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Non-uniform Small-gain Theorems for Systems with Critical and Slow Relaxations

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Abstract: We consider the problem of small-gain analysis of asymptotic behavior in interconnected nonlinear dynamic systems. Mathematical models of these systems are allowed to be uncertain and time-varying. In contrast to standard small-gain theorems that require global asymptotic stability of each interacting component in the absence of inputs, we consider interconnections of systems that can be critically stable and have infinite input-output L_∞ gains. For this class of systems we derive small-gain conditions specifying state boundedness of the interconnection. The estimates of the domain in which the system's state remains are also provided. Conditions that follow from the main results of our paper are non-uniform in space. That is they hold generally only for a set of initial conditions in the system's state space. We show that under some mild continuity restrictions this set has a non-zero volume, hence such bounded yet potentially globally unstable motions are realizable with a non-zero probability. Proposed results can be used for the design and analysis of intermittent, itinerant and meta-stable dynamics which is the case in the domains of control of chemical kinetics, biological and complex physical systems, and non-linear optimization. The main results are illustrated with simple examples, and relation of our results with the standard small-gain conditions is discussed.

Keywords: non-uniform convergence, non-uniform small-gain theorems, input-output stability

1. INTRODUCTION

Small-Gain theorems are widely recognized as effective tools for the analysis of asymptotic behavior of the cascades and interconnections of linear and nonlinear systems (Zames 1966), (Jiang, Teel & Praly 1994). They are especially advantageous in those situations when mathematical models of systems are uncertain, and only estimates of the input-output properties of each component are available. The latter property together with the notions of *input-output* and *input-to-state stability* (Zames 1966), (Sontag 1989), (Sontag & Wang 1996) makes the small-gain technique a promising instrument in the analysis of complex biological and physical systems, see for instance, (de Leenheer, Angeli & Sontag 2006), (de Leenheer, Angeli & Sontag 2005), (Sontag 2002).

Conventional small-gain results often require (global) Lyapunov asymptotic stability of unperturbed dynamics of each interacting subsystems (Sontag & Wang 1996). In addition, it is generally required that the input-output properties of interacting subsystems do not change with time.

Yet, there are physical and biological systems that fail to satisfy these requirements. This is the case, for instance, in the domain of kinetic networks where external parameters, e.g. temperature, pressure, affect the rates of reactions thus changing the input-output gains. Moreover, the target invariant sets in these systems could be unstable in the Lyapunov sense (Gorban 2004), (Gorban 1980).

Presently available small-gain results that consider non-uniformity in time and space are few (Karafyllis & Tsinias 2004), (Tyukin, Steur, Nijmeijer & van Leeuwen 2008), and none of them addresses these issues altogether. Hence new developments are needed to allow extending the power of small-gain analysis to systems with critical dynamics and time-varying input-output gains.

In the present paper we concentrate on the developing the small-gain results for a class of systems that contain the following prototype dynamics as a special case:

$$\dot{x}_1 = -\lambda_1(t)x_1 + c_1(x_2, t) + u \quad (1)$$

$$\dot{x}_2 = -\lambda_2(t)x_2 + c_2(x_1, t), \quad (2)$$

where the function $\lambda_1 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ is separated from zero, i.e. $\exists\ \lambda^* \in \mathbb{R}_{>0} :\ \lambda_1(t) \geq \lambda^*$, and $\lambda_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq} 0$ can assume zero values over $\mathbb{R}_{\geq 0}$. The functions $c_1, c_2 : \mathbb{R} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ are globally Lipschitz in x_1, x_2, and $c_2(x_1, t)$

is non-negative (non-positive) in x_1. Variable $u \in \mathbb{R}$ constitutes an external regulatory input.

The first subsystem, equation (1), represents an input-to-state stable component. The second, equation (2), is allowed to become critical over time. In other words, there exists a (potentially infinite) set of intervals $\mathcal{T}_i$ in $\mathbb{R}_{\geq 0}$ such that $0 \leq \lambda_2(\tau) \leq \rho_i$, $\rho_i \in \mathbb{R}_{\geq 0}$ for all $\tau \in \mathcal{T}_i$. Clearly, the lower bound for the input-output L_∞-gain of system (2) is proportional to $1/\rho_i$ over the time intervals $\mathcal{T}_i$. Hence the input-output gain of the second component might be arbitrary large, and conventional small-gain analysis will not be applicable.

In our present contribution we aim at developing tools for the analysis of asymptotic behavior of cascades (1), (2) in which one of the components can become critically stable with time. In contrast to previous studies our results are based around the concept of weakly attracting sets (Milnor 1985) and relaxation times (Gorban 1980, Gorban 2004) rather than the notion of uniform attraction in the state space (Guckenheimer & Holmes 2002). The machinery behind the proofs is similar to (Tyukin et al. 2008), where the singular case, i.e. when $\lambda_2(t) \equiv 0$, was considered. Here we extend these results to the non-singular and pre-critical interconnections thus allowing a substantially wider domain of applications.

The paper is organized as follows. Section 2 describes notational agreements. In Section 3 we specify the class of systems of our study and formally state the problem. Section 4 contains main results of our paper. Namely, Theorem 1 provides a set of general sufficient conditions for non-uniform convergence, and Corollary 4 shapes these conditions into the usual small-gain formulae for a wide class of nonlinear systems. Section 5 provides discussion of these results, and Section 6 concludes the paper. Proofs of the main results are provided in Appendices A and B.

2. NOTATION

Throughout the paper we use the following notational conventions.

- Symbol $\mathbb{R}$ denotes the field of real numbers, symbol $\mathbb{R}_+$ stands for the following subset of $\mathbb{R}$: $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}$; $\mathbb{N}$ and $\mathbb{Z}$ denote the set of natural numbers and its extension to the negative domain respectively.
- Symbol $\mathcal{C}^k$ denotes the space of functions that are at least k times differentiable.
- $\mathcal{K}$ denotes the class of all strictly increasing continuous functions $\kappa : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\kappa(0) = 0$. If, in addition, $\lim_{s\to\infty} \kappa(s) = \infty$ we say that $\kappa \in \mathcal{K}_\infty$.
- Symbol $\mathcal{KL}$ denotes the class of functions $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that $\beta(\cdot, s) \in \mathcal{K}$ for each $s \in \mathbb{R}_+$, and $\beta(r, \cdot)$ is monotonically decreasing to zero for each $r \in \mathbb{R}_+$.
- Let $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{x}$ can be partitioned into two vectors $\mathbf{x}_1 \in \mathbb{R}^q$, $\mathbf{x}_1 = (x_{11}, \ldots, x_{1q})^T$, $\mathbf{x}_2 \in \mathbb{R}^p$, $\mathbf{x}_2 = (x_{21}, \ldots, x_{2p})^T$ with $q + p = n$, then $\oplus$ denotes their concatenation: $\mathbf{x} = \mathbf{x}_1 \oplus \mathbf{x}_2$.
- The symbol $\|\mathbf{x}\|$ denotes the Euclidian norm in $\mathbf{x} \in \mathbb{R}^n$.
- By $L^n_\infty[t_0, T]$ we denote the space of all functions $\mathbf{f} : \mathbb{R}_+ \to \mathbb{R}^n$ such that $\|\mathbf{f}\|_{\infty,[t_0,T]} = \sup\{\|\mathbf{f}(t)\|, t \in [t_0, T]\} < \infty$, and $\|\mathbf{f}\|_{\infty,[t_0,T]}$ stands for the $L^n_\infty[t_0, T]$ norm of $\mathbf{f}(t)$.
- Let $\mathcal{A}$ be a set in $\mathbb{R}^n$ and $\|\cdot\|$ be the usual Euclidean norm in $\mathbb{R}^n$. By the symbol $\|\cdot\|_\mathcal{A}$ we denote the following induced norm:
$$\|\mathbf{x}\|_\mathcal{A} = \inf_{\mathbf{q}\in\mathcal{A}} \{\|\mathbf{x} - \mathbf{q}\|\}$$
- Let $\Delta \in \mathbb{R}_+$ then the notation $\|\mathbf{x}\|_{\mathcal{A}_\Delta}$ stands for the following equality:
$$\|\mathbf{x}\|_{\mathcal{A}_\Delta} = \begin{cases} \|\mathbf{x}\|_\mathcal{A} - \Delta, & \|\mathbf{x}\|_\mathcal{A} > \Delta \\ 0, & \|\mathbf{x}\|_\mathcal{A} \leq \Delta \end{cases}$$
- The symbol $\|\cdot\|_{\mathcal{A}_\infty,[t_0,t]}$ is defined as follows:
$$\|\mathbf{x}(\tau)\|_{\mathcal{A}_\infty,[t_0,t]} = \sup_{\tau\in[t_0,t]} \|\mathbf{x}(\tau)\|_\mathcal{A}$$

3. PROBLEM FORMULATION

Similar to (Tyukin et al. 2008), we consider a system that can be decomposed into two interconnected subsystems, $\mathcal{S}_a$ and $\mathcal{S}_w$:
$$\begin{aligned} \mathcal{S}_a &: (u_a, \mathbf{x}_0) \mapsto \mathbf{x}(t) \\ \mathcal{S}_w &: (u_w, \mathbf{z}_0) \mapsto \mathbf{z}(t) \end{aligned} \tag{3}$$
where $u_a \in \mathcal{U}_a \subseteq L_\infty[t_0, \infty]$, $u_w \in \mathcal{U}_w \subseteq L_\infty[t_0, \infty]$ are the spaces of inputs to $\mathcal{S}_a$ and $\mathcal{S}_w$, respectively $\mathbf{x}_0 \in \mathbb{R}^n$, $\mathbf{z}_0 \in \mathbb{R}^m$ represent initial conditions, and $\mathbf{x}(t) \in \mathcal{X} \subseteq L^n_\infty[t_0, \infty]$, $\mathbf{z}(t) \in \mathcal{Z} \subseteq L^m_\infty[t_0, \infty]$ are the system states.

System $\mathcal{S}_a$ represents the contracting dynamics. More precisely, we require that $\mathcal{S}_a$ is input-to-state stable (Sontag 1990) with respect to a compact set $\mathcal{A}$:

Assumption 1. (*Globally stable dynamics*).
$$\mathcal{S}_a : \quad \|\mathbf{x}(t)\|_\mathcal{A} \leq \beta(\|\mathbf{x}(t_0)\|_\mathcal{A}, t - t_0) + c\|u_a(t)\|_{\infty,[t_0,t]}, \quad \forall \quad t_0 \in \mathbb{R}_+,\ t \geq t_0 \tag{4}$$
where the function $\beta(\cdot, \cdot) \in \mathcal{KL}$, and $c > 0$ is some positive constant.

In what follows we will assume that the function $\beta(\cdot, \cdot)$ and constant c are known or can be estimated a-priori. Clearly, Assumption 1 holds for (1). In particular, when $\mathcal{A} = 0$ the function $\beta(\|\mathbf{x}(t_0)\|_\mathcal{A}, t - t_0)$ is defined as $\beta(|x_1(t_0)|, t - t_0) = e^{-\lambda^*(t-t_0)}|x_1(t_0)|$, and coefficient $c = C_1/\lambda^*$ where C_1 is the Lipschitz constant of $c_1(x_2, t)$ with respect to x_2.

The system $\mathcal{S}_w$ stands for a locally/critically stable compartment. We will restrict our attention to those systems $\mathcal{S}_w$ that satisfy the following constraints:

Assumption 2. (*Locally and/or critically stable dynamics*). The system $\mathcal{S}_w$ is forward-complete:
$$u_w(t) \in \mathcal{U}_w \Rightarrow \mathbf{z}(t) \in \mathcal{Z},$$
and there exists an "output" function $h : \mathbb{R}^m \to \mathbb{R}$, and functions $\gamma_0 \in \mathcal{K}_{\infty,e}$, $\beta_w(\cdot)$, $\alpha_w(\cdot) \in \mathcal{K}$, and constant $\beta^* \in \mathbb{R}_{\geq 0}$ such that $\forall\ t \geq t_0$, $t_0 \in \mathbb{R}_+$ the following inequality holds:
$$\begin{aligned} \mathcal{S}_w : \ & -\beta^*|h(\mathbf{z}(t_0))| \leq h(\mathbf{z}(t_0)) - h(\mathbf{z}(t)) \leq \\ & \beta_w(t - t_0)h(\mathbf{z}(t_0)) + \alpha_w(t - t_0)\gamma_0(\|u_w(\tau)\|_{\infty,[t_0,t]}). \end{aligned} \tag{5}$$

For the sake of simplicity of presentation we assume that the function $\gamma_0(\cdot)$ in (5) is Lipschitz:
$$|\gamma_0(s)| \leq D_{\gamma,0} \cdot |s| \tag{6}$$

Assumption 2, when applied to system (2) with non-positive $c_2(x_1,t)$, results in the following $\alpha_w(\cdot)$, $\beta_w(\cdot)$, β^*:

$$\alpha_w(T) = \frac{1}{\lambda_2}(1 - e^{-\lambda_2^* T}), \quad (7)$$

$$\beta_w(T) = (1 - e^{-\lambda_2^* T}), \quad (8)$$

$$\lambda_2^* = \min_{t \in \mathbb{R}_{\geq 0}} \{\lambda_2(t)\}$$

$$\beta^* = 1$$

Now consider the interconnection of (4), (5) with coupling $u_a(t) = h(\mathbf{z}(t))$, and $u_s(t) = \|\mathbf{x}(t)\|_{\mathcal{A}}$. Equations for the combined system can be written as:

$$\|\mathbf{x}(t)\|_{\mathcal{A}} \leq \beta(\|\mathbf{x}(t_0)\|_{\mathcal{A}}, t - t_0) + c\|h(\mathbf{z}(t))\|_{\infty,[t_0,t]}; \quad (9)$$

$$\begin{aligned} -\beta^*|h(\mathbf{z}(t_0))| &\leq h(\mathbf{z}(t_0)) - h(\mathbf{z}(t)) \leq \\ &\beta_w(t-t_0)h(\mathbf{z}(t_0)) + \alpha_w(t-t_0)\gamma_0(\|\mathbf{x}\|_{\mathcal{A}_\infty,[t_0,t]}). \end{aligned} \quad (10)$$

In what follows we aim to derive simple small-gain conditions for interconnection (9), (10) that can be used to determine state boundedness of the system. Given that conventional notion of the input-output gain hardly applies to subsystem $\mathcal{S}_w$ [1], we do not wish to present these conditions in the standard form, e.g. that *the loop gain is less than unit* (Zames 1966). We rather search for conditions that can be formulated as follows:

$$\frac{C_1}{\lambda_1^*} \cdot D_{\gamma,0} \cdot G(\lambda_1^*, \lambda_2^*) < 1, \quad (11)$$

where $G(\cdot)$ is a positive continuous function of λ_1^*, λ_2^*. Despite that the gains C_1/λ_1^*, $D_{\gamma,0}$ in (11) refer to the different spaces, equation (11) has familiar small-gain form. Small-gain like conditions (11) follow as a corollary (Corollary 4) from a more general statement (Theorem 1). Detailed formulations of these results are provided in the next section.

4. MAIN RESULTS

Before we formulate the main results of this section let us first comment briefly on the machinery of our analysis. First of all we introduce three sequences

$$\mathcal{S} = \{\sigma_i\}_{i=0}^\infty, \quad \Xi = \{\xi_i\}_{i=0}^\infty, \quad \mathcal{T} = \{\tau_i\}_{i=0}^\infty$$

The first sequence, $\mathcal{S}$, partitions the interval $[0, h(\mathbf{z}_0)]$, $h(\mathbf{z}_0) > 0$ into the union of shrinking subintervals H_i:

$$[0, h(\mathbf{z}_0)] = \cup_{i=0}^\infty H_i, \ H_i = [\sigma_{i+1}h(\mathbf{z}_0), \sigma_i h(\mathbf{z}_0)] \quad (12)$$

We define this property in the form of Condition 1

Condition 1. (Partition of $\mathbf{z}_0$). The sequence $\mathcal{S}$ is strictly monotone and converging

$$\{\sigma_n\}_{n=0}^\infty : \ \lim_{n\to\infty} \sigma_n = 0, \ \sigma_0 = 1 \quad (13)$$

Sequences Ξ and $\mathcal{T}$ are defined as follows:

Condition 2. (Rate of contraction, Part 1). For the given sequences Ξ, $\mathcal{T}$ and function $\beta(\cdot,\cdot) \in \mathcal{KL}$ in (4) the following inequality holds:

$$\beta(\cdot, T_i) \leq \xi_i \beta(\cdot, 0), \ \forall \ T_i \geq \tau_i \quad (14)$$

Given that $\beta(\cdot,\cdot) \in \mathcal{KL}$ such choice is always possible. Next, we introduce two systems of functions, Φ and Υ:

$$\Phi : \begin{array}{l} \phi_j(s) = \phi_{j-1} \circ \rho_{\phi,j}(\xi_{i-j} \cdot \beta(s,0)), \ j = 1,\dots,i \\ \phi_0(s) = \beta(s,0) \end{array} \quad (15)$$

[1] This is because the L_∞ gain of the loop may become infinite

$$\Upsilon : \begin{array}{l} \upsilon_j(s) = \phi_{j-1} \circ \rho_{\upsilon,j}(s), \ j = 1,\dots,i \\ \upsilon_0(s) = \beta(s,0) \end{array} \quad (16)$$

where the functions $\rho_{\phi,j}$, $\rho_{\upsilon,j} \in \mathcal{K}$ satisfy the following inequality

$$\phi_{j-1}(a+b) \leq \phi_{j-1} \circ \rho_{\phi,j}(a) + \phi_{j-1} \circ \rho_{\upsilon,j}(b) \quad (17)$$

Notice that in case $\beta(\cdot,0) \in \mathcal{K}_\infty$ the functions $\rho_{\phi,j}(\cdot)$, $\rho_{\upsilon,j}(\cdot)$ will always exist (Jiang et al. 1994). Finally, we impose the following constraints on Ξ, $\mathcal{T}$, and $\mathcal{S}$:

Condition 3. (Rate of contraction, Part 2). The sequences

$$\sigma_n^{-1} \cdot \phi_n(\|\mathbf{x}_0\|_{\mathcal{A}}), \quad \sigma_n^{-1} \cdot \left(\sum_{i=0}^n \upsilon_i(c|h(\mathbf{z}_0)|\sigma_{n-i})\right),$$

$n = 0,\dots,\infty$, are bounded from above, e.g. there exist functions $B_1(\|\mathbf{x}_0\|)$, $B_2(|h(\mathbf{z}_0)|, c)$ such that

$$\sigma_n^{-1} \cdot \phi_n(\|\mathbf{x}_0\|_{\mathcal{A}}) \leq B_1(\|\mathbf{x}_0\|_{\mathcal{A}}) \quad (18)$$

$$\sigma_n^{-1} \cdot \left(\sum_{i=0}^n \upsilon_i(c|h(\mathbf{z}_0)|\sigma_{n-i})\right) \leq B_2(|h(\mathbf{z}_0)|, c) \quad (19)$$

for all $n = 0, 1, \dots, \infty$

Conditions 1–3 are discussed in details in our earlier work (Tyukin et al. 2008) addressing the problem of asymptotic behavior in the critical, singular case when $\beta_w(\cdot) = 0$, and $\alpha_w(s) = s, \forall \ s \in \mathbb{R}_{\geq 0}$. It was also shown in (Tyukin et al. 2008) that for the functions $\beta(s,0)$ that are Lipschitz in s these conditions reduce to more transparent ones which can always be satisfied by an appropriate choice of sequences Ξ and $\mathcal{S}$. Here we show how these conditions can be used to address the issue of convergence for a wider class of interconnections (9), (10). The results are formulated in Theorem 1.

Theorem 1. (Non-uniform Small-gain Theorem 1). Let systems $\mathcal{S}_a$, $\mathcal{S}_w$ be given and satisfy Assumptions 1, 2. Consider their interconnection (9), (10) and suppose there exist sequences $\mathcal{S}$, Ξ, and $\mathcal{T}$ satisfying Conditions 1–3. In addition, suppose that the following conditions hold:

1) There exists a positive number $\Delta_0 > 0$ such that

$$\frac{\sigma_i - \sigma_{i+1}}{\sigma_i} \geq \alpha_w(\tau_i) + \Delta_0 \cdot \beta_w(\tau_i) \quad \forall \ i = 0,1,\dots,\infty \quad (20)$$

2) The set Ω_γ of all points $\mathbf{x}_0$, $\mathbf{z}_0$ satisfying the inequality

$$\begin{aligned} D_{\gamma,0}(B_1(\|\mathbf{x}_0\|_{\mathcal{A}}) + B_2(|h(\mathbf{z}_0)|, c^*) + \\ c^*|h(\mathbf{z}_0)|) \leq h(\mathbf{z}_0)\Delta_0, \end{aligned} \quad (21)$$

where $c^* = c(1+\beta^*)$ is not empty.

3) Partial sums of elements from $\mathcal{T}$ diverge:

$$\sum_{i=0}^\infty \tau_i = \infty \quad (22)$$

Then for all $\mathbf{x}_0$, $\mathbf{z}_0 \in \Omega_\gamma$ the state $\mathbf{x}(t,\mathbf{z}_0) \oplus \mathbf{z}(t,\mathbf{z}_0)$ of system (9), (10) converges into the set specified by (23)

$$\begin{aligned} \Omega_a = \{\mathbf{x} \in \mathcal{X}, \ \mathbf{z} \in \mathcal{Z} | \ \|\mathbf{x}\|_{\mathcal{A}} \leq c^* \cdot h(\mathbf{z}_0), \\ \mathbf{z} : \ h(\mathbf{z}) \in [0, h(\mathbf{z}_0)(1+\beta^*)]\} \end{aligned} \quad (23)$$

The proof of the theorem is provided in Appendix A.

Remark 2. As follows immediately from the proof, in case the dynamics of $\mathcal{S}_w$, instead of inequality (5), satisfies inequality

$$\begin{aligned}\mathcal{S}_w: \ & \beta^* h(\mathbf{z}(t_0)) \leq h(\mathbf{z}(t_0)) - h(\mathbf{z}(t)) \leq \\ & \beta_w(t-t_0)h(\mathbf{z}(t_0)) + \alpha_w(t-t_0)\gamma_0(\|u_w(\tau)\|_{\infty,[t_0,t]}),\end{aligned} \tag{24}$$

the value of c^* in (21), (23) can be set to $c^* = c$.

Remark 3. Conditions 1), 3) of the theorem can be easily checked for the given sequences $\mathcal{S}$, $\mathcal{T}$. Verifying condition 2), however, might be a nontrivial operation. Therefore, a simpler statement that does not involve explicit verification of condition 2) of Theorem 1 is desirable.

In what follows we will show that this goal can be achieved in case additional information about the function $\beta(\cdot,\cdot)$ is available. This information is the knowledge of functions $\beta_x(\cdot)$, $\beta_t(\cdot)$ in the following factorization:

$$\beta(\|\mathbf{x}\|_{\mathcal{A}}, t) \leq \beta_x(\|\mathbf{x}\|_{\mathcal{A}}) \cdot \beta_t(t), \tag{25}$$

where $\beta_x(\cdot) \in \mathcal{K}$ and $\beta_t(\cdot) \in \mathcal{C}^0$ is strictly decreasing [2] with

$$\lim_{t\to\infty} \beta_t(t) = 0 \tag{26}$$

It is shown in (Sontag 1998) (Lemma 8) that factorization (25) is always achievable for any $\mathcal{KL}$ function. In case the function $\beta_x(\cdot)$ in the factorization (25) is Lipschitz the conditions of Theorem 1 reduce to a single and easily verifiable inequality. Let us consider this case in detail.

Without loss of generality, we assume that the state $\mathbf{x}(t)$ of system $\mathcal{S}_a$ satisfies the following equation

$$\begin{aligned}\|\mathbf{x}(t)\|_{\mathcal{A}} \leq \|\mathbf{x}(t_0)\|_{\mathcal{A}} \cdot \beta_t(t-t_0) + \\ c \cdot \|h(\mathbf{z}(\tau,\mathbf{z}_0))\|_{\infty,[t_0,t]},\end{aligned} \tag{27}$$

where $\beta_t(0)$ is greater or equal to one. Given that $\beta_t(t)$ is strictly decreasing and continuous, there is a (continuous) mapping $\beta_t^{-1} : [0, \beta_t(0)] \mapsto \mathbb{R}_+$:

$$\beta_t^{-1} \circ \beta_t(t) = t, \ \forall\, t > 0 \tag{28}$$

The small-gain criterion for interconnection (9), (10) in which the dynamics of $\mathcal{S}_a$ is governed by (27) is provided below:

Corollary 4. (Non-Uniform Small-gain Theorem 2). Let interconnection (9), (10) be given, system $\mathcal{S}_a$ satisfy (27), and furthermore the following hold:

$$\begin{aligned}D_{\gamma,0} \cdot c^* \cdot \mathcal{G}(\kappa, d) < 1 \\ c^* = c(1+\beta^*)\end{aligned} \tag{29}$$

for some $\kappa \in (1,\infty)$, $d \in (0,1)$, where

$$\begin{aligned}&\mathcal{G}(\kappa,d) = \left(\beta_t(0)\left(1+\frac{k}{1-d}\right)+1\right)\Delta_0^{-1}(\kappa,d)\\ &\Delta_0(\kappa,d) =\\ &\alpha_w\left(\beta_t^{-1}\left(\frac{d}{\kappa}\right)\right)^{-1}\left[\frac{\kappa-1}{\kappa} - \beta_w\left(\beta_t^{-1}\left(\frac{d}{\kappa}\right)\right)\right],\end{aligned} \tag{30}$$

and $\Delta_0(\kappa,d) > 0$. Then

1) there is a non-empty domain Ω_γ such that for all initial conditions $\mathbf{x}_0$ $\mathbf{z}_0$ $\in \Omega_\gamma$ the state $\mathbf{x}(t,\mathbf{x}_0) \oplus \mathbf{z}(t,\mathbf{z}_0)$ of interconnection (9), (10) converges into the set (23);

2) domain Ω_γ, contains the set of points $\mathbf{x}_0$, $\mathbf{z}_0$ specified by the following inequality:

$$D_{\gamma,0} \leq \Delta_0(\kappa,d)h(\mathbf{z}_0) \times \frac{1}{\beta_t(0)\left(\|\mathbf{x}_0\|_{\mathcal{A}} + c^*\cdot|h(\mathbf{z}_0)|\left(1+\frac{\kappa}{1-d}\right)\right) + c^*|h(\mathbf{z}_0)|} \tag{31}$$

3) in case the function $h(\mathbf{z})$ in (9), (10) is continuous, the volume of the set Ω_γ is nonzero in $\mathbb{R}^n \oplus \mathbb{R}^m$.

Proof of the corollary is provided in Appendix B.

Remark 5. Conditions (29), (30) of Corollary 4 guarantee that trajectories starting in the domain specified by (31) converge into the set (23). In order to obtain less conservative estimates of the system parameters ensuring such convergence the following substitute

$$G = \min_{\kappa\in(1,\infty),\ d\in(0,1)} \{\mathcal{G}(\kappa,d) | \mathcal{G}(\kappa,d) \geq 0\}$$

can be used in (29) instead of $\mathcal{G}(\kappa,d)$.

Remark 6. It is also worth mentioning that estimate (30) has been derived for the case when the sequence $\mathcal{T}$ in the formulation of Theorem 1 consists of repeating constants $\tau_i = \tau^* \in \mathbb{R}$. It is therefore possible that these conditions may be further improved if we repeat this analysis in the broader classes of sequences $\mathcal{T}$ and $\mathcal{S}$.

In case the function $\beta_t(\cdot)$ in (27) is exponential, i.e. $\beta_t(t-t_0) = e^{-\lambda_1(t-t_0)}$, and h is generated by the following differential equation

$$\dot{h} = -\lambda_2 h - \gamma_0(\|\mathbf{x}(t)\|_{\mathcal{A}}), \tag{32}$$

conditions (29), (30) can be substantially simplified. In this case $\beta_t(0) = 1$, and

$$\beta_t^{-1}\left(\frac{d}{\kappa}\right) = -\frac{1}{\lambda_1}\ln\left(\frac{d}{\kappa}\right)$$

Hence

$$\begin{aligned}&\mathcal{G}(\kappa,d) = \left(2+\frac{k}{1-d}\right)\Delta_0^{-1}(\kappa,d)\\ &\Delta_0(\kappa,d) =\\ &\alpha_w\left(-\frac{1}{\lambda_1}\ln\left(\frac{d}{\kappa}\right)\right)^{-1}\left[\frac{\kappa-1}{\kappa} - \beta_w\left(-\frac{1}{\lambda_1}\ln\left(\frac{d}{\kappa}\right)\right)\right].\end{aligned}$$

According to (7), (8), functions $\alpha_w(\cdot)$, $\beta_w(\cdot)$ are given by

$$\alpha_w\left(\beta_t^{-1}\left(\frac{d}{\kappa}\right)\right) = \frac{1}{\lambda_2}\left(1-\left(\frac{d}{k}\right)^{\frac{\lambda_2}{\lambda_1}}\right)$$

$$\beta_w\left(\beta_t^{-1}\left(\frac{d}{\kappa}\right)\right) = 1-\left(\frac{d}{k}\right)^{\frac{\lambda_2}{\lambda_1}}.$$

Therefore we can conclude that

$$\Delta_0(\kappa,d) = \lambda_2\left[\frac{k-1}{k}\left(1-\left(\frac{d}{k}\right)^{\frac{\lambda_2}{\lambda_1}}\right)^{-1} - 1\right]$$

Thus equation (30) transforms into:

$$\mathcal{G}(\kappa,d) = \left(2+\frac{k}{1-d}\right)\frac{1}{\lambda_2}\left[\frac{k-1}{k}\left(1-\left(\frac{d}{k}\right)^{\frac{\lambda_2}{\lambda_1}}\right)^{-1} - 1\right]^{-1} \tag{33}$$

Furthermore, according to (32) the following holds:

[2] If $\beta_t(\cdot)$ is not strictly monotone, it can always be majorized by a strictly decreasing function

$$h(t_0) - h(t) = (1 - e^{-\lambda_2(t-t_0)})h(t_0) + \int_{t_0}^{t} e^{-\lambda_2(t-\tau)}\gamma_0(\|\mathbf{x}(\tau)\|_{\mathcal{A}})d\tau \geq (1 - e^{-\lambda_2(t-t_0)})h(t_0).$$

Hence for all nonnegative $h(t_0)$ we have that $h(t_0) - h(t) \geq 0$, and the value of c^* in (29) can be set to $c^* = c$. In this case equation (29) reduces to:

$$D_{\gamma,0} \cdot c \cdot \mathcal{G}(\kappa, d) < 1. \tag{34}$$

5. DISCUSSION

Let us analyze how the results of Theorem 1 and Corollary 4 are linked to the conventional small-gain statements. For this purpose we consider a simple example when the input-to-state stable component is defined as

$$\dot{x}_1 = -\lambda_1 x_1 + c_1 x_2, \ \lambda_1 \in \mathbb{R}_{>0}, \ c_1 \in \mathbb{R}, \tag{35}$$

and the "critical" subsystem is specified by

$$\dot{x}_2 = -\lambda_2 x_2 - D_{\gamma,0}|x_1|, \ \lambda_2 \in \mathbb{R}_{\geq 0}, \ D_{\gamma,0} \in \mathbb{R}_{\geq 0}, \tag{36}$$

The main difference between (35) and (36) is in that the value of λ_1 is strictly positive whereas the value of λ_2 can be set to zero.

Standard small-gain condition for the interconnection (35), (36) can be written as follows:

$$\begin{aligned} &\frac{|c_1|}{\lambda_1} \cdot \frac{D_{\gamma,0}}{\lambda_2} = D_{\gamma,0} \cdot G(\lambda_1, \lambda_2) < 1 \\ &G(\lambda_1, \lambda_2) = \frac{|c_1|}{\lambda_1}\frac{1}{\lambda_2} \end{aligned} \tag{37}$$

The non-uniform small-gain conditions that follow from Corollary 4 and (33), (34) are

$$D_{\gamma,0} \cdot G(\lambda_1, \lambda_2) < 1 \tag{38}$$

$$G(\lambda_1, \lambda_2) = \frac{|c_1|}{\lambda_1} \min_{\kappa \in (1,\infty), \ d \in (0,1)} \{\mathcal{G}(\kappa, d) | \mathcal{G}(\kappa, d) \geq 0\}$$

The smaller the value of $G(\lambda_1, \lambda_2)$ the wider the range of admissible values for $D_{\gamma,0}$.

Let us consider the case when the value of variable λ_2, the relaxation constant in (36), becomes small while all other parameters remain the same. In this case the value of $G(\lambda_1, \lambda_2)$ in (37) monotonically increases, and $\lim_{\lambda_2 \to 0} G(\lambda_1, \lambda_2) = \infty$. Hence condition (37) will eventually fail for small λ_2 and any fixed c_1, $D_{0,\gamma}$, λ_1.

In contrast to this, as can be easily derived from (38), (33), the non-uniform small-gain conditions will always hold, even for arbitrarily small λ_2 (see also fig. 1). And the set of initial conditions leading to the bounded solutions can be estimated by (31). Therefore, we can conclude that our results are advantageous over conventional small-gain statements when one of the interconnected systems operates in a vicinity of its critical state. Whether this tendency holds for all values of λ_2?

In order to answer to this question we plotted the diagrams of $G(\lambda_1, \lambda_2)/D_{0,\gamma}$ defined by (37) and (38) respectively (fig. 1). As we increase the value of λ_2 we find a point $\lambda_2 = \lambda^*$ at which the curves intersect, and for all $\lambda_2 > \lambda_2^*$ conventional small-gain theorems produce less conservative estimates. The latter suggests that neither of the tools should have a preference over the whole parameter space. Rather their combination shall be used to approach the problem of asymptotic behavior most effectively.

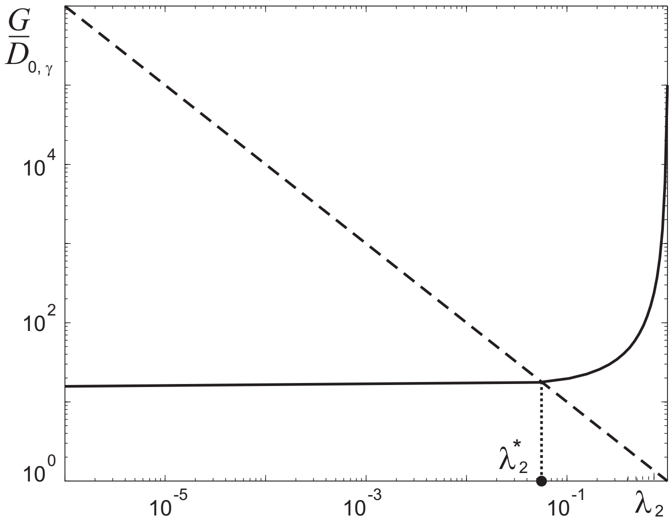

Fig. 1. The diagram of the *uniform* vs *non-uniform* relative loop gains $G(\lambda_1, \lambda_2)/D_{0,\gamma}$ as functions of λ_2. Solid thick line corresponds to the estimate of the non-uniform gain given by (38), and dashed thick line depicts conventional small-gain condition (37).

6. CONCLUSION

We proposed tools for the analysis of asymptotic behavior of a class of dynamical systems. In particular, we consider an interconnection of an input-to-state stable system with a system of which the dynamics is critically stable or in a vicinity of the critical regime. Our results allow not only to establish the fact of convergence to a given set from a set of initial conditions, they are also constructive. In particular, we provide the estimates of domains of initial conditions from which such convergence is guaranteed.

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Appendix A. PROOF OF THEOREM 1

Let the conditions of the theorem be satisfied for given $t_0 \in \mathbb{R}_+$: $\mathbf{x}(t_0) = \mathbf{x}_0$, $\mathbf{z}(t_0) = \mathbf{z}_0$. Notice that in this case $h(\mathbf{z}_0) \geq 0$, otherwise requirement (21) will be violated. Consider the following sequence of sets Ω_i induced by $\mathcal{S}$:

$$\Omega_i = \{\mathbf{x} \in \mathcal{X}, \ \mathbf{z} \in \mathcal{Z} | \ h(\mathbf{z}(t)) \in H_i\} \tag{A.1}$$

To prove the theorem we show that $0 \leq h(\mathbf{z}(t)) \leq h(\mathbf{z}_0)$ for all $t \geq t_0$. For the given partition (A.1) we consider two alternatives.

First, in the degenerative case, the state $\mathbf{x}(t) \oplus \mathbf{z}(t)$ enters the domain Ω_i, $i \geq 0$ at the time instant t_i and does not enter the set Ω_{i+1} for all $t \geq t_i$, e.g. $h(\mathbf{z}(t)) \geq \sigma_{i+1} h(\mathbf{z}(t_0))$ $\forall \ t \geq t_i$. Then according to (9), (10), this automatically implies that for all $t \geq t_i$ the following holds:

$$\sigma_{i+1} h(\mathbf{z}_0) \leq h(\mathbf{z}(t)) \leq h(\mathbf{z}_0)\sigma_i + \beta^* |h(\mathbf{z}_0)|\sigma_i \tag{A.2}$$

Hence, according to (4) trajectory $\mathbf{x}(t)$ satisfies the following inequality:

$$\|\mathbf{x}(t)\|_{\mathcal{A}} \leq \beta(\|\mathbf{x}_0\|_{\mathcal{A}}, t-t_0) + c(1+\beta^*)|h(\mathbf{z}_0)| \quad (A.3)$$

Thus

$$\limsup_{t\to\infty} \|\mathbf{x}(t)\|_{\mathcal{A}} = c(1+\beta^*)|h(\mathbf{z}_0)| \quad (A.4)$$

and the statements of the theorem hold.

Let us consider the second alternative, where the state $\mathbf{x}(t) \oplus \mathbf{z}(t)$ visits all domains Ω_j, $j = 1, \dots, \infty$. By t_j we denote the time instance of the first entry of $\mathbf{x}(t) \oplus \mathbf{z}(t)$ into the domain Ω_j. Clearly, t_j form an ordered sequence:

$$t_0 > t_1 > t_2 \cdots t_j > t_{j+1} \cdots, \quad (A.5)$$

and for every t_i we have that

$$h(\mathbf{z}(t_i)) = \sigma_i h(\mathbf{z}_0) \quad (A.6)$$

Hence to prove the theorem we must show that the sequence $\{t_i\}_{i=0}^{\infty}$ does not converge. In other words, the boundary $\sigma_\infty h(\mathbf{z}_0) = 0$ will not be reached in finite time.

In order to do this let us estimate the upper bounds for the following differences

$$T_i = t_{i+1} - t_i$$

Consider the following function

$$\varrho(T_i, \Delta_0) = \beta_w(T_i) + \Delta_0 \cdot \alpha_w(T_i). \quad (A.7)$$

Notice that the function $\varrho(\cdot,\cdot)$ is monotone in T_i and Δ_0. Taking into account (5), (A.7), and that $h(\mathbf{z}(t_i)) \neq 0$ we can derive that

$$\begin{aligned} &h(\mathbf{z}(t_i)) - h(\mathbf{z}(t_{i+1})) \leq \beta_w(T_i) h(\mathbf{z}(t_i)) + \\ &\alpha_w(T_i) D_{0,\gamma} \|\mathbf{x}(\tau)\|_{\mathcal{A}_\infty,[t_i,t_{i+1}]} \leq \\ &h(\mathbf{z}(t_i)) \varrho(T_i, h(\mathbf{z}(t_i))^{-1} D_{0,\gamma} \|\mathbf{x}(\tau)\|_{\mathcal{A}_\infty,[t_i,t_{i+1}]}) \end{aligned} \quad (A.8)$$

Given that $h(\mathbf{z}(t_i)) - h(\mathbf{z}(t_{i+1})) = (\sigma_i - \sigma_{i+1}) h(\mathbf{z}_0)$, inequality (A.8) results in the following estimate:

$$\begin{aligned} &\varrho(T_i, \sigma_i^{-1} h(\mathbf{z}_0)^{-1} D_{0,\gamma} \|\mathbf{x}(\tau)\|_{\mathcal{A}_\infty,[t_i,t_{i+1}]}) \geq \\ &\geq \frac{h(\mathbf{z}_0)(\sigma_i - \sigma_{i+1})}{h(\mathbf{z}_0)\sigma_i} = \frac{\sigma_i - \sigma_{i+1}}{\sigma_i} \end{aligned} \quad (A.9)$$

Taking into account that for all $t \in [t_i, t_{i+1}]$ estimate (A.2) holds, and using (4) we can bound the norm $\|\mathbf{x}(\tau)\|_{\mathcal{A}_\infty,[t_i,t_{i+1}]}$ as follows

$$\begin{aligned} &\|\mathbf{x}(\tau)\|_{\mathcal{A}_\infty,[t_i,t_{i+1}]} \leq \beta(\|\mathbf{x}(t_i)\|_{\mathcal{A}}, 0) + \|h(\mathbf{z}(\tau))\|_{\infty,[t_i,t_{i+1}]} \\ &\leq \beta(\|\mathbf{x}(t_i)\|_{\mathcal{A}}, 0) + c(1+\beta^*) \cdot \sigma_i h(\mathbf{z}_0) \end{aligned} \quad (A.10)$$

Hence, combining (A.9), (A.10), and taking into account that function $\varrho(\cdot, s)$ is non-decreasing in s, we obtain that

$$\begin{aligned} &\varrho(T_i, \sigma_i^{-1} h(\mathbf{z}_0)^{-1} D_{0,\gamma}(\beta(\|\mathbf{x}(t_i)\|_{\mathcal{A}}, 0) + \\ &c(1+\beta^*) \cdot \sigma_i h(\mathbf{z}_0))) \geq \\ &\varrho(T_i, \sigma_i^{-1} h(\mathbf{z}_0)^{-1} D_{0,\gamma} \|\mathbf{x}(\tau)\|_{\mathcal{A}_\infty,[t_i,t_{i+1}]}) \geq \frac{\sigma_i - \sigma_{i+1}}{\sigma_i} \end{aligned} \quad (A.11)$$

Regrouping the terms in (A.11) yields

$$\begin{aligned} &\varrho(T_i, h(\mathbf{z}_0)^{-1} D_{0,\gamma}(\sigma_i^{-1}\beta(\|\mathbf{x}(t_i)\|_{\mathcal{A}}, 0) + \\ &c(1+\beta^*) \cdot h(\mathbf{z}_0))) \geq \frac{\sigma_i - \sigma_{i+1}}{\sigma_i} \end{aligned} \quad (A.12)$$

Taking into account condition (22) of the theorem, the theorem will be proven if we assure that

$$T_i \geq \tau_i \quad (A.13)$$

for all $i = 0, 1, 2, \dots, \infty$. Noticing that the function $\varrho(T_i, \cdot)$ is nondecreasing in T_i we can conclude that inequality (A.13) is ensured if

$$\varrho(T_i, s) \geq \varrho(\tau_i, s) \quad (A.14)$$

for some $s \in \mathbb{R}_{\geq 0}$.

We prove this claim by induction with respect to the index $i = 0, 1, \dots, \infty$. We start with $i = 0$, and then show that for all $i > 0$ the following implication holds

$$T_i \geq \tau_i \Rightarrow \varrho(T_{i+1}, \Delta_0) \geq \varrho(\tau_{i+1}, \Delta_0) \quad (A.15)$$

Hence according to (A.14), $T_{t+1} \geq \tau_{i+1}$ and conclusions of the theorem follow.

Let us prove that (A.13) holds for $i = 0$. To this purpose consider the term $(\sigma_i - \sigma_{i+1})/\sigma_i$. As follows immediately from equation (20), we have that

$$\frac{\sigma_i - \sigma_{i+1}}{\sigma_i} \geq \beta_w(\tau_i) + \Delta_0 \cdot \alpha_w(\tau_i) \ \forall\ i \geq 0 \quad (A.16)$$

In particular

$$\frac{\sigma_0 - \sigma_1}{\sigma_0} \geq \beta_w(\tau_0) + \Delta_0 \cdot \alpha_w(\tau_0)$$

Therefore, inequality (A.12) reduces to

$$\begin{aligned} &\varrho(T_0, h(\mathbf{z}_0)^{-1} D_{0,\gamma}(\sigma_0^{-1}\beta(\|\mathbf{x}(t_0)\|_{\mathcal{A}}, 0) + \\ &c(1+\beta^*) \cdot h(\mathbf{z}_0))) \geq \frac{\sigma_0 - \sigma_1}{\sigma_0} \geq \\ &\beta_w(\tau_0) + \Delta_0 \cdot \alpha_w(\tau_0) = \varrho(\tau_0, \Delta_0) \end{aligned} \quad (A.17)$$

Taking into account Condition 3, (15), (16), and notation $c^* = c(1+\beta^*)$ we can derive the following estimate:

$$\begin{aligned} &\sigma_0^{-1}\beta(\|\mathbf{x}(t_0)\|_{\mathcal{A}}, 0) \leq \sigma_0^{-1}\phi_0(\|\mathbf{x}(t_0)\|_{\mathcal{A}}) + \\ &\sigma_0^{-1}\upsilon_0(c^* \cdot |h(\mathbf{z}_0)|\sigma_0) \leq B_1(\|\mathbf{x}_0\|_{\mathcal{A}}) + B_2(|h(\mathbf{z}_0)|, c^*) \end{aligned}$$

According to the theorem conditions $\mathbf{x}_0$ and $\mathbf{z}_0$ satisfy inequality (21). This in turn implies that

$$\begin{aligned} &D_{0,\gamma}(\sigma_0^{-1}\beta(\|\mathbf{x}(t_0)\|_{\mathcal{A}}, 0) + c^* \cdot h(\mathbf{z}_0)) \leq \\ &D_{0,\gamma}(B_1(\|\mathbf{x}_0\|_{\mathcal{A}}) + B_2(|h(\mathbf{z}_0)|, c^*) + \\ &c^* \cdot h(\mathbf{z}_0)) \leq \Delta_0 \cdot h(\mathbf{z}_0) \end{aligned} \quad (A.18)$$

Combining (A.17) and (A.18) we obtain the desired inequality

$$\begin{aligned} &\varrho(T_0, h(\mathbf{z}_0)^{-1}\Delta_0 h(\mathbf{z}_0)) = \varrho(T_0, \Delta_0) \geq \\ &\varrho(T_0, h(\mathbf{z}_0)^{-1} D_{0,\gamma}(\sigma_0^{-1}\beta(\|\mathbf{x}(t_0)\|_{\mathcal{A}}, 0) + \\ &c^* \cdot h(\mathbf{z}_0))) \geq \varrho(\tau_0, \Delta_0) \end{aligned}$$

Thus the basis of induction is proven.

Let us assume that (A.13) holds for all $i = 0, \dots, n$, $n \geq 0$. We shall prove now that implication (A.15) holds for $i = n+1$. Consider the term $\beta(\|\mathbf{x}(t_{n+1})\|_{\mathcal{A}}, 0)$:

$$\begin{aligned} &\beta(\|\mathbf{x}(t_{n+1})\|_{\mathcal{A}}, 0) \leq \beta(\beta(\|\mathbf{x}(t_n)\|_{\mathcal{A}}, T_n) + \\ &c\|h(\mathbf{z}(\tau))\|_{\infty,[t_n,t_{n+1}]}, 0) \leq \beta(\beta(\|\mathbf{x}(t_n)\|_{\mathcal{A}}, T_n) + \\ &c^* \cdot \sigma_n \cdot h(\mathbf{z}_0), 0) \end{aligned}$$

Taking into account Condition 2 (specifically, inequality (14)) and (15)–(17) we can derive that

$$\begin{aligned} &\beta(\|\mathbf{x}(t_{n+1})\|_{\mathcal{A}}, 0) \leq \beta(\xi_n \cdot \beta(\|\mathbf{x}(t_n)\|_{\mathcal{A}}), 0) + \\ &c^* \cdot \sigma_n \cdot h(\mathbf{z}_0), 0) \\ &\leq \phi_1(\|\mathbf{x}(t_n)\|_{\mathcal{A}}) + \upsilon_1(c^* \cdot |h(\mathbf{z})_0| \cdot \sigma_n) \end{aligned} \quad (A.19)$$

Notice that, according to the inductive hypothesis ($T_i \geq \tau_i$), the following holds

$$\|\mathbf{x}(t_{i+1})\|_{\mathcal{A}} \leq \beta(\|\mathbf{x}(t_i)\|_{\mathcal{A}}, T_i) + c^* \cdot \sigma_i \cdot h(\mathbf{z}_0)$$
$$\leq \xi_i \beta(\|\mathbf{x}(t_i)\|_{\mathcal{A}}, 0) + c^* \cdot \sigma_i \cdot h(\mathbf{z}_0) \quad \text{(A.20)}$$

for all $i = 0, \ldots, n$. Then (A.19), (A.20), (15)–(17) imply

$$\beta(\|\mathbf{x}(t_{n+1})\|_{\mathcal{A}}, 0) \leq$$
$$\phi_1(\xi_i \beta(\|\mathbf{x}(t_{n-1})\|_{\mathcal{A}}, 0) + c^* \cdot \sigma_{n-1} \cdot h(\mathbf{z}_0))$$
$$+\upsilon_1((c + \beta^*) \cdot |h(\mathbf{z})_0| \cdot \sigma_n) \leq \phi_2(\|\mathbf{x}(t_{n-1})\|_{\mathcal{A}}) +$$
$$\upsilon_2(c^* \cdot |h(\mathbf{z}_0)| \cdot \sigma_{n-1}) +$$
$$\upsilon_1(c^* \cdot |h(\mathbf{z}_0)| \cdot \sigma_n) \leq \phi_{n+1}(\|\mathbf{x}_0\|_{\mathcal{A}}) +$$
$$\sum_{i=1}^{n+1} \upsilon_i(c^* \cdot |h(\mathbf{z}_0)| \sigma_{n+1-i}) \leq \quad \text{(A.21)}$$
$$\phi_{n+1}(\|\mathbf{x}_0\|_{\mathcal{A}}) + \sum_{i=0}^{n+1} \upsilon_i(c^* \cdot |h(\mathbf{z}_0)| \sigma_{n+1-i})$$

According to Condition 3, term

$$\sigma_{n+1}^{-1}\left(\phi_{n+1}(\|\mathbf{x}_0\|_{\mathcal{A}}) + \sum_{i=0}^{n+1} \upsilon_i(c^* \cdot |h(\mathbf{z}_0)| \sigma_{n+1-i})\right)$$

is bounded from above by the sum

$$B_1(\|\mathbf{x}_0\|_{\mathcal{A}}) + B_2(|h(\mathbf{z}_0)|, c^*)$$

Therefore, estimate (A.21), and inequality (21) lead to the following inequality

$$D_{0,\gamma}(\sigma_{n+1}^{-1}\beta(\|\mathbf{x}(t_{n+1}\|_{\mathcal{A}}), 0) + c^* \cdot h(\mathbf{z}_0)) \leq$$
$$D_{0,\gamma}(B_1(\|\mathbf{x}_0\|_{\mathcal{A}}) + B_2(|h(\mathbf{z}_0)|, c^*) +$$
$$c^* \cdot h(\mathbf{z}_0)) \leq h(\mathbf{z}_0)\Delta_0$$

Hence, according to (A.12), (A.16) we have:

$$\varrho(T_{n+1}, \Delta_0) \geq \varrho(T_{n+1}, h(\mathbf{z}_0)^{-1} D_{0,\gamma}(\sigma_{n+1}^{-1}\beta(\|\mathbf{x}(t_{n+1}\|_{\mathcal{A}}), 0)$$
$$+ c^* \cdot h(\mathbf{z}_0))) \geq \frac{\sigma_{n+1} - \sigma_{n+2}}{\sigma_{n+1}} \geq \varrho(\tau_{n+1}, \Delta_0)$$

Thus implication (A.15) is proven. This implies that $h(\mathbf{z}(t)) \in [0, h(\mathbf{z}_0)(1+\beta^*)]$ for all $t \geq t_0$ and, consequently, that (A.4) holds. *The theorem is proven.*

Appendix B. PROOF OF COROLLARY 4

As follows from Theorem 1, the corollary will be proven if Conditions 1 – 3 are satisfied and also (20), (21), and (22) hold. In order to satisfy Condition 1 we select the following sequence $\mathcal{S}$:

$$\mathcal{S} = \{\sigma_i\}_{i=0}^{\infty}, \ \sigma_i = \frac{1}{\kappa^i}, \ \kappa \in \mathbb{R}_+, \ \kappa > 1 \quad \text{(B.1)}$$

Let us chose sequences $\mathcal{T}$ and Ξ as follows:

$$\mathcal{T} = \{\tau_i\}_{i=0}^{\infty}, \ \tau_i = \tau^*, \quad \text{(B.2)}$$
$$\Xi = \{\xi_i\}_{i=0}^{\infty}, \ \xi_i = \xi^*, \quad \text{(B.3)}$$

where τ^*, ξ^* are positive constants yet to be defined. Notice that choosing $\mathcal{T}$ as in (B.2) automatically fulfills condition (22) of Theorem 1. On the other hand, taking into account (14), (27) and that $\beta_t(t)$ is monotonically decreasing in t, this choice defines a constant ξ^* as follows:

$$\beta_t(\tau^*) \leq \xi^* \beta_t(0) < \beta_t(0), \ 0 \leq \xi^* < 1 \quad \text{(B.4)}$$

Given that the inverse β_t^{-1} exists, (28), this choice is always possible. In particular, (B.4) will be satisfied for the following values of τ^*:

$$\tau^* \geq \beta_t^{-1}(\xi^* \beta_t(0)) \quad \text{(B.5)}$$

Let us now find the values for τ^* and ξ^* such that Condition 3 is also satisfied. To this purpose consider systems of functions Φ, Υ specified by equations (15), (16). Notice that function $\beta(s, 0)$ in (15), (16) is linear for system (27)

$$\beta(s, 0) = s \cdot \beta_t(0),$$

and therefore the functions $\rho_{\phi,j}(\cdot)$, $\rho_{\upsilon,j}$ are identity maps. Hence, Φ, Υ reduce to the following

$$\Phi: \begin{array}{rl} \phi_j(s) & = \phi_{j-1} \cdot \xi^* \cdot \beta(s, 0) \\ & = \xi^* \cdot \beta_t(0) \cdot \phi_{j-1}(s), \ j = 1, \ldots, i \\ \phi_0(s) & = \beta_t(0) \cdot s \end{array} \quad \text{(B.6)}$$

$$\Upsilon: \begin{array}{rl} \upsilon_j(s) & = \phi_{j-1}(s), \ j = 1, \ldots, i \\ \upsilon_0(s) & = \beta_t(0) \cdot s \end{array} \quad \text{(B.7)}$$

Taking into account (B.1), (B.6), (B.7) let us explicitly formulate requirements (18), (19) in Condition 3. These conditions are equivalent to the boundedness of the following functions

$$\|\mathbf{x}(t_0)\|_{\mathcal{A}} \cdot \beta_t(0) \cdot \kappa^n (\xi^* \cdot \beta_t(0))^n; \quad \text{(B.8)}$$

$$\kappa^n \left(\beta_t(0)\frac{c^*|h(\mathbf{z}_0)|}{\kappa^n} + \frac{\beta_t(0)c^*|h(\mathbf{z}_0)|}{\kappa^{n-1}} + \right.$$
$$\left. \beta_t(0) \sum_{i=2}^{n} c^*|h(\mathbf{z}_0)| \frac{1}{k^{n-i}} (\xi^* \cdot \beta_t(0))^{i-1} \right)$$
$$= \beta_t(0)c^*|h(\mathbf{z}_0)| + \beta_t(0)c^*|h(\mathbf{z}_0)|\kappa \times$$
$$\left(1 + \sum_{i=2}^{n} \kappa^{i-1} (\xi^* \cdot \beta_t(0))^{i-1}\right) \quad \text{(B.9)}$$

Boundedness of the functions $B_1(\|\mathbf{x}_0\|_{\mathcal{A}})$, $B_2(|h(\mathbf{z}_0)|, c^*)$ is ensured if ξ^* satisfies the following inequality

$$\xi^* \leq \frac{d}{\kappa \cdot \beta_t(0)} \quad \text{(B.10)}$$

for some $0 \leq d < 1$. Notice that $\kappa > 1$, $\beta_t(0) \geq 1$ imply that $\xi^* \leq 1$ and therefore constant τ^* satisfying (B.5) will always be defied. Hence, according to (B.8), (B.9), the functions $B_1(\|\mathbf{x}_0\|_{\mathcal{A}})$ and $B_2(|h(\mathbf{z}_0)|, c^*)$ satisfying Condition 3 can be chosen as

$$B_1(\|\mathbf{x}_0\|_{\mathcal{A}}) = \beta_t(0)\, \|\mathbf{x}_0\|_{\mathcal{A}};$$
$$B_2(|h(\mathbf{z}_0)|, c) = \beta_t(0) \cdot c^* \cdot |h(\mathbf{z}_0)| \left(1 + \frac{\kappa}{1-d}\right) \quad \text{(B.11)}$$

In order to apply Theorem 1 we have to check the remaining conditions (20) and (21). According to (6) the function $\gamma_0(\cdot)$ is Lipschitz:

$$|\gamma_0(s)| \leq D_{\gamma,0} \cdot |s|$$

Condition (20), therefore, is equivalent to solvability of the following inequality:

$$\left(\frac{1}{\kappa^i} - \frac{1}{\kappa^{i+1}}\right) \kappa^i \geq \beta_w(\tau^*) + \Delta_0 \alpha_w(\tau^*) \quad \text{(B.12)}$$

Taking into account inequalities (B.5), (B.10) we can derive that existence of the positive solutions for Δ_0 of the following equality

$$\Delta_0(\kappa, d) = \alpha_w \left(\beta_t^{-1}\left(\frac{d}{\kappa}\right)\right)^{-1} \times$$
$$\left[\frac{\kappa - 1}{\kappa} - \beta_w \left(\beta_t^{-1}\left(\frac{d}{\kappa}\right)\right)\right] \quad \text{(B.13)}$$

implies existence of $\Delta_0 > 0$ satisfying (B.12) and, consequently, condition (20) of Theorem 1. Hence, the proof will be complete and the claim is non-vacuous if the domain

$$D_{\gamma,0} \leq \Delta_0(\kappa, d) h(\mathbf{z}_0) \frac{1}{\beta_t(0)\left(\|\mathbf{x}_0\|_{\mathcal{A}} + c^* \cdot |h(\mathbf{z}_0)| \left(1 + \frac{\kappa}{1-d}\right)\right) + c^*|h(\mathbf{z}_0)|} \tag{B.14}$$

is not empty.

Let us rewrite (B.14) as follows:

$$D_{\gamma,0}\beta_t(0)\|\mathbf{x}_0\|_{\mathcal{A}} + D_{\gamma,0}\beta_t(0) \cdot c^* \cdot |h(\mathbf{z}_0)| \left(1 + \frac{\kappa}{1-d}\right) + D_{\gamma,0}c^*|h(\mathbf{z}_0)| \leq \Delta_0(\kappa, d) \cdot h(\mathbf{z}_0) \tag{B.15}$$

Hence, without loss of generality, assuming that $h(\mathbf{z}_0) > 0$ we can rewrite (B.15) in the following way:

$$D_{\gamma,0} \cdot \beta_t(0)\|\mathbf{x}_0\|_{\mathcal{A}} \leq \left(\Delta_0(\kappa, d) - D_{\gamma,0} \cdot c^* \left(\beta_t(0) \cdot \left(1 + \frac{\kappa}{1-d}\right) + 1\right)\right) h(\mathbf{z}_0) \tag{B.16}$$

Solutions to (B.16) exist, however, if the inequality

$$\Delta_0(\kappa, d) \geq D_{\gamma,0} \cdot c^* \left(\beta_t(0) \cdot \left(1 + \frac{\kappa}{1-d}\right) + 1\right)$$

or, equivalently

$$D_{\gamma,0} \cdot c^* \cdot \mathcal{G}(\kappa, d) < 1 \tag{B.17}$$
$$\mathcal{G}(\kappa, d) = \left(\beta_t(0) \cdot \left(1 + \frac{\kappa}{1-d}\right) + 1\right) \cdot \Delta_0(\kappa, d)^{-1}$$

is satisfied. The estimate of the trapping region follows from (B.16).

Let us finally show that continuity of $h(\mathbf{z})$ implies that the volume of Ω_γ is nonzero in $\mathbb{R}^n \oplus \mathbb{R}^m$. For the sake of compactness we rewrite inequality (B.16) in the following form:

$$\|\mathbf{x}_0\|_{\mathcal{A}} \leq C_\gamma h(\mathbf{z}_0), \tag{B.18}$$

where C_γ is a constant depending on d, κ, $\beta_t(0)$, and $D_{\gamma,0}$. Given that (B.17) holds we can conclude that $C_\gamma > 0$. According to (B.18), domain Ω_γ contains the following set:

$$\{\mathbf{x}_0 \in \mathbb{R}^n,\ \mathbf{z}_0 \in \mathbb{R}^m |\ h(\mathbf{z}_0) > D_z \in \mathbb{R}_+,\ \|\mathbf{x}_0\|_{\mathcal{A}} \leq C_\gamma D_z\}$$

Consider the following domain: $\Omega_{\mathbf{x},\gamma} = \{\mathbf{x}_0 \in \mathbb{R}^n |\ \|\mathbf{x}_0\|_{\mathcal{A}} \leq C_\gamma D_z\}$. Clearly, it contains a point $\mathbf{x}_{0,1} \in \mathbb{R}^n :\ \|\mathbf{x}_{0,1}\|_{\mathcal{A}} = \frac{C_\gamma D_z}{2}$. For the point $\mathbf{x}_{0,1}$ and for all $\boldsymbol{\varepsilon}_1 \in \mathbb{R}^n :\ \|\boldsymbol{\varepsilon}_1\| \leq \frac{C_\gamma D_z}{4}$ we have that $\|\mathbf{x}_{0,1} + \boldsymbol{\varepsilon}_1\|_{\mathcal{A}} = \inf_{\mathbf{q}\in\mathcal{A}} \|\mathbf{x}_{0,1} + \boldsymbol{\varepsilon}_1 - \mathbf{q}\| \leq \inf_{\mathbf{q}\in\mathcal{A}}\{\|\mathbf{x}_{0,1} - \mathbf{q}\| + \|\boldsymbol{\varepsilon}_1\|\} \leq \frac{3C_\gamma D_z}{4}$. On the other hand $\|\mathbf{x}_{0,1} + \boldsymbol{\varepsilon}_1\|_{\mathcal{A}} = \inf_{\mathbf{q}\in\mathcal{A}} \|\mathbf{x}_{0,1} + \boldsymbol{\varepsilon}_1 - \mathbf{q}\| \geq \inf_{\mathbf{q}\in\mathcal{A}}\{\|\mathbf{x}_{0,1} - \mathbf{q}\| - \|\boldsymbol{\varepsilon}_1\|\} \geq \frac{C_\gamma D_z}{4}$. This implies that there exists a set of points $\mathbf{x}_{0,2} = \mathbf{x}_{0,1} + \boldsymbol{\varepsilon}_1 \in \mathbb{R}^n$: $\|\mathbf{x}_{0,1} - \mathbf{x}_{0,2}\| \leq \frac{C_\gamma D_z}{4}$, $\mathbf{x}_{0,2} \notin \mathcal{A}$, $\|\mathbf{x}_{0,2}\|_{\mathcal{A}} \leq C_\gamma D_z$.

Consider now the following domain: $\Omega_{\mathbf{z},\gamma} = \{\mathbf{z}_0 \in \mathbb{R}^m |\ h(\mathbf{z}_0) > D_z\}$. Let us pick $\mathbf{z}_{0,1} \in \Omega_{\mathbf{z},\gamma}$: $h(\mathbf{z}_{0,1}) = 2D_z$. Because $h(\cdot)$ is continuous we have that

$$\forall\, \varepsilon > 0,\ \exists\, \delta > 0 :\ \|\mathbf{z}_{0,1} - \mathbf{z}_{0,2}\| < \delta \Rightarrow |h(\mathbf{z}_{0,1}) - h(\mathbf{z}_{0,2})| < \varepsilon$$

Let $\varepsilon = D_z$, then $-D_z < h(\mathbf{z}_{0,1}) - h(\mathbf{z}_{0,2}) < D_z$ and therefore $h(\mathbf{z}_{0,2}) > D_z$. Hence there exists a set of points $\mathbf{z}_{0,2} \in \mathbb{R}^m$: $\|\mathbf{z}_{0,1} - \mathbf{z}_{0,2}\| < \delta$, $\mathbf{z}_{0,2} \in \Omega_{\mathbf{z},\gamma}$.

Consider the following set

$$\Omega_{\mathbf{xz},\gamma} = \left\{\mathbf{x}' \in \mathbb{R}^n,\ \mathbf{z}' \in \mathbb{R}^m |\ \|\mathbf{x}_{0,1} - \mathbf{x}'\|^2 + \|\mathbf{z}_{0,1} - \mathbf{z}'\|^2 \leq r^2,\ r = \min\left\{\delta, \frac{C_\gamma D_z}{4}\right\}\right\}$$

For all $\mathbf{x}_0, \mathbf{z}_0 \in \Omega_{\mathbf{xz},\gamma}$ we have that $\mathbf{x}_0 \in \Omega_{\mathbf{x},\gamma}$, $\mathbf{z}_0 \in \Omega_{\mathbf{z},\gamma}$. Hence, inequality (B.18) holds, and $\mathbf{x}_0 \oplus \mathbf{z}_0 \in \Omega_\gamma$. The volume of the set $\Omega_{\mathbf{xz},\gamma}$ is defined by the volume of the interior of a sphere in $\mathbb{R}^{n+m}$ with nonzero radius. Thus the volume of $\Omega_\gamma \supset \Omega_{\mathbf{xz},\gamma}$ is also nonzero. *The corollary is proven.*

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