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Global robust output regulation of nonlinear strict feedforward systems ⋆

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Abstract: This paper studies the global robust output regulation of nonlinear strict feedforward systems. By utilizing the general framework for tackling the output regulation problem , the output regulation problem is converted into a global robust stabilization problem for a general class of nonlinear feedforward systems that is subject to both dynamic uncertainty and time-varying static uncertainty.

1. INTRODUCTION

Output regulation problem of nonlinear systems has been one of the central control problems for nearly two decades Byrnes et al. [1997], Chen and Huang [2004], Ding [2001, 2006], Huang [2004], Huang and Chen [2004], Huang and Lin [1991], Huang and Rugh [1990, 1992], Isidori [1997], Isidori and Byrnes [1990], Khalil [1994, 2000], Pavlov et al. [2004, 2006], Serrani and Isidori [2000], Serrani et al. [2001]. The research was first focused on the local version of the problem where all the initial conditions and uncertain parameter are assumed to be sufficiently small Byrnes et al. [1997], Huang and Lin [1991], Huang and Rugh [1990, 1992], Isidori and Byrnes [1990], Pavlov et al. [2004]. The research on the nonlocal version of the problem started in the late 1990s Chen and Huang [2004], Ding [2001, 2006], Huang and Chen [2004], Isidori [1997], Khalil [1994, 2000], Pavlov et al. [2006], Serrani and Isidori [2000], Serrani et al. [2001]. It is now well known that the robust output regulation problem can be approached in two steps Huang and Chen [2004]. In the first step, the problem is converted into a robust stabilization problem of a so-called augmented system which consists of the original plant and a suitably defined dynamic system called an internal model candidate, and in the second step, the robust stabilization problem of the augmented system is further pursued. The success of the first step depends on whether or not an internal model candidate exists which can usually be ascertained by the property of the solution of the regulator equations. Even though the first step is succeeded, the success of the second step is by no means guaranteed due to at least two obstacles. First, the stabilizability of the augmented system is dictated not only by the given plant but also by the particular internal model candidate employed. An internal model candidate can be chosen from an infinite set of dynamic systems and a suitable internal model candidate is usually obtained from the past experience and some trial and error. Second, the structure of the augmented system may be much more complex than that of the original plant. Therefore, even though the stabilization of the original plant with the exogenous signal set to 0 is solvable, the stabilization of the augmented system may still be untractable. Perhaps, it is because of these difficulties, so far almost all papers on semi-global or global robust output regulation problem are focused on the lower triangular systems Chen and Huang [2004], Huang and Chen [2004], Isidori [1997], Khalil [1994, 2000], Serrani et al. [2001] and output feedback systems Ding [2001, 2006], Serrani and Isidori [2000].

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In this paper, we study the global robust output regulation problem of nonlinear systems in strict feedforward form:

$$\begin{aligned} \dot{x}_i &= f_i(x_1, \cdots, x_{i-1}, u, v, w),\ i = n, \cdots, 2 \\ \dot{x}_1 &= cu + f_1(v, w) \\ e &= x_1 - q_d(v, w) \end{aligned} \tag{1}$$

where $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ the control, $e \in \mathbb{R}$ the tracking error, $w \in \mathbb{R}^{n_w}$ the uncertain constant parameter, $v \in \mathbb{R}^q$ the state of the exosystem

$$\dot{v} = Sv \tag{2}$$

where all eigenvalues of the matrix S are simple with zero real parts, c is a known nonzero constant, and for $i = 1, \cdots, n$, the functions f_i and q_d are globally defined *smooth* functions satisfying $f_i(0, \cdots, 0, w) = 0$ and $q_d(0, w) = 0$ for all $w \in \mathbb{R}^{n_w}$.

Global robust output regulation problem (GRORP): For any compact set $V_0 \subset \mathbb{R}^q$ with a known bound and any compact set $W \subset \mathbb{R}^{n_w}$ with a known bound, design for system (1) a dynamic state feedback controller in the following form

$$u = \mathcal{K}(\eta, x, e),\ \dot{\eta} = \mathcal{F}(\eta, x, e) \tag{3}$$

where η is the compensator state and $\mathcal{K}, \mathcal{F}$ are locally *Lipschitz* functions vanishing at the origin, such that the closed-loop system composed of (1) and (3) has the following properties:

(a) For all $v(0) \in V_0, w \in W$ and for all initial state $x(0), \eta(0)$, the trajectory of the closed-loop system exists and is bounded for all $t \geq 0$;
(b) The tracking error converges to zero as t tends to infinity, i.e., $\lim_{t\to\infty} e(t) = 0$.

To our knowledge, the only papers that are relevant to the problem described above are Astolfi et al. [2005] and Marconi et al. [2002]. An approximate and restricted tracking problem for a class of block feedforward systems is studied in Astolfi et al. [2005] via dynamic output feedback control. The term approximate refers to the approximate regulation Huang and Rugh [1992] which is achieved by utilizing the k-fold internal model Huang [2004]. The term restricted refers to the fact that the state of the exosystem should be sufficiently small. In Marconi et al. [2002], the authors deal with the input disturbance suppression problem (IDSP) via dynamic state feedback control for a special case of (1) described by

$$\begin{aligned} \dot{x}_i &= w_{i-1}x_{i-1} + g_i(\dot{x}_1, \cdots, \dot{x}_{i-1}, w),\ i = n, \cdots, 2 \\ \dot{x}_1 &= u - g_1(v) \end{aligned} \tag{4}$$

where $w = (w_1, \cdots, w_{n-1})$ is the uncertain (possibly time varying) parameter and the functions $g_i, i = 2, \cdots, n$ are vanishing at $(0, \cdots, 0, w)$. The goal of IDSP is to achieve property (a) of GRORP and $\lim_{t\to\infty} x(t) = 0$. There is distinct difference between IDSP and GRORP. Roughly speaking (See Remark 3.4 for more specific comparison between IDSP and GRORP), for the IDSP, only one internal model associated with the input u needs to be constructed. The IDSP of system (4) can be converted into a global robust stabilization problem of a class of feedforward systems subject to input unmodeled dynamics. Several results about this robust stabilization problem have been reported, such as Arcak et al. [2001], Krstic [2004], Marconi and Isidori [2001], Sepulchre et al. [1997]. In contrast, for the GRORP, n internal models associated with the first $n-1$ state variables and the input u need to be constructed. The GRORP of system (1) can be converted into a global robust stabilization problem of an augmented system consisting of system (1) and the internal model. For this class of systems, even if an internal model candidate exists, how to choose a suitable internal model candidate and appropriate transformations, such that the system is stabilizable and the stabilization problem is solvable, is still a challenging problem.

In this paper, we will present a set of solvability conditions on the global robust output regulation problem of strict feedforward system (1). In order to obtain our results, we have to overcome the difficulties outlined above. First, we identify the structural properties of the functions q_d and f_i in (1) so that an internal model candidate exists. Then, by looking for a suitable internal model and performing appropriate transformations on the augmented system consisting of system (1) and the internal model, we succeed in converting the problem into a global robust stabilization problem of the following system

$$\begin{aligned} \dot{\bar{x}}_i &= \tilde{f}_i(\xi_1, \bar{x}_1, \cdots, \bar{x}_{i-1}, \xi_i, \bar{u}, d) \\ \dot{\xi}_i &= g_i(\xi_1, \bar{x}_1, \cdots, \bar{x}_{i-1}, \xi_i, \bar{u}, d),\ i = n, \cdots, 2 \\ \dot{\bar{x}}_1 &= \tilde{f}_1(\xi_1, \bar{u}, d) \\ \dot{\xi}_1 &= g_1(\xi_1, \bar{u}, d) \end{aligned} \tag{5}$$

where for $i = 1, \cdots, n$, $\bar{x}_i \in \mathbb{R}$, $\xi_i \in \mathbb{R}^{n_{\xi_i}}$, $d \in \mathbb{R}^{n_d}$, $\bar{u} \in \mathbb{R}$, $\tilde{f}_i$ and g_i are globally defined *smooth* functions satisfying $\tilde{f}_i(0, \cdots, 0, d) = 0$ and $g_i(0, \cdots, 0, d) = 0$ for all $d \in \mathcal{D}$, $d : [0, \infty) \to \mathcal{D}$ is a continuous function with its range $\mathcal{D}$ a compact subset of $\mathbb{R}^{n_d}$, where n_{ξ_i} and n_d are dimensions of ξ_i and d respectively. System (5) contains two types of uncertainties, i.e., static uncertainty represented by external disturbance d, and dynamic uncertainty represented by dynamics governing $\xi_1, \xi_2, \cdots, \xi_n$. The dynamics governing $\xi_1, \xi_2, \cdots, \xi_n$ are called dynamic uncertainty because the state of the dynamics are not allowed for feedback. The global robust stabilization problem of system (5) had not been studied before until recently Chen and Huang [2007] in which, a bottom-up recursive design procedure is presented to deal with the global robust stabilization problem of system (5). Two types of the small gain theorem with restrictions adapted from Teel [1996] is applied to establish the local stability and global attractiveness of the closed-loop system at the origin respectively.

2. PRELIMINARIES

Throughout the paper, we let $\mathcal{L}_\infty^m$ be the set of all piecewise continuous functions $u : [0, \infty) \to \mathbb{R}^m$ with a finite supremum norm $\|u\|_\infty = \sup_{t\geq 0} \|u(t)\|$, and let $\|u\|_a = \limsup_{t\to\infty} \|u(t)\|$ denote the asymptotic $\mathcal{L}_\infty$ norm of u, where $\|\cdot\|$ denotes the standard Euclidean norm. A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is called a gain function if it is continuous, nondecreasing, and satisfies $\gamma(0) = 0$. Let Id denote the gain function $\gamma(s) = s$.

We first review some terminologies introduced in Chen and Huang [2007] and Teel [1996] for nonlinear systems of the following form:

$$\dot{x} = f(x, u, d),\ y = h(x, u, d) \tag{6}$$

where $x \in \mathbb{R}^n$ is the plant state, $y \in \mathbb{R}^p$ the output, $u \in \mathbb{R}^m$ the piecewise continuous input, $f(x, u, d)$ and $h(x, u, d)$ are locally *Lipschitz* functions satisfying $f(0, 0, d) = 0$ and $h(0, 0, d) = 0$ for all $d \in \mathcal{D}$, and $d : [0, \infty) \to \mathcal{D}$ is a continuous function with its range $\mathcal{D}$ a compact subset of $\mathbb{R}^{n_d}$. Let $x(t)$ denote the solution of system (6) with initial state $x(0)$, input u and d.

Definition 2.1. Chen and Huang [2007] The output y of system (6) is said to satisfy a robust $\mathcal{L}_\infty$ stability bound (RLB) with restrictions X_s, Δ on $x(0), u$ and gains γ^0, γ respectively, if there exist open set X_s of the origin of $\mathbb{R}^n$, positive real number Δ, gain functions γ^0, γ, all independent of d, such that, for each $x(0) \in X_s$, $d \in \mathcal{D}$, $\|u\|_\infty < \Delta$, the solution of (6) exists for all $t \geq 0$ and

$$\|y\|_\infty \leq \max\{\gamma^0(\|x(0)\|), \gamma(\|u\|_\infty)\} \tag{7}$$

Definition 2.2. Teel [1996] The output y of system (6) is said to satisfy a robust asymptotic bound (RAB) with restriction X_a on $x(0)$, restriction Δ on u and gain γ, if there exist open set X_a of the origin of $\mathbb{R}^n$, non-negative real number Δ, gain function γ, all independent of d, such that, for each $x(0) \in X_a$, $d \in \mathcal{D}$ and piecewise continuous u satisfying $\|u\|_a \leq \Delta$, the solution of (6) exists for all $t \geq 0$ and

$$\|y\|_a \leq \gamma(\|u\|_a) \tag{8}$$

In the following, we restate the robust stabilization result obtained in Chen and Huang [2007] for system (5).

Define $A_1(d) = \frac{\partial g_1}{\partial \xi_1}|_{(0,0,d)}, B_1(d) = \frac{\partial g_1}{\partial \bar{u}}|_{(0,0,d)}, c_1(d) = \frac{\partial \tilde{f}_1}{\partial \bar{u}}|_{(0,0,d)}, D_1(d) = \frac{\partial \tilde{f}_1}{\partial \xi_1}|_{(0,0,d)}$ and for $i = 2, \cdots, n$,

$A_i(d) = \frac{\partial g_i}{\partial \xi_i}|_{(0,\cdots,0,d)}, B_i(d) = \frac{\partial g_i}{\partial \bar{x}_{i-1}}|_{(0,\cdots,0,d)}, c_i(d) = \frac{\partial \tilde{f}_i}{\partial \bar{x}_{i-1}}|_{(0,\cdots,0,d)}, D_i(d) = \frac{\partial \tilde{f}_i}{\partial \xi_i}|_{(0,\cdots,0,d)}$. To simplify the notation, we drop the argument d in the matrices defined above. Then system (5) can be rewritten in the following form:

$$\begin{aligned}\dot{\bar{x}}_i &= D_i\xi_i + c_i\bar{x}_{i-1} + \tilde{f}_i^r(\xi_1, \bar{x}_1, \cdots, \bar{x}_{i-1}, \xi_i, \bar{u}, d)\\ \dot{\xi}_i &= A_i\xi_i + B_i\bar{x}_{i-1} + g_i^r(\xi_1, \bar{x}_1, \cdots, \bar{x}_{i-1}, \xi_i, \bar{u}, d),\\ &\qquad i = n, \cdots, 2 \\ \dot{\bar{x}}_1 &= D_1\xi_1 + c_1\bar{u} + \tilde{f}_1^r(\xi_1, \bar{u}, d)\\ \dot{\xi}_1 &= A_1\xi_1 + B_1\bar{u} + g_1^r(\xi_1, \bar{u}, d)\end{aligned} \tag{9}$$

where $\tilde{f}_i^r, g_i^r$ are suitably defined *smooth* functions.

Assumption 2.1. For $i = 1, \cdots, n$, $\mu_i = c_i - D_iA_i^{-1}B_i$ is nonzero and does not change its sign for all $d \in \mathcal{D}$.

Assumption 2.2. ξ_1 satisfies RLB and RAB with no restriction on $\xi_1(0)$, both with nonzero restriction Δ_1 on u and gain $\bar{N}_1 \cdot Id$, and for $i = 2, \cdots, n$, ξ_i satisfies RLB and RAB with no restriction on $\xi_i(0)$, both with nonzero restriction Δ_i on $(\bar{x}_{i-1}, \xi_{i-1}, \cdots, \bar{x}_1, \xi_1, \bar{u})$ and gain $\bar{N}_i \cdot Id$.

Then, we can obtain the following result.

Theorem 2.1. Chen and Huang [2007] Consider system (5). Under Assumptions 2.1-2.2, there exist $\lambda_i > 0$ and nonzero k_i with the same sign as $\theta_i, i = 1, \cdots, n$ where $\theta_1 = \mu_1$ and $\theta_i = \mu_i/k_{i-1}, i = 2, \cdots, n$, such that, under the control

$$u = -\sigma_1(k_1\bar{x}_1 + \sigma_2(k_2\bar{x}_2 + \cdots + \sigma_n(k_n\bar{x}_n))) \tag{10}$$

where for $i = 1, \cdots, n$, σ_i is a saturation function with level λ_i, the closed-loop system at the origin is globally asymptotically stable, for all $d \in \mathcal{D}$.

3. GLOBAL ROBUST OUTPUT REGULATION

As pointed out in Introduction, the robust output regulation problem of (1) can be converted into a robust stabilization problem of a well defined augmented system in the form of (5). To introduce this conversion, we assume the following:

Assumption 3.1. There exist smooth functions $\mathbf{x}(v, w) = (\mathbf{x}_1(v, w), \cdots, \mathbf{x}_n(v, w))$ and $\mathbf{u}(v, w)$ with $\mathbf{x}(0, 0) = 0$ and $\mathbf{u}(0, 0) = 0$ satisfying for all $v \in \mathbb{R}^q, w \in \mathbb{R}^{n_w}$, the following equations:

$$\begin{aligned}\dot{\mathbf{x}}_i(v, w) &= f_i(\mathbf{x}_1(v, w), \cdots, \mathbf{x}_{i-1}(v, w), \mathbf{u}(v, w), v, w),\\ &\qquad i = n, \cdots, 2\\ \dot{\mathbf{x}}_1(v, w) &= c\mathbf{u}(v, w) + f_1(v, w)\\ \mathbf{x}_1(v, w) &= q_d(v, w)\end{aligned} \tag{11}$$

Assumption 3.2. There exist sufficiently smooth functions $\tau_i : \mathbb{R}^q \times \mathbb{R}^{n_w} \to \mathbb{R}^{r_i}$ for $i = 1, \cdots, n$, vanishing at $(0, 0)$, such that

$$\dot{\tau}_i(v, w) = \Phi_i\tau_i(v, w),\ \pi_i(v, w) = \Psi_i\tau_i(v, w) \tag{12}$$

where the pair (Ψ_i, Φ_i) is observable and all the eigenvalues of Φ_i are simple with zero real parts, and $\pi_1(v, w) = \mathbf{u}(v, w)$, $\pi_i(v, w) = \mathbf{x}_i(v, w)$ for $i = 2, \cdots, n$.

Remark 3.1. Equation (11) is called regulator equations and solvability of these equations is necessary but not sufficient for the solvability of the robust output regulation problem Byrnes et al. [1997], Huang [2004], Huang and Chen [2004], Isidori and Byrnes [1990]. Assumption 3.2 is made for the existence of appropriate linear internal models. Both Assumption 3.1 and 3.2 are quite standard in literature. Under Assumption 3.2, given a pair of controllable matrices (M_i, N_i) with $M_i \in \mathbb{R}^{r_i \times r_i}$ Hurwitz and N_i a column vector, for $i = 1, \cdots, n$, there exists a unique nonsingular matrix $T_i \in \mathbb{R}^{r_i \times r_i}$ satisfying the Sylvester equation

$$T_i\Phi_i - M_iT_i = N_i\Psi_i \tag{13}$$

since the spectra of M_i and Φ_i are disjoint and the pair (Ψ_i, Φ_i) is observable. We can define the following system

$$\begin{aligned}\dot{\eta}_1 &= M_1\eta_1 + N_1u - \tfrac{M_1N_1}{c}e,\\ \dot{\eta}_i &= M_i\eta_i + N_ix_i,\ i = 2, \cdots, n.\end{aligned} \tag{14}$$

which is called internal model of (1) with output $(u, x_2, \cdots, x_n)$.

Next, we will convert the robust output regulation problem for system (1) into a robust stabilization problem for the augmented system composed of the original plant (1) and the internal model (14). Performing the following coordinate and input transformation

$$\begin{aligned}\bar{x}_1 &= x_1 - \mathbf{x}_1(v, w) = e\\ \bar{x}_i &= x_i - \Psi_iT_i^{-1}\eta_i,\ i = 2, \cdots, n\\ \bar{\eta}_i &= \eta_i - T_i\tau_i,\ i = 1, \cdots, n\\ \hat{u} &= u - \Psi_1T_1^{-1}\eta_1\end{aligned} \tag{15}$$

on the augmented system gives the following system

$$\begin{aligned}\dot{\bar{x}}_i &= -\Psi_iT_i^{-1}[(M_i + N_i\Psi_iT_i^{-1})\bar{\eta}_i + N_i\bar{x}_i]\\ &\quad + \hat{f}_i(\bar{\eta}_1, \bar{x}_1, \cdots, \bar{\eta}_{i-1}, \bar{x}_{i-1}, \hat{u}, v, w)\\ \dot{\bar{\eta}}_i &= (M_i + N_i\Psi_iT_i^{-1})\bar{\eta}_i + N_i\bar{x}_i,\ i = n, \cdots, 2\\ \dot{\bar{x}}_1 &= c\Psi_1T_1^{-1}\bar{\eta}_1 + c\hat{u}\\ \dot{\bar{\eta}}_1 &= (M_1 + N_1\Psi_1T_1^{-1})\bar{\eta}_1 + N_1\hat{u} - \frac{M_1N_1}{c}\bar{x}_1\end{aligned} \tag{16}$$

where $\hat{f}_2(\bar{\eta}_1, \bar{x}_1, \hat{u}, v, w) = -f_2(\mathbf{x}_1, \mathbf{u}, v, w) + f_2(\bar{x}_1 + \mathbf{x}_1, \hat{u} + \Psi_1T_1^{-1}\bar{\eta}_1 + \mathbf{u}, v, w)$ and for $i = 3, \cdots, n$,

$$\begin{aligned}\hat{f}_i(\bar{\eta}_1, \bar{x}_1, \cdots, \bar{\eta}_{i-1}, \bar{x}_{i-1}, \hat{u}, v, w) &= -f_i(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_{i-1},\\ \mathbf{u}, v, w) + f_i(\bar{x}_1 + \mathbf{x}_1, \cdots, \bar{x}_{i-1} &+ \Psi_{i-1}T_{i-1}^{-1}\bar{\eta}_{i-1} + \mathbf{x}_{i-1},\\ &\hat{u} + \Psi_1T_1^{-1}\bar{\eta}_1 + \mathbf{u}, v, w)\end{aligned}$$

It is known from Huang [2004], Huang and Chen [2004] that the global robust output regulation problem of system (1) will be solved if we can make the equilibrium of system (16) at $(\bar{x}, \bar{\eta}) = (0, 0)$ globally asymptotically stable for all trajectories $v(t)$ starting from V_0 and all $w \in W$. A system of the form (16) has never been encountered and there is no clue whether or not the equilibrium of this system at the origin is stabilizable. Nevertheless, by performing some further coordinate and input transform on (16), it is possible to convert (16) to the form of (5) with all desirable properties. For this purpose, we introduce two more assumptions.

Assumption 3.3. For $i = 2, \cdots, n$, Φ_i is invertible.

Assumption 3.4. For $i = 2, \cdots, n$,,

$$\frac{\partial f_i}{\partial x_{i-1}}|_{(x_1,\cdots,x_{i-1},u)=(\mathbf{x}_1(v,w),\cdots,\mathbf{x}_{i-1}(v,w),\mathbf{u}(v,w))}$$

is nonzero and does not change its sign for all $v \in \mathbb{R}^q, w \in \mathbb{R}^{n_w}$.

Now define the following coordinate and input transformation

$$\begin{aligned} \xi_1 &= c\bar{\eta}_1 - N_1\bar{x}_1 \\ \xi_i &= (M_i + N_i\Psi_i T_i^{-1})\bar{\eta}_i + N_i\bar{x}_i,\ i = 2, \cdots, n \\ \bar{u} &= \hat{u} + \frac{\Psi_1 T_1^{-1} N_1}{c}\bar{x}_1 \end{aligned} \quad (17)$$

From (13), $M_i + N_i\Psi_i T_i^{-1} = T_i\Phi_i T_i^{-1}$, and then from Assumption 3.3 and $c \neq 0$, the transformation (17) is globally invertible.

Performing the transformation (17) on (16) yields, for $i = 2, \cdots, n$

$$\begin{aligned} \dot{\bar{x}}_i &= -\Psi_i T_i^{-1}\xi_i \\ &+ \hat{f}_i(\frac{\xi_1 + N_1\bar{x}_1}{c}, \bar{x}_1, \cdots, T_{i-1}\Phi_{i-1}^{-1}T_{i-1}^{-1}(\xi_{i-1} - N_{i-1}\bar{x}_{i-1}), \\ &\bar{x}_{i-1}, \bar{u} - \frac{\Psi_1 T_1^{-1} N_1}{c}\bar{x}_1, v, w) \\ &= -\Psi_i T_i^{-1}\xi_i + \bar{f}_i(\xi_1, \bar{x}_1, \cdots, \xi_{i-1}, \bar{x}_{i-1}, \bar{u}, v, w) \end{aligned}$$

and

$$\begin{aligned} \dot{\xi}_i &= (M_i + N_i\Psi_i T_i^{-1})\dot{\bar{\eta}}_i + N_i\dot{\bar{x}}_i \\ &= (M_i + N_i\Psi_i T_i^{-1})\xi_i + N_i\dot{\bar{x}}_i \\ &= (M_i + N_i\Psi_i T_i^{-1})\xi_i + N_i[-\Psi_i T_i^{-1}\xi_i \\ &\quad + \bar{f}_i(\xi_1, \bar{x}_1, \cdots, \xi_{i-1}, \bar{x}_{i-1}, \bar{u}, v, w)] \\ &= M_i\xi_i + N_i\bar{f}_i(\xi_1, \bar{x}_1, \cdots, \xi_{i-1}, \bar{x}_{i-1}, \bar{u}, v, w) \end{aligned}$$

where $\bar{f}_2(\xi_1, \bar{x}_1, \bar{u}, v, w) = -f_2(\mathbf{x}_1, \mathbf{u}, v, w) + f_2(\bar{x}_1 + \mathbf{x}_1, \bar{u} + \frac{1}{c}\Psi_1 T_1^{-1}\xi_1 + \mathbf{u}, v, w)$ and for $i = 3, \cdots, n$,

$$\begin{aligned} &\bar{f}_i(\xi_1, \bar{x}_1, \cdots, \xi_{i-1}, \bar{x}_{i-1}, \bar{u}, v, w) = -f_i(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_{i-1}, \\ &\mathbf{u}, v, w) + f_i(\bar{x}_1 + \mathbf{x}_1, \cdots, (1 - \Psi_{i-1}\Phi_{i-1}^{-1}T_{i-1}^{-1}N_{i-1})\bar{x}_{i-1} \\ &\quad + \Psi_{i-1}\Phi_{i-1}^{-1}T_{i-1}^{-1}\xi_{i-1} + \mathbf{x}_{i-1}, \bar{u} + \frac{1}{c}\Psi_1 T_1^{-1}\xi_1 + \mathbf{u}, v, w). \end{aligned}$$

and

$$\begin{aligned} \dot{\bar{x}}_1 &= \Psi_1 T_1^{-1}\xi_1 + c\bar{u} \\ \dot{\xi}_1 &= c\dot{\bar{\eta}}_1 - N_1\dot{\bar{x}}_1 \\ &= c[(M_1 + N_1\Psi_1 T_1^{-1})\bar{\eta}_1 + N_1\hat{u} - \frac{M_1 N_1}{c}\bar{x}_1] \\ &\quad - cN_1(\Psi_1 T_1^{-1}\bar{\eta}_1 + \hat{u}) \\ &= cM_1\bar{\eta}_1 - M_1 N_1\bar{x}_1 \\ &= M_1\xi_1 \end{aligned}$$

Now, let $d = (v, w)$. Then, (16) becomes

$$\begin{aligned} \dot{\bar{x}}_i &= -\Psi_i T_i^{-1}\xi_i + \bar{f}_i(\xi_1, \bar{x}_1, \cdots, \xi_{i-1}, \bar{x}_{i-1}, \bar{u}, d) \\ \dot{\xi}_i &= M_i\xi_i + N_i\bar{f}_i(\xi_1, \bar{x}_1, \cdots, \xi_{i-1}, \bar{x}_{i-1}, \bar{u}, d), \\ &\quad i = n, \cdots, 2 \\ \dot{\bar{x}}_1 &= \Psi_1 T_1^{-1}\xi_1 + c\bar{u} \\ \dot{\xi}_1 &= M_1\xi_1 \end{aligned} \quad (18)$$

which is in the form of (5). Further, let $c_1 = c$, and

$$\begin{aligned} c_2 &= \frac{\partial \bar{f}_2}{\partial \bar{x}_1}|_{(0,0,0,d)} = \frac{\partial f_2}{\partial x_1}|_{(x_1,u)=((\mathbf{x}_1(v,w),\mathbf{u}(v,w)))} \\ c_i &= \frac{\partial \bar{f}_i}{\partial \bar{x}_{i-1}}|_{(0,\cdots,0,d)} = (1 - \Psi_{i-1}\Phi_{i-1}^{-1}T_{i-1}^{-1}N_{i-1}) \times \\ &\quad \frac{\partial f_i}{\partial x_{i-1}}|_{(x_1,\cdots,x_{i-1},u)=(\mathbf{x}_1(v,w),\cdots,\mathbf{x}_{i-1}(v,w),\mathbf{u}(v,w))}. \end{aligned} \quad (19)$$

Then, (18) can be put in the form of (9) as follows:

$$\begin{aligned} \dot{\bar{x}}_i &= -\Psi_i T_i^{-1}\xi_i + c_i\bar{x}_{i-1} + \bar{f}_i^r(\xi_1, \bar{x}_1, \cdots, \xi_{i-1}, \bar{x}_{i-1}, \bar{u}, d) \\ \dot{\xi}_i &= M_i\xi_i + N_i c_i\bar{x}_{i-1} + N_i\bar{f}_i^r(\xi_1, \bar{x}_1, \cdots, \xi_{i-1}, \bar{x}_{i-1}, \bar{u}, d), \\ &\quad i = n, \cdots, 2 \\ \dot{\bar{x}}_1 &= \Psi_1 T_1^{-1}\xi_1 + c_1\bar{u} \\ \dot{\xi}_1 &= M_1\xi_1 \end{aligned} \quad (20)$$

where $\bar{f}_i^r$, $i = 2, \cdots, n$ are suitably defined *smooth* functions. We will now show that (20) satisfies Assumptions 2.1 to 2.2, thus obtaining the main result of this paper.

Theorem 3.1. Suppose system (1) satisfies Assumptions 3.1 to 3.4. Then, the global robust output regulation problem can be solved by a dynamic state feedback controller of the form

$$\begin{aligned} u &= \Psi_1 T_1^{-1}(\eta_1 - \tfrac{N_1}{c}e) - \sigma_1(k_1 e + \cdots \\ &\quad + \sigma_n(k_n x_n - k_n\Psi_n T_n^{-1}\eta_n)), \\ \dot{\eta}_1 &= M_1\eta_1 + N_1 u - \tfrac{M_1 N_1}{c}e, \\ \dot{\eta}_i &= M_i\eta_i + N_i x_i,\ i = 2, \cdots, n. \end{aligned} \quad (21)$$

Proof: Due to the space limit, we omit the proof here.

Remark 3.2. For the class of nonlinear systems which only involves polynomial nonlinearities, Assumptions 3.1 to 3.3 can be easily testified. Let us first review some facts which can be found in Huang [2004]. Let $v^{[1]} = v \in \mathbb{R}^q$, and, for $i \geq 2$, $v^{[l]} = (v_1^l, v_1^{l-1}v_2, \cdots, v_1^{l-1}v_q, v_1^{l-2}v_2^2, \cdots, v_1^{l-2}v_2v_q, \cdots, v_q^l)^T$. Then from Section 4.2 of Huang [2004], there exists a matrix denoted by $S^{[l]}$ such that

$$\frac{\partial v^{[l]}}{\partial v}Sv = S^{[l]}v^{[l]} \quad (22)$$

Moreover, all the eigenvalues of $S^{[l]}$ are given by

$$\begin{aligned} \lambda = l_1\lambda_1 + \cdots + l_q\lambda_q, l_1 + \cdots + l_q = l, \\ l_1, \cdots, l_q = 0, 1, \cdots, l \end{aligned} \quad (23)$$

where $\lambda_1, \cdots, \lambda_q$ are eigenvalues of S. As a result, when all eigenvalues of S are simple with zero real parts, $S^{[l]}$ is nonsingular if and only if q is even and l is odd. Moreover, the roots of the minimal polynomials of $S^{[l]}$ coincide with all distinct eigenvalues of $S^{[l]}$.

In the following, we call $f : \mathbb{R}^q \times \mathbb{R}^{n_w} \to \mathbb{R}$ a polynomial in v, if it takes the form

$$f(v,w) = \sum_{l=1}^{\kappa} F_l(w) v^{[l]} \tag{24}$$

where $F_l(w)$, $l = 1, \cdots, \kappa$, are row vectors of appropriate dimensions.

Proposition 3.1. Assume $f : \mathbb{R}^q \times \mathbb{R}^{n_w} \to \mathbb{R}$ is a polynomial function in v and takes the form (24). If q is even and $F_l(w) = 0$ when l is even, then there exists a polynomial solution $x(v,w)$ in v for the following partial differential equation

$$\frac{\partial x(v,w)}{\partial v} Sv = f(v,w) \tag{25}$$

Moreover, there exist an integer r and matrices $\Phi \in \mathbb{R}^{r\times r}, \Psi \in \mathbb{R}^{1\times r}$, where Φ is nonsingular with all its eigenvalues simple and on the imaginary axis and the pair (Ψ, Φ) is observable, such that $\tau(v,w) = (x(v,w), \dot{x}(v,w), \cdots, \frac{d^{(r-1)}x(v,w)}{dt^{(r-1)}})$ satisfies

$$\dot{\tau}(v,w) = \Phi\tau(v,w),\ x(v,w) = \Psi\tau(v,w). \tag{26}$$

Proof: Due to the space limit, we omit the proof here.

Remark 3.3. We call $x(v,w)$ an odd polynomial in v if it takes the special form

$$x(v,w) = \sum_{i=0}^{k} X_{2i+1}(w) v^{[2i+1]} \tag{27}$$

for some integer k and for all $v \in \mathbb{R}^n, w \in \mathbb{R}^{n_w}$. Assume q is even and $q_d(v,w)$ is an odd polynomial in v. Clearly, $\mathbf{x}_1(v,w)$ is an odd polynomial in v. From Proposition 3.1 and (11), if $f_1(v,w)$ is an odd polynomial in v, $\mathbf{u}(v,w)$ exists and is an odd polynomial in v. For $i = 2, \cdots, n$, if $f_i(x_1, \cdots, x_{i-1}, u, v, w)$ is an odd polynomial in $(x_1, \cdots, x_{i-1}, u, v)$, then an odd polynomial solution $\mathbf{x}_i(v,w)$ in v of (11) exists. As a result, Assumptions 3.1 to 3.3 are satisfied if $f_1(v,w)$ is an odd polynomial in v and for $i = 2, \cdots, n$, $f_i(x_1, \cdots, x_{i-1}, u, v, w)$ is an odd polynomial in $(x_1, \cdots, x_{i-1}, u, v)$.

Remark 3.4. When $q_d(v,w) = 0$ and the functions $f_i, i = 2, \cdots, n$, in (1) are independent of u and vanishing at $(0, \cdots, 0, v, w)$, the GRORP of system (1) reduces to the IDSP studied in Marconi et al. [2002]. For this special case, $\mathbf{u}(v,w) = -\frac{1}{c} f_1(v,w), \mathbf{x}(v,w) = 0$ and thus Assumptions 3.1 is satisfied automatically. Moreover, since $\mathbf{x}(v,w) = 0$, there is no need to estimate $\mathbf{x}(v,w)$. It suffices to use one single system $\dot{\eta}_1 = M_1\eta_1 + N_1 u - \frac{M_1 N_1}{c} x_1$ to define the internal model which is essentially the same as what has been done in Marconi et al. [2002]. Clearly, Assumption 3.3 is not needed anymore and thus Assumptions 3.2 with $i = 1$ and 3.4 become the assumptions to the IDSP of system (1) which is a more general class of feedforward systems than system (4). The IDSP of system (1) can be converted into a global robust stabilization problem of a general class of feedforward systems with input unmodeled dynamics.

4. EXAMPLE

We study the global robust output regulation problem of the following system

$$\begin{aligned}
\dot{x}_2 &= (1 + 0.05wv_1^2 v_2)x_1 + 0.05x_1 u + w(v_1 - v_1^3)\\
\dot{x}_1 &= 10u + 7wv_1^2 v_2\\
\dot{v}_1 &= -v_2,\ \dot{v}_2 = v_1\\
e &= x_1 - wv_1^3
\end{aligned} \tag{28}$$

where $|w| \le 1$ is the uncertain parameter and $v_1^2(t) + v_2^2(t) \le 1$ for all $t \ge 0$.

System (28) is in the form of (1). Let us first verify that (28) satisfies Assumptions 3.1 to 3.4. Firstly, the regulator equations of (28) have a globally defined solution as follows:

$$\mathbf{x}_1(v,w) = wv_1^3, \mathbf{x}_2(v,w) = wv_2, \mathbf{u}(v,w) = -wv_1^2 v_2$$

which also implies that Assumption 3.2 is satisfied. Simple calculation shows that $\Psi_1 = [1\ 0\ 0\ 0]$, $\Psi_2 = [1\ 0]$, and

$$\Phi_1 = \begin{bmatrix} 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -9 & 0 & -10 & 0 \end{bmatrix}, \Phi_2 = \begin{bmatrix} 0 & 1\\ -1 & 0 \end{bmatrix}. \tag{29}$$

From (29) and $\frac{\partial f_2}{\partial x_1}|_{(x_1,u)=(\mathbf{x}_1(v,w),\mathbf{u}(v,w))} = 1$, Assumptions 3.3 and 3.4 are both satisfied.

To design an internal model, let $N_1 = (1,1,1,1), N_2 = (1,1)$ and

$$M_1 = \begin{bmatrix} -4 & 0 & 0 & 0\\ 0 & -3 & 0 & 0\\ 0 & 0 & -2 & 0\\ 0 & 0 & 0 & -1 \end{bmatrix}, M_2 = \begin{bmatrix} -2 & 0\\ 0 & -1 \end{bmatrix}.$$

Solving the Sylvester equation (13) gives $T_2 = \begin{bmatrix} 0.4 & -0.2\\ 0.5 & -0.5 \end{bmatrix}$ and

$$T_1 = \begin{bmatrix} 0.2447 & -0.0612 & 0.0094 & -0.0024\\ 0.3167 & -0.1056 & 0.0167 & -0.0056\\ 0.4308 & -0.2154 & 0.0308 & -0.0154\\ 0.5500 & -0.5500 & 0.0500 & -0.0500 \end{bmatrix}.$$

The internal model takes the following form:

$$\dot{\eta}_1 = M_1\eta_1 + N_1 u - 0.1M_1N_1 e,\ \dot{\eta}_2 = M_2\eta_2 + N_2 x_2 \tag{30}$$

Then Theorem 3.1 can be applied to solve the global robust output regulation problem for system (28). The augmented system consisting of (28) and (30) can be put into the following form (for convenience, we retain the original coordinates on the right hand side of the following equation)

$$\begin{aligned}
\dot{z}_2 &= \theta_2\bar{u} + 0.05(1 + \Psi_2 T_2^{-1} M_2^{-1} N_2)(\bar{x}_1 + wv_1^3)\\
&\quad \times (\bar{u} + 0.1\Psi_1 T_1^{-1}\xi_1) + \theta_2 k_1 \bar{x}_1\\
\dot{\xi}_2 &= M_2\xi_2 + N_2\bar{x}_1 + 0.05N_2(\bar{x}_1 + wv_1^3)(\bar{u} + 0.1\Psi_1 T_1^{-1}\xi_1)\\
\dot{z}_1 &= \theta_1\bar{u}\\
\dot{\xi}_1 &= M_1\xi_1
\end{aligned}$$

where $\theta_1 = \mu_1 = 10, \theta_2 = \mu_2/k_1 = 0.5/k_1$. Since k_i has the same sign with θ_i, k_1, k_2 are both positive in this case.

We set $k_1 = 0.4, k_2 = 0.0017, \lambda_1 = 10$ and $\lambda_2 = 0.038$, and the designed controller takes the following form

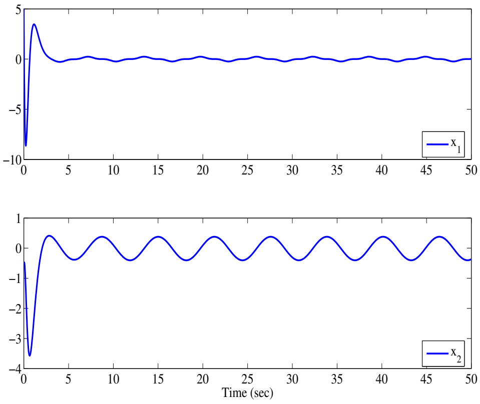

Fig. 1. State trajectory

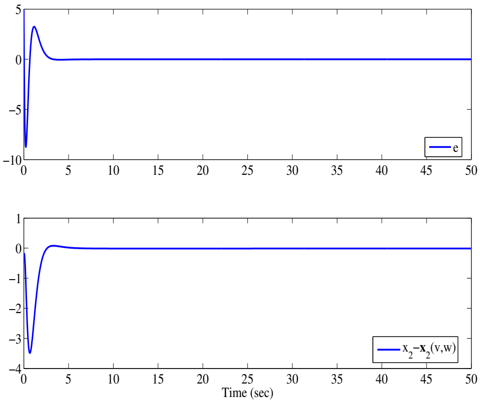

Fig. 2. Tracking error

$$\begin{aligned} u = & \Psi_1 T_1^{-1}(\eta_1 - 0.1N_1 e) \\ & -\sigma_1(0.4e + \sigma_2(0.0017(x_2 - \Psi_2 T_2^{-1}\eta_2))), \end{aligned} \tag{31}$$

where σ_1, σ_2 are saturation functions with level 10 and 0.038 respectively.

As an illustration, Fig.1 and Fig.2 shows the simulation result of system (28) under the control (31) with initial state $(x_1(0), x_2(0), v(0), \eta_1(0), \eta_2(0)) = (5, -0.5, (0.5, -0.6), (0.5, 1, 1.5, 1), (5, 5))$ and $w = 0.5$.

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