\tau), \hat{\boldsymbol{\lambda}}(\tau), \tau) d\tau\\ &\geq (\mu - 2BDLM\gamma^*) I\end{aligned} \quad (33)$$

Then choosing γ_i such that

$$\gamma^* = \frac{\mu}{4BDLM} \quad (34)$$

we can ensure that

$$\int_t^{t+L} \bar{\boldsymbol{\phi}}(x_0(\tau), \hat{\boldsymbol{\lambda}}(\tau), \tau)\bar{\boldsymbol{\phi}}^T(x_0(\tau), \hat{\boldsymbol{\lambda}}(\tau), \tau)d\tau \geq \frac{\mu}{2} I \; \forall\, t \geq 0.$$

In other words the function $\bar{\boldsymbol{\phi}}(x_0(t), \hat{\boldsymbol{\lambda}}(t), t)$ is persistently exciting.

According to (Morgan & Narendra 1992), this implies that system

$$\dot{\mathbf{q}} = \mathbf{A}(\hat{\boldsymbol{\lambda}}(t), t)\mathbf{q}$$

is uniformly exponentially stable for all $\hat{\boldsymbol{\lambda}}(t)$ satisfying conditions (29), (34). Hence there exists $\rho \in \mathbb{R}_{>0}$ such that the following inequality holds

$$\|\Phi(t)\mathbf{q}(t_0)\| \leq e^{-\rho(t-t_0)}\|\mathbf{q}(t_0)\| \tag{35}$$

for all $\mathbf{q}(t_0) \in \mathbb{R}^n$, $t \geq t_0$, $t_0 \in \mathbb{R}$. Therefore, taking (28), (35) into account we can conclude that for any bounded and continuous function $u(t)$ solutions of system (26) satisfy the following estimate:

$$\begin{aligned}\|\mathbf{q}(t)\| &= \left\|\Phi(t)\mathbf{q}(t_0) + \int_{t_0}^t \Phi(t-\tau)\mathbf{b}u(\tau)d\tau\right\| \\ &\leq e^{-\rho(t-t_0)}\|\mathbf{q}(t_0)\| + \int_{t_0}^t e^{-\rho(t-\tau)}\|\mathbf{b}\||u(\tau)|d\tau. \end{aligned} \tag{36}$$

Taking notational agreement (22) into account and using (36) we can obtain:

$$\|\hat{\boldsymbol{\theta}}(t) - \boldsymbol{\theta}\| \leq e^{-\rho t}\|\mathbf{q}_0\| + \int_0^t e^{-\rho(t-\tau)}|u(\tau)|d\tau. \tag{37}$$

Notice that according to (18), (25) function $u(t)$ in (37) satisfies the following inequality:

$$|u(t)| \leq \|\boldsymbol{\theta}\|D\|\hat{\boldsymbol{\lambda}}(t) - \boldsymbol{\lambda}\| + \|\boldsymbol{\theta}\|\Delta + |\epsilon(t)|.$$

Hence

$$\|\hat{\boldsymbol{\theta}}(t) - \boldsymbol{\theta}\| \leq \|\boldsymbol{\theta}\|D\int_0^t e^{-\rho(t-\tau)}\|\hat{\boldsymbol{\lambda}}(\tau) - \boldsymbol{\lambda}\|d\tau + \frac{\|\boldsymbol{\theta}\|\Delta}{\rho} + \epsilon_1(t), \tag{38}$$

where $\epsilon_1(t)$ is an exponentially decaying term. Denoting $B_1 = \sup_{\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2 \in \Omega_\lambda} \|\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2\|$ we obtain the following estimate from (38):

$$\begin{aligned}\|\hat{\boldsymbol{\theta}}(t) - \boldsymbol{\theta}\| \leq & \frac{\|\boldsymbol{\theta}\|D}{\rho}\|\hat{\boldsymbol{\lambda}}(\tau) - \boldsymbol{\lambda}\|_{\infty,[t-T_0,t]} + \frac{\|\boldsymbol{\theta}\|\Delta}{\rho} \\ & + \frac{\|\boldsymbol{\theta}\|DB_1}{\rho}e^{-\rho T_0} + \epsilon_2(t),\end{aligned} \tag{39}$$

where $\epsilon_2(t)$ is an exponentially decaying function, and $T_0 \in \mathbb{R}_{>0}$. Equations (39), (34), and (29) imply existence of γ_0 such that for all $\gamma^* \leq \min\{\gamma_0, \mu/4BDLM\}$ the following holds:

$$\|\hat{\boldsymbol{\theta}}(t) - \boldsymbol{\theta}\| \leq \frac{\|\boldsymbol{\theta}\|D}{\rho}\|\hat{\boldsymbol{\lambda}}(t) - \boldsymbol{\lambda}\| + 2\frac{\|\boldsymbol{\theta}\|\Delta}{\rho} + \epsilon_2(t). \tag{40}$$

Combining the first equation in (24) with (40) we obtain desired estimate (20) (see, for example, (Tyukin, Tyukina & van Leeuwen n.d.) for details). *The lemma is proven.*

According to Lemma 3 there exists a non-empty interval $(0, \bar{\gamma})$ such that for all $\gamma_j \in (0, \bar{\gamma})$ solutions of system (6), (14), (16) satisfy inequality (20). On the other hand solutions of system

$$\begin{aligned}\dot{\hat{x}}_{1,j} &= \gamma_j\left(\hat{x}_{1,j} - \hat{x}_{2,j} - \hat{x}_{1,j}\left(\hat{x}_{1,j}^2 + \hat{x}_{2,j}^2\right)\right) \\ \dot{\hat{x}}_{2,j} &= \gamma_j\left(\hat{x}_{1,j} + \hat{x}_{2,j} - \hat{x}_{2,j}\left(\hat{x}_{1,j}^2 + \hat{x}_{2,j}^2\right)\right) \\ \hat{\lambda}_j(\hat{x}_{1,j}) &= \lambda_{j,\min} + \frac{\lambda_{j,\max} - \lambda_{j,\min}}{2}(\hat{x}_{1,j} + 1)\end{aligned} \tag{41}$$

with initial conditions (17) are forward-invariant on $\hat{x}_{1,j}^2(t) + \hat{x}_{2,j}^2(t) = 1$ and can be expressed as $\hat{x}_{1,j}(t) = \sin(\gamma_j t)$, $\hat{x}_{2,j}(t) = \cos(\gamma_j t)$. Taking into account that γ_j are rationally-independent we can conclude that trajectories $\hat{x}_{1,j}(t)$ densely fill an invariant n-dimensional tori (Arnold 1978), and system (41) with initial conditions (17) is Poisson-stable in $\Omega_x = \{\hat{x}_{1,j}, \hat{x}_{2,j} \in \mathbb{R} | \hat{x}_{1,j} \in [-1, 1]\}$. Therefore, according to (Tyukin et al. 2008*b*) there exists γ^* such that

$$\lim_{t\to\infty} \hat{\boldsymbol{\lambda}}(t) = \boldsymbol{\lambda}^*, \;\; \boldsymbol{\lambda}^* \in \Omega_\lambda \tag{42}$$

$$\lim_{t\to\infty} \|x_0(t) - \hat{x}_0(t)\|_\varepsilon = 0. \tag{43}$$

for all $\gamma_j \in (0, \gamma^*)$. Taking (40) into account and denoting $\kappa = \|\boldsymbol{\theta}\|/\rho$, $\|\boldsymbol{\lambda} - \boldsymbol{\lambda}^*\| = \delta/2$ we can conclude that

$$\exists\, t' > 0 : \; \|\hat{\boldsymbol{\theta}}(t) - \boldsymbol{\theta}\| \leq \kappa(D\delta + 3\Delta), \; \|\hat{\boldsymbol{\lambda}}(t) - \boldsymbol{\lambda}\| \leq \delta \; \forall\, t \geq t'.$$

The theorem is proven.

Theorem 2 assures that the estimates $\hat{\boldsymbol{\theta}}(t)$, $\hat{\boldsymbol{\lambda}}(t)$ converge to a neighborhood of the actual values $\boldsymbol{\theta}$, $\boldsymbol{\lambda}$ asymptotically. It does not specify, however, how precise these estimates are. Yet, taking properties (19) into account we can conclude that precision of estimating $\boldsymbol{\theta}$, $\boldsymbol{\lambda}$ depends merely on the precision of estimating the values of $\boldsymbol{\lambda}$.

To specify conditions ensuring that $\hat{\boldsymbol{\lambda}}$ are sufficiently close to $\boldsymbol{\lambda}$ consider dynamics of variable q_1:

$$\dot{q}_1 = -\alpha q_1 + f(\boldsymbol{\lambda}, \boldsymbol{\lambda}^*, t) + \varepsilon_\lambda(t),$$

where the function $f(\boldsymbol{\lambda}, \boldsymbol{\lambda}^*, t)$ is defined as

$$\begin{aligned} f(\boldsymbol{\lambda}, \boldsymbol{\lambda}^*, t) = & \bar{\boldsymbol{\phi}}(x_0, \boldsymbol{\lambda}^*, t)^T H \times \\ & \int_{t_0}^t \Phi(t-\tau)\mathbf{b}u(\boldsymbol{\lambda}, \boldsymbol{\lambda}^*, \tau)d\tau + u(\boldsymbol{\lambda}, \boldsymbol{\lambda}^*, t)\end{aligned} \tag{44}$$

$$H = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ & & \dots & & & \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

and $\varepsilon_\lambda(t)$ converges to zero asymptotically according to the conclusions of Theorem 2. When the value of $\boldsymbol{\lambda}$ coincides with that of $\boldsymbol{\lambda}^*$ the following equivalence holds (see (25)):

$$u(\boldsymbol{\lambda}, \boldsymbol{\lambda}^*, t) = \boldsymbol{\theta}^T\boldsymbol{\eta}(t, \boldsymbol{\lambda}) + \varepsilon(t). \tag{45}$$

Due to the presence of nonlinearly parameterized terms equivalence (45) may hold for multiple values of $\boldsymbol{\lambda}^*$. By symbol $E(\boldsymbol{\lambda})$ we denote the set of all $\boldsymbol{\lambda}^* \in \prod[\lambda_{i,\min}, \lambda_{i,\max}]$ for which equivalence (45) holds. Hence reconstruction of $\boldsymbol{\lambda}$ is achievable up to their equivalence classes $E(\boldsymbol{\lambda})$ at most.

Similar to systems with linearly parameterized regressors, in order to determine the possibility to distinguish between two vectors in nonlinearly parameterized regressors we a modified notion of nonlinear persistent excitation (Cao, Annaswamy & Kojic 2003):

$$\begin{aligned}&\exists\, T \in \mathbb{R}_{>0}, \; \delta \in \mathbb{R}_{>0} : \; \forall\, t_0 \geq 0 \;\exists\, t' \in [t_0, t_0 + T] \\ &|f(\boldsymbol{\lambda}, \boldsymbol{\lambda}^*, t')| \geq \delta\|\boldsymbol{\lambda}^*\|_{E(\boldsymbol{\lambda})} + \Delta^*, \;\; \Delta^* \in \mathbb{R}_{\geq 0}\end{aligned} \tag{46}$$

where $\|\boldsymbol{\lambda}^*\|_{E(\boldsymbol{\lambda})} = \inf_{\bar{\boldsymbol{\lambda}} \in E(\boldsymbol{\lambda})} \|\boldsymbol{\lambda}^* - \bar{\boldsymbol{\lambda}}\|$, and Δ^* is determined by term $\boldsymbol{\theta}^T\boldsymbol{\eta}(t, \boldsymbol{\lambda}) + \varepsilon(t)$. Given that $\varepsilon(t)$ is exponentially decaying, and that $\boldsymbol{\eta}(t, \boldsymbol{\lambda})$ can be made arbitrarily small we shall regard Δ^* as small.

According to (Tyukin, Prokhorov & van Leeuwen 2008*a*) (Theorem 1, Lemma 1, see also (Loria, Panteley, Popovic & Teel 2003)), variable $q_1(t)$ is persistently exciting for sufficiently small Δ^*. On the other hand, the value of ε in (19) can be made arbitrarily small as well (the smaller the Δ, γ_j the smaller the ε). Then, applying the same argument as in (Tyukin et al. 2008*a*) we can conclude that $\|\boldsymbol{\lambda}^*\|_{E(\boldsymbol{\lambda})}$ is bounded from above by a function of Δ^*, ε. Furthermore, the smaller the Δ^*, ε the smaller the distance $\|\boldsymbol{\lambda}^*\|_{E(\boldsymbol{\lambda})}$, and so is the for the estimate $\hat{\boldsymbol{\lambda}}(t)$ at $t \rightarrow \infty$.

5. DISCUSSION

In the previous section we provided a system that can track unknown parameters of the "master", equation (6), subject to the conditions of linear/nonlinear persistency of excitation for the corresponding regressors. State variables of the original system can be estimated by the solutions of

$$\dot{x}_i = -\hat{\lambda}_{2_i}\hat{x}_i + \hat{\theta}_i^T(t)\boldsymbol{\phi}_i(x_0, \hat{\lambda}_{2_{i-1}}, t). \qquad (47)$$

Because $\lambda_i(t) \rightarrow \lambda_i^* \in \mathbb{R}$, $\lambda_{2_i}^* \in \mathbb{R}_{>0}$ at $t \rightarrow \infty$ solutions of (47) will approach $x_i(t)$ at $t \rightarrow \infty$ in case the residual terms (45) are kept small.

In our effort to achieve our goal we have not used Lyapunov stability theory to ensure convergence of the estimates to the neighborhoods of the actual values of the parameters. The set to which the estimates converge is not stable in the sense of Lyapunov. It is, nevertheless, attracting and Poisson stable.

We would also like to stress that our present approach share much in spirit with the earlier works on the topics of universal adaptive stabilization (Pomet 1992), (Ilchman 1997), (Martensson 1985), (Martensson & Polderman 1993) and global nonlinear optimization (Shang & Wah 1996). In these references it was proposed to apporach the problems of adaptation and optimization using controllers and minimizers combining advantages of exponentially stable and searching dynamics. We also employ these ideas here, although targeting different problem and using different mathematical machinery.

6. CONCLUSION

In the present article we provided a solution to the problem of state and parameter reconstruction for uncertain dynamical systems that cannot be transformed into the canonical adaptive observer form. We established that uniform persistency of excitation of a linearly parameterized regressor in the observer combined with a nonlinear persistency of excitation condition for the nonlinearly parameterized part are sufficient to ensure successful reconstruction.

For the sake of simplicity we restricted our consideration to the case when nonlinear parameters in each equation of the original system were scalars. The method, however, can straightforwardly be extended to the cases in which these parameters are vectors.

We should comment that the amount of time required for convergence depends substantially on the number of uncertain time constants and nonlinear parameters. It does not depend on the dimension of the linearly parameterized part (subject to ensuring sufficient level of excitation). This implies that the method is most successful for systems in which dimension of the nonlinear uncertainties is low. The latter seems to be a natural price for reasonable generality of our approach. Whether there are less demanding and conservative solutions remains the subject for our future study.

ACKNOWLEDGMENT

The authors are grateful to Prof. A.N. Gorban, the University of Leicester, and V.A. Terekhov, Staint-Petersburg State University of Electrical Engineering, for their useful comments and numerous technical discussions during preparation of this work. The authors are also thankful to anonymous Referees for helpful remarks. The research was supported by a Royal Society International Joint Project grant, and partially supported by a grant from Royal Academy of Engineering and RFBR grant 8-08-00103-a.

REFERENCES

Arnold, V. (1978), *Mathematical Methods in Classical Mechanics*, Springer-Verlag.

Bastin, G. & Dochain, D. (1990), *On-line Estimation and Adaptive Control of Bioreactors*, Elseveir.

Bastin, G. & Gevers, M. (1988), 'Stable adaptive observers for nonlinear time-varying systems', *IEEE Trans. on Automatic Control* **33**(7), 650–658.

Bykov, V. I., Volokitin, E. P. & Treskov, S. A. (1997), 'Parametric analysis of the mathematical model of a nonisothermal well-stirred reactor', *Combustion, Explosion and Shock Waves* **33**(3), 294–300.

Cao, C., Annaswamy, A. & Kojic, A. (2003), 'Parameter convergence in nonlinearly parametrized systems', *IEEE Trans. on Automatic Control* **48**(3), 397–411.

Deistler, M. (1989), Linear system identification – a survey, *in* 'From Data to Model', Springer-Verlag.

Eykhoff, P. (1975), *System Identification. Parameter and State Estimation*, Univ. of Techn. Eindhoven.

Gevers, M. & Ljung, L. (1986), 'Optimal experiment designs with respect to the intended model application', *Automatica* **22**, 543–554.

Gorban, A. (1980), Slow relaxations and bifurcations of omega-limit sets of dynamical systems, PhD thesis, Kuibyshev, Russia.

Gorban, A. (2004), 'Singularities of transition processes in dynamical systems: Qualitative theory of critical delays electron', *Electr. J. Diff. Eqns. Monograph 5.* http://ejde.math.txstate.edu/Monographs/05/.

Gorban, A. & Karlin, I. (2005), *Invariant Manifolds for Physical and Chemical Kinetics*, Lecture Notes in Physics, Springer.

Hildebrand, R. & Gevers, M. (2003), 'Identification for control: Optimal input design with respect to a worst case ν-gap cost function', *SIAM Journal on Control and Optimization* **41**(5), 1586–1608.

Hindmarsh, J. & Rose, R. (1984), 'A model of neuronal bursting using three coupled first order differential equations', *Proc. R. Soc. Lond.* **B 221**(1222), 87–102.

Ilchman, A. (1997), 'Universal adaptive stabilization of nonlinear systems', *Dynamics and Control* (7), 199–213.

Isidori, A. (1989), *Nonlinear control systems: An Introduction*, second edn, Springer–Verlag.

Koch, C. (2002), *Biophysics of Computation. Information Processing in Signle Neurons*, Oxford University Press.

Kreisselmeier, G. (1977), 'Adaptive obsevers with exponential rate of convergence', *IEEE Trans. Automatic Control* **AC-22**, 2–8.

Ljung, L. (1987), *System Identification – Theory for the User*, Prentice Hall, Englewood Cliffs, New Jersy, USA.

Loria, A. & Panteley, E. (2003), 'Uniform exponential stability of linear time-varying systems: revisited', *Systems and Control Letters* **47**(1), 13–24.

Loria, A., Panteley, E., Popovic, D. & Teel, A. (2003), 'Persistency of excitation for uniform convergence in nonlinear control systems', *http://arxiv.org/abs/math/0301335*.

Marino, R. (1990), 'Adaptive observers for single output nonlinear systems', *IEEE Trans. Automatic Control* **35**(9), 1054–1058.

Marino, R. & Tomei, P. (1993*a*), 'Global adaptive output-feedback control of nonlinear systems, part I: Linear parameterization', *IEEE Trans. Automatic Control* **38**(1), 17–32.

Marino, R. & Tomei, P. (1993*b*), 'Global adaptive output-feedback control of nonlinear systems, part II: Nonlinear parameterization', *IEEE Trans. Automatic Control* **38**(1), 33–48.

Marino, R. & Tomei, P. (1995), 'Adaptive observers with arbitrary exponential rate of convergence for nonlinear systems', *IEEE Trans. Automatic Control* **40**(7), 1300–1304.

Martensson, B. (1985), 'The order of any stabilizing regulator is sufficient a priori information for adaptive stabilization', *Systems and Control Letters* **6**(2), 87–91.

Martensson, B. & Polderman, J. (1993), 'Correction and simplification to "the order of any stabilizing regulator is sufficient a priori information for adaptive stabilization"', *Systems and Control Letters* **20**(6), 465–470.

Milnor, J. (1985), 'On the concept of attractor', *Commun. Math. Phys.* **99**, 177–195.

Morgan, A. P. & Narendra, K. S. (1992), 'On the stability of nonautonomous differential equations $\dot{\mathbf{x}} = [\mathbf{A} + \mathbf{B}(t)]\mathbf{x}$ with skew symmetric matrix $\mathbf{B}(t)$', *SIAM J. Control and Optimization* **37**(9), 1343–1354.

Nijmeijer, H. & van der Schaft, A. (1990), *Nonlinear Dynamical Control Systems*, Springer–Verlag.

O'Reilly, J. (1983), *Observers for Linear Systems*, Academic Press.

Pomet, J.-B. (1992), Remarks on sufficient information for adaptive nonlinear regulation, *in* 'Proceedings of the 31-st IEEE Control and Decision Conference', pp. 1737–1739.

Sastry, S. & Bodson, M. (1989), *Adaptive Control: Stability, Convergense, and Robustness*, Prentice Hall.

Shang, Y. & Wah, B. W. (1996), 'Global optimization for neural network training', *Computer* **29**(3), 45–54.

Soderstrom, T. & Stoica, P. (1988), *System Identification*, Prentice Hall, Englewood Cliffs, New Jersy, USA.

Tyukin, I., Prokhorov, D. V. & van Leeuwen, C. (2008*a*), 'Adaptive classification of temporal signals in fixed-weights recurrent neural networks: an existence proof', *Neural Computation*. (prov. accepted, preprint available at http://arxiv.org/abs/0705.3370v1).

Tyukin, I., Steur, E., Nijmeijer, H. & van Leeuwen, C. (2008*b*), 'Non-uniform small-gain theorems for systems with unstable invariant sets', *SIAM Journal on Control and Optimization* **47**(2), 849–882.

Tyukin, I., Tyukina, T. & van Leeuwen, C. (n.d.), 'Invariant template matching in systems with spatiotemporal coding: a vote for instability', *http://arxiv.org/abs/cs.CV/0702082*.