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Transformation from Real Homogeneous Systems of Degree ℓ to Complex Homogeneous Systems of Degree $(\ell, 0)$

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Abstract: In eigenvalue analysis, transformation from real systems to complex systems is very important. First, we clarify a necessary and sufficient condition that solutions of real nonlinear systems coincide with solutions of transformed complex nonlinear systems in the real subspace. Moreover, we propose a complex transformation such that a) real homogeneous systems of degree ℓ with respect to r are transformed to complex homogeneous systems of degree $(\ell, 0)$ with respect to r and b) solutions of real systems coincide with solutions of transformed complex systems in the real subspace. Then, we show examples.

1. INTRODUCTION

1.1 Eigenvalue Analysis for Linear Systems

For linear systems, eigenvalue analysis is a very useful technique to determine the characteristics. First, a real linear system

$$\dot{x} = Ax \qquad (x \in \mathbb{R}^n) \tag{1}$$

is transformed to a complex linear system

$$\dot{z} = Az \qquad (z \in \mathbb{C}^n). \tag{2}$$

Notice that

1) System (1) is real homogeneous of degree 0 with respect to $r = (1, \dots, 1)^T$. And system (2) is complex homogeneous of degree $(0, 0)$ with respect to $r = (1, \dots, 1)^T$.
2) $x(t)$ is a solution of system (1) if and only if $x(t)e^{i \cdot 0}$ is a solution of system (2).
3) System (1) is (asymptotically) stable if and only if system (2) is (asymptotically) stable.
4) Both systems (1) and (2) are analytic.

System (2) has n eigenvalues and n linearly independent eigenvectors. Moreover, solutions on eigenvectors can be written by using corresponding eigenvalues and eigenvectors. Since system (2) satisfies the superposition principle, general solutions can be written by using eigenvalues and eigenvectors. Furthermore, the stability of (2) can be identified by eigenvalues. Hence, the stability of (1) also can be identified by eigenvalues.

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1.2 Eigenvalue Analysis for Homogeneous Systems

Our ultimate purpose is to extend eigenvalue analysis for homogeneous systems.

In [1], homogeneous eigenvalues and homogeneous eigenvectors are defined for real homogeneous systems. While the number of homogeneous eigenvalues is only partly elucidated [2], solutions on homogeneous eigenvectors can be written by using corresponding homogeneous eigenvalues and homogeneous eigenvectors [1].

In [4], complex homogeneous systems are defined. Moreover, homogeneous eigenvalues and homogeneous eigenvectors are defined for complex homogeneous systems of degree $(\ell, 0)$ or degree (ℓ, ℓ) [4] [5]. For these complex homogeneous systems, solutions on homogeneous eigenvectors also can be written by using corresponding homogeneous eigenvalues and homogeneous eigenvectors [4] [5].

In [5], we have shown a transformation from a real nonlinear system

$$\dot{x} = f(x) \qquad (x \in \mathbb{R}^n) \tag{3}$$

to a complex nonlinear systems

$$\dot{z} = F(z) \qquad (z \in \mathbb{C}^n). \tag{4}$$

This complex transformation holds the following conditions:

1') System (3) is real homogeneous of degree ℓ with respect to r. And system (4) is complex homogeneous of degree (ℓ, ℓ) with respect to r.
2') $x(t)$ is a solution of system (3) if and only if $x(t)e^{i \cdot 0}$ is a solution of system (4).

3') System (4) may not be (asymptotically) stable even if system (3) is (asymptotically) stable.
4') Both systems (3) and (4) are analytic.

For general homogeneous systems, a transformation satisfying 2'), 4') and

3") System (3) is (asymptotically) stable if and only if system (4) is (asymptotically) stable

does not exist because a transformation satisfying conditions 2') and 4') exists uniquely [5].

Example 1. The real analytic homogeneous system of degree 2 with respect to $r = 1$

$$\dot{x} = -x^3 \qquad (x \in \mathbb{R}) \tag{5}$$

is transformed to the complex analytic homogeneous system of degree $(2, 2)$ with respect to $r = 1$

$$\dot{z} = -z^3 \qquad (z \in \mathbb{C}). \tag{6}$$

While system (5) is asymptotically stable, system (6) is unstable. Both systems (5) and (6) are analytic. □

Example 2. The real analytic homogeneous system of degree 1 with respect to $r = (1, 2)$

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -x_1^3 \end{pmatrix} \qquad (x_1, x_2 \in \mathbb{R}) \tag{7}$$

is transformed to the complex analytic homogeneous system of degree $(1, 1)$ with respect to $r = (1, 2)$

$$\frac{d}{dt}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} z_2 \\ -z_1^3 \end{pmatrix} \qquad (z_1, z_2 \in \mathbb{C}). \tag{8}$$

While system (7) is stable, system (8) is unstable. Both systems (7) and (8) are analytic. □

1.3 The Purpose of This Paper

In this paper, we propose another complex transformation for homogeneous systems. In Section 2, we introduce some definitions and results as a preliminary. In Section 3.1, we clarify a necessary and sufficient condition for condition 2'). In Section 3.2, we propose a complex transformation satisfying 2') and

1") System (3) is real homogeneous of degree ℓ with respect to r. And system (4) is complex homogeneous of degree $(\ell, 0)$ with respect to r.

In Section 4, we show examples and in Section 5 conclude this paper.

2. PRELIMINARY

Let $\mathbb{R}$, $\mathbb{R}_{>0}$, and $\mathbb{C}$ denote the set of all real numbers, the set of all positive real numbers, and the set of all complex numbers, respectively.

2.1 Category and Functor

Definition 1. (Category). A category C consists of

C1) a set O of objects
C2) a set A of arrows
C3) domain $\mathrm{dom} : A \to O : f \mapsto \mathrm{dom}\, f$
C4) codomain $\mathrm{cod} : A \to O : f \mapsto \mathrm{cod}\, f$
C5) identity $\mathrm{id} : O \to A : c \mapsto \mathrm{id}_c$
C6) composition $\circ : A \times_O A \to A : \langle g, f\rangle \mapsto g \circ f$ where $A \times_O A := \{\langle g, f\rangle |\ g, f \in A \ \wedge\ \mathrm{dom}\, g = \mathrm{cod}\, f\}$

such that

$$\begin{aligned}
&\mathrm{dom}(\mathrm{id}_c) = c = \mathrm{cod}(\mathrm{id}_c), \quad \forall c \in O \\
&\mathrm{dom}(g \circ f) = \mathrm{dom}\, f, \quad \forall \langle g, f\rangle \in A \times_O A \\
&\mathrm{cod}(g \circ f) = \mathrm{cod}\, g, \quad \forall \langle g, f\rangle \in A \times_O A \\
&h \circ (g \circ f) = (h \circ g) \circ f, \\
&\qquad\qquad \forall \langle h, g\rangle, \langle g, f\rangle \in A \times_O A \\
&1_b \circ f = f, \quad \forall f : a \mapsto b \\
&g \circ 1_b = g, \quad \forall g : b \mapsto c.
\end{aligned}$$

□

Definition 2. (Functor). For categories C and B, a functor $T : C \to B$ with domain C and codomain B consists of

F1) a object function T which assigns to each object c of C an object Tc of B
F2) an arrow function T which assigns to each arrow $f : c \mapsto c'$ of C an arrow $Tf : Tc \mapsto Tc'$ of B such that

$$T(1_c) = 1_{Tc}, \quad T(g \circ f) = Tg \circ Tf.$$

□

2.2 Real Homogeneous Systems

We consider the following nonlinear system:

$$\dot{x} = f(x), \tag{9}$$

where $x \in \mathbb{R}^n$ is a state vector and $t \in \mathbb{R}$.

Definition 3. (dilation). A mapping

$$\Delta^r_\varepsilon x = (\varepsilon^{r_1} x_1, \ldots, \varepsilon^{r_n} x_n)^T, \qquad \forall \varepsilon > 0 \tag{10}$$

is said to be a dilation on $\mathbb{R}^n$, where $r = (r_1, \ldots, r_n)$, $0 < r_j < \infty$ $(j = 1, \ldots, n)$ and $x \in \mathbb{R}^n$. □

Definition 4. (Euler vector field). A vector field

$$\nu(x) = (r_1 x_1, \ldots, r_n x_n)^T. \tag{11}$$

is said to be a Euler vector field with respect to the dilation exponent $r = (r_1, \ldots, r_n)$. □

Definition 5. (homogeneous ray). Solution curves of $\dot{x} = \nu(x)$ $(t \in \mathbb{R})$ are said to be homogeneous rays. □

For fixed $x \in \mathbb{R}^n$, a dilation $\Delta^r_\varepsilon x$ $(\varepsilon > 0)$ coincides with an homogeneous ray by definitions.

Definition 6. (homogeneous function). A function $V : \mathbb{R}^n \to \mathbb{R}$ is said to be homogeneous of degree $k \in \mathbb{R}$ with respect to the dilation $\Delta^r_\varepsilon x$ if

$$V(\Delta^r_\varepsilon x) = \varepsilon^k V(x). \tag{12}$$

□

Definition 7. (homogeneous vector field). A vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be homogeneous of degree $\ell \in \mathbb{R}$ with respect to the dilation $\Delta^r_\varepsilon x$ if

$$f(\Delta_\varepsilon^r x) = \varepsilon^\ell \Delta_\varepsilon^r f(x). \tag{13}$$

□

Definition 8. (homogeneous system). Nonlinear system (9) satisfying (13) is said to be homogeneous of degree $\ell \in \mathbb{R}$ with respect to the dilation $\Delta_\varepsilon^r x$. □

Lemma 1. A C^1 function $V : \mathbb{R}^n \to \mathbb{R}$ satisfies (12) if and only if

$$\frac{\partial V}{\partial x}\nu(x) = kV(x). \tag{14}$$

□

Lemma 2. If a C^1 function $V : \mathbb{R}^n \to \mathbb{R}$ satisfies (12),

$$\frac{\partial V}{\partial x_j}(\Delta_\varepsilon^r x) = \varepsilon^{k-r_j}\frac{\partial V}{\partial x_j}(x). \tag{15}$$

□

Lemma 3. A C^1 vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfies (13) if and only if

$$\frac{\partial f}{\partial x}\nu(x) - \frac{\partial \nu}{\partial x}f(x) = \ell f(x). \tag{16}$$

□

Lemma 4. We consider homogeneous system (9) of degree ℓ with respect to a dilation $\Delta_\varepsilon^r x$. If $x(t)$ is the solution for an initial state $x(0)$, $\Delta_\varepsilon^r x\left(\varepsilon^\ell t\right)$ is the solution for an initial state $\Delta_\varepsilon^r x(0)$. □

2.3 Complex Homogeneous Systems

In this paper, we consider complex functions defined on a Riemmann surface for avoiding discontinuity and multi-value problems of complex functions [9]. Namely, we consider complex functions defined on

$$\begin{aligned} &z = (z_1, \dots, z_n)^T \\ &z_j = r_j e^{i\theta_j} \qquad (r_j, \theta_j \in \mathbb{R},\ j = 1, \dots, n). \end{aligned} \tag{17}$$

We define the complex conjugate of $z = re^{i\theta}$ by $\bar{z} := re^{-i\theta}$.

Definition 9. (complex dilation). A mapping

$$\begin{aligned} \Delta_\varepsilon^r z = \Delta_{\{\xi,\eta\}}^r z = (\xi^{r_1} e^{i\eta r_1} z_1, \dots, \xi^{r_n} e^{i\eta r_n} z_n)^T, \\ \forall \varepsilon = \xi e^{i\eta} \in \mathbb{C}\backslash\{0\} \quad (\forall \xi \in \mathbb{R}_{>0},\ \forall \eta \in \mathbb{R}) \end{aligned} \tag{18}$$

is said to be a complex dilation on $\mathbb{C}^n$, where $r = (r_1, \dots, r_n)$, $0 < r_j < \infty$ $(j = 1, \dots, n)$ and $z \in \mathbb{C}^n$. □

Definition 10. (complex Euler vector field). A vector field

$$\nu(z) = (r_1 z_1, \dots, r_n z_n)^T \tag{19}$$

is said to be a complex Euler vector field with respect to the dilation exponent $r = (r_1, \dots, r_n)$. □

Definition 11. (complex homogeneous ray). Solution curves of $\dot{z} = \nu(z)$ $(t \in \mathbb{C})$ are said to be complex homogeneous rays. □

For fixed $z \in \mathbb{C}^n$, a complex dilation $\Delta_\varepsilon^r z$ $(\varepsilon \in \mathbb{C}\backslash\{0\})$ coincides with a complex homogeneous ray by definitions.

Definition 12. (complex homogeneous function). A function $V : \mathbb{C}^n \to \mathbb{C}$ is said to be complex homogeneous of degree $(k_1, k_2) \in \mathbb{R}^2$ with respect to the complex dilation $\Delta_{\{\xi,\eta\}}^r z$ if

$$V(\Delta_{\{\xi,\eta\}}^r z) = \xi^{k_1} e^{i\eta k_2} V(z). \tag{20}$$

□

Definition 13. (complex homogeneous vector field). A vector field $F : \mathbb{C}^n \to \mathbb{C}^n$ is said to be complex homogeneous of degree $(\ell_1, \ell_2) \in \mathbb{R}^2$ with respect to the complex dilation $\Delta_{\{\xi,\eta\}}^r z$ if

$$F(\Delta_{\{\xi,\eta\}}^r z) = \xi^{\ell_1} e^{i\eta\ell_2} \Delta_{\{\xi,\eta\}}^r F(z). \tag{21}$$

□

Definition 14. (complex homogeneous system). A nonlinear system

$$\dot{z} = F(z) \tag{22}$$

satisfying (21) is said to be complex homogeneous of degree $(\ell_1, \ell_2) \in \mathbb{R}^2$ with respect to the complex dilation $\Delta_{\{\xi,\eta\}}^r z$, where $t \in \mathbb{R}$. □

Lemma 5. An analytic function $V : \mathbb{C}^n \to \mathbb{C}$ satisfies (20) if and only if

$$\frac{\partial V}{\partial z}\nu(z) = k_1 V(z) = k_2 V(z). \tag{23}$$

□

Lemma 6. If an analytic function $V : \mathbb{C}^n \to \mathbb{C}$ satisfies (20),

$$\frac{\partial V}{\partial z_j}(\Delta_{\{\xi,\eta\}}^r z) = \xi^{k_1 - r_j} e^{i\eta(k_2 - r_j)}\frac{\partial V}{\partial z_j}(z). \tag{24}$$

□

Lemma 7. An analytic vector field $F : \mathbb{C}^n \to \mathbb{C}^n$ satisfies (21) if and only if

$$\frac{\partial F}{\partial z}\nu(z) - \frac{\partial \nu}{\partial z}F(z) = \ell_1 F(z) = \ell_2 F(z). \tag{25}$$

□

Lemma 8. We consider complex homogeneous system (22) of degree $(\ell_1, 0)$ with respect to a complex dilation $\Delta_{\{\xi,\eta\}}^r z$. If $z(t)$ is the solution for an initial state $z(0)$, $\Delta_{\{\xi,\eta\}}^r z\left(\xi^{\ell_1} t\right)$ is the solution for an initial state $\Delta_{\{\xi,\eta\}}^r z(0)$. □

2.4 Eigenvalue Analysis for Complex Homogeneous System of Degree $(\ell, 0)$

Definition 15. (complex homogeneous norm). The following complex homogeneous function of degree $(1, 0)$ with respect to a complex dilation $\Delta_\varepsilon^r z$ is said to be a complex homogeneous p-norm:

$$\|z\|_{\{r,p\}} = \left(\sum_{j=1}^{n} |z_j|^{\frac{p}{r_j}}\right)^{\frac{1}{p}}. \tag{26}$$

□

Definition 16. (Homogeneous eigenvalue). We consider complex homogeneous system (22) of degree $(\ell, 0)$ with respect to a complex dilation $\Delta_\varepsilon^r z$. If there exist $\lambda \in \mathbb{C}$ and $z_\lambda \in \mathbb{C}^n$ such that

$$F(z_\lambda) = \lambda \|z_\lambda\|_{\{r,2\}}^\ell \nu(z_\lambda), \tag{27}$$

λ and z_λ are said to be an homogeneous eigenvalue and an homogeneous eigenvector, respectively. □

Lemma 9. We consider complex homogeneous system (22) of degree $(\ell, 0)$ with respect to a complex dilation $\Delta_\varepsilon^r z$. If there exist $\lambda \in \mathbb{C}$ and $z_\lambda \in \mathbb{C}^n$ with (27),

$$z(t) = \Delta_{Q(t)}^r z_\lambda$$

$$Q(t) = \begin{cases} \exp\left(\lambda \|z_\lambda\|_{\{r,2\}}^\ell t\right) & (\ell\Re(\lambda) = 0) \\ \left(1 - \ell\Re(\lambda)\|z_\lambda\|_{\{r,2\}}^\ell t\right)^{-\frac{1}{\ell}} & \\ \cdot \exp\left\{-\dfrac{i\Im(\lambda)}{\ell\Re(\lambda)} \ln\left(1 - \ell\Re(\lambda)\|z_\lambda\|_{\{r,2\}}^\ell t\right)\right\} & \\ & (\ell\Re(\lambda) \neq 0) \end{cases} \tag{28}$$

is the solution for an initial state $z(0) = z_\lambda$. □

Theorem 1. For complex homogeneous system (22) of degree $(\ell, 0)$, the followings are true:

1) If an homogeneous eigenvalue with positive real part exists, the system is unstable.
2) If an homogeneous eigenvalue with zero real part exists, the system is not asymptotically stable.
3) If an homogeneous eigenvalue with negative real part exists, the system has a solution that converges to the origin.

□

Lemma 10. We consider complex nonlinear system (22) with $\overline{F(z)} = F(\bar{z})$. If $z(t)$ is the solution for an initial state $z(0)$, $\bar{z}(t)$ is the solution for an initial state $\bar{z}(0)$. □

Lemma 11. We consider complex homogeneous system (22) with $\overline{F(z)} = F(\bar{z})$. If z_0 is an homogeneous eigenvector corresponding to an homogeneous eigenvalue λ, $\bar{z}_0$ is an homogeneous eigenvector corresponding to an homogeneous eigenvalue $\bar{\lambda}$. □

3. COMPLEX TRANSFORMATION

3.1 Necessary and Sufficient Condition

We obtain a necessary and sufficient condition that solutions of real nonlinear system (9) coincide with solutions of transformed complex nonlinear system (9) in the real subspace as follows:

Lemma 12. Let T_c be a functor that assigns $xe^{i\cdot 0} \in \mathbb{C}^n$ to $x \in \mathbb{R}^n$ and $F : \mathbb{C}^n \to \mathbb{C}^n$ to $f : \mathbb{R}^n \to \mathbb{R}^n$. Then, the followings are equivalent:

1) The following diagram

$$\begin{array}{ccc} x & \xrightarrow{\;f\;} & f(x) \\ {\scriptstyle T_c}\downarrow & & \downarrow{\scriptstyle T_c} \\ z & \xrightarrow{\;F\;} & F(z) \end{array} \tag{29}$$

is commutative.

2) $x(t)$ is a solution of real nonlinear system (9) if and only if $x(t)e^{i\cdot 0}$ is a solution of complex nonlinear system (22).

Proof 1. First, we show 1)⇒2).

$$\{\dot{x} - f(x)\}e^{i\cdot 0} = T_c(\dot{x}) - T_c(f(x)) \tag{30}$$

$$\frac{d}{dt}(xe^{i\cdot 0}) - F(xe^{i\cdot 0}) = T_c(\dot{x}) - F(T_c(x)). \tag{31}$$

If condition 1) is satisfied, the right-hand side of (30) coincides with the right-hand side of (31). Hence, condition 2) is held.

Then, we show 2)⇒1). If condition 2) is satisfied, the left-hand side of (30) and the left-hand side of (31) becomes 0. Hence, condition 1) is held. □

3.2 Complex Transformation Functor for Homogeneous Systems

We consider real homogeneosu system (9) of degree ℓ with respect to $\Delta_\varepsilon^r x$ and assume that

H1) f is a C^1 mapping.
H2) $\ell + r_j \neq 0$ $(j = 1, \ldots, n)$.

We propose a functor that transforms this system (9) to complex homogeneous system (22) of degree $(\ell, 0)$ with respect to $\Delta_\varepsilon^r z$ as follows:

Theorem 2. We consider real homogeneosu system (9) of degree ℓ with respect to $\Delta_\varepsilon^r x$. We assume that H1) and H2) are satisfied. Let T_c be a functor that assigns $xe^{i\cdot 0} \in \mathbb{C}^n$ to $x \in \mathbb{R}^n$ and

$$F : \begin{pmatrix} x_1 e^{iy_1} \\ \vdots \\ x_n e^{iy_n} \end{pmatrix} \mapsto \begin{pmatrix} \dfrac{1}{\ell + r_1} \displaystyle\sum_{j=1}^n \frac{\partial f_1}{\partial x_j}(x) r_j x_j e^{i\frac{r_1}{r_j} y_j} \\ \vdots \\ \dfrac{1}{\ell + r_n} \displaystyle\sum_{j=1}^n \frac{\partial f_n}{\partial x_j}(x) r_j x_j e^{i\frac{r_n}{r_j} y_j} \end{pmatrix} \tag{32}$$

$$(x_j, y_j \in \mathbb{R})$$

to $f : x \mapsto f(x)$. Then, T_c transforms from real homogeneous system (9) to complex homogeneous system (22) of degree $(\ell, 0)$ with respect to $\Delta_\varepsilon^r z$. Moreover, $x(t)$ is a solution of real homogeneous system (9) if and only if $x(t)e^{i\cdot 0}$ is a solution of complex homogeneous system (22). Furthermore, $\overline{F(z)} = F(\bar{z})$. □

Proof 2. By the homogeneity of f and Lemma 2, we obtain

$$F(\Delta_\varepsilon^r z) = \begin{pmatrix} \dfrac{1}{\ell + r_1} \displaystyle\sum_{j=1}^n \frac{\partial f_1}{\partial x_j}(\Delta_\xi^r x) r_j \xi^{r_j} x_j e^{i\frac{r_1}{r_j}(y_j + r_j\eta)} \\ \vdots \\ \dfrac{1}{\ell + r_n} \displaystyle\sum_{j=1}^n \frac{\partial f_n}{\partial x_j}(\Delta_\xi^r x) r_j \xi^{r_j} x_j e^{i\frac{r_n}{r_j}(y_j + r_j\eta)} \end{pmatrix}$$

$$= \xi^\ell \Delta_\varepsilon^r F(z),$$

where $z_j = x_j e^{iy_j}$ $(j = 1, \ldots, n)$ and $\varepsilon = \xi e^{i\eta}$ $(\xi > 0)$. Since (21) with $(\ell_1, \ell_2) = (\ell, 0)$ is satisfied, F is a complex homogeneous vector field of degree $(\ell, 0)$ with respect to $\Delta_\varepsilon^r z$.

By the homogeneity of f and Lemma 1, we get

$$\begin{aligned} F(T_c(x)) &= F(xe^{i\cdot 0}) \\ &= \begin{pmatrix} \dfrac{1}{\ell + r_1}\dfrac{\partial f_1}{\partial x}\nu(x)e^{i\cdot 0} \\ \vdots \\ \dfrac{1}{\ell + r_n}\dfrac{\partial f_n}{\partial x}\nu(x)e^{i\cdot 0} \end{pmatrix} \\ &= T_c(f(x)). \end{aligned}$$

By the equation and Lemma 12, $x(t)$ is a solution of real homogeneous system (9) if and only if $x(t)e^{i\cdot 0}$ is a solution of complex homogeneous system (22). By (32), $\overline{F(z)} = F(\bar{z})$ is clearly satisfied. □

4. EXAMPLE

Example 3. By Theorem 2, real homogeneous system (5) of degree 2 with respect to $r = 1$ is transformed to the following complex homogeneous system of degree $(2, 0)$ with respect to $r = 1$:

$$\dot{z} = -|z|^2 z \qquad (z \in \mathbb{C}). \tag{33}$$

Both systems (5) and (33) are asymptotically stable. While system (5) is analytic, system (33) is not analytic. □

Example 4. By Theorem 2, real homogeneous system (7) of degree 1 with respect to $r = (1, 2)$ is transformed to the follwoing complex homogeneous system of degree $(1, 0)$ with respect to $r = (1, 2)$:

$$\frac{d}{dt}\begin{pmatrix} x_1 e^{iy_1} \\ x_2 e^{iy_2} \end{pmatrix} = \begin{pmatrix} x_2 e^{i\frac{y_2}{2}} \\ -x_1^3 e^{i2y_1} \end{pmatrix} \quad (x_1, x_2, y_1, y_2 \in \mathbb{R}). \tag{34}$$

While system (7) is stable and analytic, system (34) is unstable and nonanalytic. □

5. CONCLUSION

We have clarified a necessary and sufficient condition that solutions of real nonlinear systems coincide with solutions of transformed complex nonlinear systems in the real subspace. Moreover, we have proposed a complex transformation such that a) real homogeneous systems of degree ℓ with respect to r are transformed to complex homogeneous systems of degree $(\ell, 0)$ with respect to r and b) solutions of real systems coincide with solutions of transformed complex systems in the real subspace. Then, we have shown examples.

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