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Constrained control of discrete-time stochastic systems

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Abstract: The purpose of the paper is to present a class of algorithms using to solve the optimization problem concerning with H_∞ feedback control for linear discrete-time stochastic systems with stochastic parameter uncertainties, as well as a method for the optimization problem reducing this to a standard formulation used two convex inequalities, which can be solved by linear matrix inequality (LMI) methodology. Some generalized considerations for the algorithm procedures are given and the problem of LMI set-up, using the technique based on Schur complement, for calculating the terminal weight matrix $\mathbf{P}$ is outlined. Finally, obtained results are adapted by that way to design constrained H_∞ feedback control for system which state variable satisfy equality constraints in the mean. The procedure results the constant feedback for the linear controller defined in terms of matrix inequalities and a matrix equality.

1. INTRODUCTION

Many real systems operate in a stochastic environment where they are subject to unknown disturbances and in addition, the controller has to rely, in practice, on imperfect measurements. One of the principal reasons for introducing feedback into a control system is to obtain relative insensibility to changes in plant parameters and to disturbances.

In the last years many significant results have spurred interest in problem of determining a H_∞ feedback control for discrete-time linear systems with stochastic uncertainties. One solution based on state-multiplicative noise can be obtained by solving multiple convex matrix inequalities using a linear matrix inequality (LMI).

On the other hand, it is necessary to determine control laws for systems with constraints. One approach to the problem of finding the constrained control results is to deal with system constraints directly. A special form of this constrained problem can be formulated with the goal to design state feedback controller parameters while the system state variables satisfy the equality constraints.

In this note a new class of algorithms to solve both above mentioned problems for linear systems with stochastic uncertainties is introduced. Starting with standard selection of a Lyapunov function, which leads to sufficient stability condition of the state-feedback control, the solution is carried out along the transform approach to develop a modified Lyapunov function for equivalent constrained system description. Finally, two LMIs are set-up using Schur complement.

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These techniques can be viewed as an extension to the methods applied for uncertain stochastic discrete-time systems with multiplicative noise in Gershon, Shaked and Yaesh (2001) and for constrained linear discrete-time systems in Ko and Bitmead (2007), with full references presented therein. An example is given to demonstrate the role of LMI in the design procedure, where any solution is easily achieved by solving for a set of two LMIs.

2. CONTROL TASK FORMULATION

Through this paper the task will be concerned with the computation of a state feedback $\boldsymbol{u}$(i), which control a stochastic uncertain discrete-time linear dynamic system, given by set of equations

$$\boldsymbol{q}(i+1) = (\boldsymbol{F} + \boldsymbol{F}_a a(i))\boldsymbol{q}(i) + \boldsymbol{G}\boldsymbol{u}(i) + \boldsymbol{G}_v\boldsymbol{v}(i) \quad (1)$$

$$\boldsymbol{z}(i) = \boldsymbol{C}\boldsymbol{q}(i) + \boldsymbol{D}\boldsymbol{u}(i) \quad (2)$$

where $\boldsymbol{q}(i) \in I\!R^n$ is the state vector, $\boldsymbol{u}(i) \in I\!R^r$ is the control input signal vector, $\boldsymbol{z}(i) \in I\!R^p$ is the objective vector, $\boldsymbol{v}(i) \in I\!R^p$ is an exogenous stochastic disturbance vector, and matrices $\boldsymbol{F} \in I\!R^{n\times n}$, $\boldsymbol{G} \in I\!R^{n\times r}$, $\boldsymbol{C} \in I\!R^{m\times n}$, $\boldsymbol{D} \in I\!R^{m\times r}$,$\boldsymbol{G}_v \in I\!R^{n\times p}$, $\boldsymbol{F}_a \in I\!R^{n\times n}$ are all finite valued.

It is assumed, that the uncertainty variable $a(j)$, $0 \le j \le i$ satisfies condition

$$E\{a(j)\} = 0, \quad E\{a(h)a(j)\} = \delta_{jh} \quad (3)$$

where $E\{\cdot\}$ denotes expectation and δ_{jh} is the Kronecker delta function. Disturbance vector is a non-anticipative precess, where $\{\boldsymbol{v}(i)\} \in l_2\langle 0, N-1\rangle;\ I\!R^p\}$.

For system (1), (2) the optimal control design task is to determine the control

$$\boldsymbol{u}(i) = -\boldsymbol{K}(i)\boldsymbol{q}(i) \quad (4)$$

that minimizes, for given $\gamma > 0$, the quadratic cost function

$$J = \sum_{i=0}^{N-1} \underset{a}{E}\{||\boldsymbol{z}(i)||_2^2 - \gamma^2 ||\boldsymbol{v}(i)||_2^2\} + + \boldsymbol{q}^T(N)\boldsymbol{Q}^{\bullet}\boldsymbol{q}(N) - \gamma^2 \boldsymbol{q}^T(0)\boldsymbol{Q}^{*}\boldsymbol{q}(0) < 0 \tag{5}$$

for all nonzero $\{\boldsymbol{v}(i)\}$, where N is finite, $\boldsymbol{Q}^{\bullet} \in \mathbb{R}^{n\times n}$ and $\boldsymbol{Q}^{*} \in \mathbb{R}^{n\times n}$ are symmetric positive semi-definite matrices, $\boldsymbol{K}(i) \in \mathbb{R}^{n\times r}$ is the optimal control gain matrix, and $||\,.\,||_2$ denotes the standard l_2-norm.

3. CONSTRAINED CONTROL

Using control law (4) the steady-state equilibrium control equation takes the form

$$\boldsymbol{q}(i+1) = (\boldsymbol{F} + \boldsymbol{F}_a a(i) - \boldsymbol{G}\boldsymbol{K})\boldsymbol{q}(i) \tag{6}$$

Considering a design constraint

$$E\{\boldsymbol{q}(i)\} \in \mathcal{Q} = \{\boldsymbol{q} : \boldsymbol{L}\boldsymbol{q} = \boldsymbol{0}\} \tag{7}$$

the state-variable vectors have to satisfy equalities

$$\boldsymbol{L}E\{\boldsymbol{q}(i+1)\} = \boldsymbol{L}(\boldsymbol{F}+\boldsymbol{F}_a E\{a(i)\}-\boldsymbol{G}\boldsymbol{K})E\{\boldsymbol{q}(i)\} = \boldsymbol{0} \tag{8}$$

$$\boldsymbol{L}(\boldsymbol{F} - \boldsymbol{G}\boldsymbol{K}) = \boldsymbol{0} \tag{9}$$

$$\boldsymbol{L}\boldsymbol{F} = \boldsymbol{L}\boldsymbol{G}\boldsymbol{K} \tag{10}$$

respectively. All solutions of $\boldsymbol{K}$ are

$$\boldsymbol{K} = (\boldsymbol{L}\boldsymbol{G})^{\ominus 1}\boldsymbol{L}\boldsymbol{F} + \boldsymbol{K}^{\circ} - (\boldsymbol{L}\boldsymbol{G})^{\ominus 1}\boldsymbol{L}\boldsymbol{G}\boldsymbol{K}^{\circ} \tag{11}$$

where $\boldsymbol{K}^{\circ}$ is an arbitrary matrix with consistent dimension and

$$(\boldsymbol{L}\boldsymbol{G})^{\ominus 1} = (\boldsymbol{L}\boldsymbol{G})^T\big(\boldsymbol{L}\boldsymbol{G}(\boldsymbol{L}\boldsymbol{G})^T\big)^{-1} \tag{12}$$

is the pseudoinverse of $\boldsymbol{L}\boldsymbol{G}$. One can therefore express (11) as

$$\boldsymbol{K} = \boldsymbol{M} + \boldsymbol{N}\boldsymbol{K}^{\circ} \tag{13}$$

where

$$\boldsymbol{M} = (\boldsymbol{L}\boldsymbol{G})^{\ominus 1}\boldsymbol{L}\boldsymbol{F} \tag{14}$$

and

$$\boldsymbol{N} = \boldsymbol{I}_m - (\boldsymbol{L}\boldsymbol{G})^T\big(\boldsymbol{L}\boldsymbol{G}(\boldsymbol{L}\boldsymbol{G})^T\big)^{-1}\boldsymbol{L}\boldsymbol{G} \tag{15}$$

is the projection matrix (the orthogonal projector onto the null space $\mathcal{N}(\boldsymbol{L}\boldsymbol{G})$ of $\boldsymbol{L}\boldsymbol{G}$). This results in

$$\begin{aligned}\boldsymbol{u}(i) &= -\boldsymbol{M}\boldsymbol{q}(i) + \boldsymbol{N}\big(-\boldsymbol{K}^{\circ}\boldsymbol{q}(i)\big) = \\ &= -\boldsymbol{M}\boldsymbol{q}(i) + \boldsymbol{N}\boldsymbol{u}^{\circ}(i)\end{aligned} \tag{16}$$

where

$$\boldsymbol{u}^{\circ}(i) = -\boldsymbol{K}^{\circ}\boldsymbol{q}(i) \tag{17}$$

(see e.g. Ko and Bitmead (2007), Krokavec and Filasová (2007)).

4. STOCHASTIC LYAPUNOV FUNCTION

Theorem 1. For a system given in (1), (2), $r < n$, and for a Lyapunov function

$$V(\boldsymbol{q}(i)) = \boldsymbol{q}^T(i)\boldsymbol{P}(i-1)\boldsymbol{q}(i) \tag{18}$$

where $\boldsymbol{P}(-1) = \boldsymbol{P}(0)$, $\boldsymbol{P}(i) \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix, the mean value of diference of this Lyapunov function is

$$\begin{aligned}&\underset{a}{E}\{\Delta V(\boldsymbol{q}(i))\} = \underset{a}{E}\{V(\boldsymbol{q}(i+1)) - V(\boldsymbol{q}(i))\} = \\ &= \underset{a}{E}\left\{\begin{array}{c}\|\boldsymbol{u}(i) - \boldsymbol{u}^*(i)\|^2_{\boldsymbol{U}(i)} - \|\boldsymbol{v}(i) - \boldsymbol{v}^*(i)\|^2_{\boldsymbol{V}(i)} + \\ +\boldsymbol{q}^T(i)\boldsymbol{T}(i)\boldsymbol{q}(i) + 2\boldsymbol{q}^T(i)\boldsymbol{F}^T\boldsymbol{P}(i)\boldsymbol{G}\boldsymbol{u}(i) + \\ +\gamma^2\boldsymbol{v}^T(i)\boldsymbol{v}(i) - \boldsymbol{z}^T(i)\boldsymbol{z}(i)\end{array}\right\}\end{aligned} \tag{19}$$

with identity matrices $\boldsymbol{I}_p \in \mathbb{R}^{p\times p}$, $\boldsymbol{I} \in \mathbb{R}^{n\times n}$ and with

$$\boldsymbol{U}(i) = \boldsymbol{R} + \boldsymbol{G}^T\boldsymbol{H}(i)\boldsymbol{G} \tag{20}$$

$$\boldsymbol{H}(i) = \boldsymbol{P}(i)(\boldsymbol{I} - \gamma^{-2}\boldsymbol{G}_v\boldsymbol{G}_v^T\boldsymbol{P}(i))^{-1} \tag{21}$$

$$\boldsymbol{V}(i) = \gamma^2\boldsymbol{I}_p - \boldsymbol{G}_v^T\boldsymbol{P}(i)\boldsymbol{G}_v > 0 \tag{22}$$

$$\begin{aligned}\boldsymbol{T}(i) = \boldsymbol{F}^T\boldsymbol{P}(i)\boldsymbol{F} + \boldsymbol{F}_a^T\boldsymbol{P}(i)\boldsymbol{F}_a + \boldsymbol{C}^T\boldsymbol{C} + \\ +\boldsymbol{K}_q^T(i)\boldsymbol{V}(i)\boldsymbol{K}_q(i) - \boldsymbol{K}^T(i)\boldsymbol{U}(i)\boldsymbol{K}(i) - \boldsymbol{P}(i-1)\end{aligned} \tag{23}$$

$$\boldsymbol{u}^*(i) = -\boldsymbol{K}(i)\boldsymbol{q}(i) \tag{24}$$

$$\boldsymbol{v}^*(i) = \boldsymbol{K}_q(i)\boldsymbol{q}(i) + \boldsymbol{K}_u(i)\boldsymbol{u}(i) \tag{25}$$

$$\boldsymbol{K}_u(i) = \boldsymbol{V}^{-1}(i)\boldsymbol{G}_v^T\boldsymbol{P}(i)\boldsymbol{G} \tag{26}$$

$$\boldsymbol{K}_q(i) = \boldsymbol{V}^{-1}(i)\boldsymbol{G}_v^T\boldsymbol{P}(i)\boldsymbol{F} \tag{27}$$

Proof. For (2) product $\boldsymbol{z}^T(i)\boldsymbol{z}(i)$ can be written as

$$\begin{aligned}\boldsymbol{z}^T(i)\boldsymbol{z}(i) &= \begin{bmatrix}\boldsymbol{q}^T(i) & \boldsymbol{u}^T(i)\end{bmatrix}\begin{bmatrix}\boldsymbol{C}^T \\ \boldsymbol{D}^T\end{bmatrix}\begin{bmatrix}\boldsymbol{C} & \boldsymbol{D}\end{bmatrix}\begin{bmatrix}\boldsymbol{q}(i) \\ \boldsymbol{u}(i)\end{bmatrix} = \\ &= \begin{bmatrix}\boldsymbol{q}(i) & \boldsymbol{u}(i)\end{bmatrix}^T\begin{bmatrix}\boldsymbol{C}^T\boldsymbol{C} & \boldsymbol{C}^T\boldsymbol{D} \\ \boldsymbol{D}^T\boldsymbol{C} & \boldsymbol{D}^T\boldsymbol{D}\end{bmatrix}\begin{bmatrix}\boldsymbol{q}(i) \\ \boldsymbol{u}(i)\end{bmatrix}\end{aligned} \tag{28}$$

Supposing, that $\boldsymbol{C}^T\boldsymbol{D} = \boldsymbol{0}$ and $\boldsymbol{D}^T\boldsymbol{D} = \boldsymbol{R} > 0$, then diference (19) can be written as

$$\begin{aligned}&\Delta V(\boldsymbol{q}(i)) = \\ &= \boldsymbol{q}^T(i)\left\{\begin{array}{c}\big(\boldsymbol{F}+\boldsymbol{F}_a a(i)\big)^T\boldsymbol{P}(i)\big(\boldsymbol{F}+\boldsymbol{F}_a a(i)\big) + \\ +\boldsymbol{C}^T\boldsymbol{C} - \boldsymbol{P}(i-1)\end{array}\right\}\boldsymbol{q}(i) + \\ &+\boldsymbol{u}^T(i)(\boldsymbol{G}^T\boldsymbol{P}(i)\boldsymbol{G} + \boldsymbol{R})\boldsymbol{u}(i) + \\ &+\boldsymbol{v}^T(i)\boldsymbol{G}_v^T\boldsymbol{P}(i)\boldsymbol{G}_v\boldsymbol{v}(i) + \\ &+\boldsymbol{q}^T(i)(\boldsymbol{F} + \boldsymbol{F}_a a(i))^T\boldsymbol{P}(i)\boldsymbol{G}\boldsymbol{u}(i) + \\ &+\boldsymbol{u}^T(i)\boldsymbol{G}^T\boldsymbol{P}(i)(\boldsymbol{F} + \boldsymbol{F}_a a(i))\boldsymbol{q}(i) + \\ &+\boldsymbol{q}^T(i)(\boldsymbol{F} + \boldsymbol{F}_a a(i))^T\boldsymbol{P}(i)\boldsymbol{G}_v\boldsymbol{v}(i) + \\ &+\boldsymbol{v}^T(i)\boldsymbol{G}_v^T\boldsymbol{P}(i)(\boldsymbol{F} + \boldsymbol{F}_a a(i))\boldsymbol{q}(i) + \\ &+\boldsymbol{u}^T(i)\boldsymbol{G}^T\boldsymbol{P}(i)\boldsymbol{G}_v\boldsymbol{v}(i) + \boldsymbol{v}^T(i)\boldsymbol{G}_v^T\boldsymbol{P}(i)\boldsymbol{G}\boldsymbol{u}(i) + \\ &+\gamma^2\boldsymbol{v}^T(i)\boldsymbol{v}(i) - \gamma^2\boldsymbol{v}^T(i)\boldsymbol{v}(i) - \boldsymbol{z}^T(i)\boldsymbol{z}(i)\end{aligned} \tag{29}$$

Since

$$\underset{a}{E}\left\{\begin{array}{c} \boldsymbol{q}^T(i)\boldsymbol{F}_a^T a(i)\boldsymbol{P}(i)\boldsymbol{F}\boldsymbol{q}(i)+ \\ +\boldsymbol{q}^T(i)\boldsymbol{F}^T\boldsymbol{P}(i)\boldsymbol{F}_a a(i)\boldsymbol{q}(i)+ \\ +\boldsymbol{q}^T(i)\boldsymbol{F}_a^T a(i)\boldsymbol{P}(i)\boldsymbol{G}\boldsymbol{u}(i)+ \\ +\boldsymbol{u}^T(i)\boldsymbol{G}^T\boldsymbol{P}(i)\boldsymbol{F}_a a(i)\boldsymbol{q}(i)+ \\ +\boldsymbol{q}^T(i)\boldsymbol{F}_a^T a(i)\boldsymbol{P}(i)\boldsymbol{G}_v\boldsymbol{v}(i)+ \\ +\boldsymbol{v}^T(i)\boldsymbol{G}_v^T\boldsymbol{P}(i)\boldsymbol{F}_a a(i)\boldsymbol{q}(i) \end{array}\right\} = 0 \tag{30}$$

one can obtain

$$\begin{array}{c} \underset{a}{E}\{\Delta V(\boldsymbol{q}(i))\} = \\ = \underset{a}{E}\{\boldsymbol{q}^T(i)\left\{\begin{array}{c}\boldsymbol{F}^T\boldsymbol{P}(i)\boldsymbol{F}+\boldsymbol{F}_a^T\boldsymbol{P}(i)\boldsymbol{F}_a+ \\ +\boldsymbol{C}^T\boldsymbol{C}-\boldsymbol{P}(i-1)\end{array}\right\}\boldsymbol{q}(i)+ \\ +\boldsymbol{u}^T(i)(\boldsymbol{G}^T\boldsymbol{P}(i)\boldsymbol{G}+\boldsymbol{R})\boldsymbol{u}(i)- \\ -\boldsymbol{v}^T(i)(\gamma^2\boldsymbol{I}_p-\boldsymbol{G}_v^T\boldsymbol{P}(i)\boldsymbol{G}_v)\boldsymbol{v}(i)+ \\ +\boldsymbol{q}^T(i)\boldsymbol{F}^T\boldsymbol{P}(i)\boldsymbol{G}\boldsymbol{u}(i)+\boldsymbol{u}^T(i)\boldsymbol{G}^T\boldsymbol{P}(i)\boldsymbol{F}\boldsymbol{q}(i)+ \\ +\boldsymbol{q}^T(i)\boldsymbol{F}^T\boldsymbol{P}(i)\boldsymbol{G}_v\boldsymbol{v}(i)+\boldsymbol{v}^T(i)\boldsymbol{G}_v^T\boldsymbol{P}(i)\boldsymbol{F}\boldsymbol{q}(i)+ \\ +\boldsymbol{u}^T(i)\boldsymbol{G}^T\boldsymbol{P}(i)\boldsymbol{G}_v\boldsymbol{v}(i)+\boldsymbol{v}^T(i)\boldsymbol{G}_v^T\boldsymbol{P}(i)\boldsymbol{G}\boldsymbol{u}(i)+ \\ +\gamma^2\boldsymbol{v}^T(i)\boldsymbol{v}(i)-\boldsymbol{z}^T(i)\boldsymbol{z}(i)\} \end{array} \tag{31}$$

Using notations (26), (27) with $\boldsymbol{V}(i)$ noted in (22), and completing to squares for $\boldsymbol{v}(i)$ from (31) one can obtain

$$\begin{array}{c} -\boldsymbol{v}^T(i)\boldsymbol{V}(i)\boldsymbol{v}(i)+ \\ +\boldsymbol{u}^T(i)\boldsymbol{K}_u^T(i)\boldsymbol{V}(i)\boldsymbol{v}(i)+\boldsymbol{v}^T(i)\boldsymbol{V}(i)\boldsymbol{K}_u(i)\boldsymbol{u}(i)+ \\ +\boldsymbol{q}^T(i)\boldsymbol{K}_q^T(i)\boldsymbol{V}(i)\boldsymbol{v}(i)+\boldsymbol{v}^T(i)\boldsymbol{V}(i)\boldsymbol{K}_q(i)\boldsymbol{q}(i) = \\ = -[\boldsymbol{v}(i)-\boldsymbol{v}^*(i)]^T\boldsymbol{V}(i)[\boldsymbol{v}(i)-\boldsymbol{v}^*(i)]+ \\ +\boldsymbol{v}^{*T}(i)\boldsymbol{V}(i)\boldsymbol{v}^*(i) = \\ = -\|\boldsymbol{v}(i)-\boldsymbol{v}^*(i)\|^2_{\boldsymbol{V}(i)}+\boldsymbol{v}^{*T}(i)\boldsymbol{V}(i)\boldsymbol{v}^*(i) \end{array} \tag{32}$$

with

$$\begin{array}{c} \boldsymbol{v}^{*T}(i)\boldsymbol{V}(i)\boldsymbol{v}^*(i) = \\ = \boldsymbol{u}^T(i)\boldsymbol{K}_u^T(i)\boldsymbol{V}(i)\boldsymbol{K}_u(i)\boldsymbol{u}(i)+ \\ +\boldsymbol{q}^T(i)\boldsymbol{K}_q^T(i)\boldsymbol{V}(i)\boldsymbol{K}_q(i)\boldsymbol{q}(i)+ \\ +\boldsymbol{u}^T(i)\boldsymbol{K}_u^T(i)\boldsymbol{V}(i)\boldsymbol{K}_q(i)\boldsymbol{q}(i)+ \\ +\boldsymbol{q}^T(i)\boldsymbol{K}_q^T(i)\boldsymbol{V}(i)\boldsymbol{K}_u(i)\boldsymbol{u}(i) \end{array} \tag{33}$$

$$\boldsymbol{K}_q^T(i)\boldsymbol{V}(i)\boldsymbol{K}_q(i) = \boldsymbol{F}^T\boldsymbol{P}(i)\boldsymbol{G}_v\boldsymbol{V}^{-1}(i)\boldsymbol{G}_v^T\boldsymbol{P}(i)\boldsymbol{F} \tag{34}$$

$$\boldsymbol{K}_q^T(i)\boldsymbol{V}(i)\boldsymbol{K}_u(i) = \boldsymbol{F}^T\boldsymbol{P}(i)\boldsymbol{G}_v\boldsymbol{V}^{-1}(i)\boldsymbol{G}_v^T\boldsymbol{P}(i)\boldsymbol{G} \tag{35}$$

$$\boldsymbol{K}_u^T(i)\boldsymbol{V}(i)\boldsymbol{K}_u(i) = \boldsymbol{G}^T\boldsymbol{P}(i)\boldsymbol{G}_v\boldsymbol{V}^{-1}(i)\boldsymbol{G}_v^T\boldsymbol{P}(i)\boldsymbol{G} \tag{36}$$

respectively.

Let

$$\begin{array}{c} \boldsymbol{W}(i) = \boldsymbol{G}^T\boldsymbol{P}(i)\boldsymbol{G}+\boldsymbol{K}_u^T(i)\boldsymbol{V}(i)\boldsymbol{K}_u(i) = \\ = \boldsymbol{G}^T\big(\boldsymbol{P}(i)+\boldsymbol{P}(i)\boldsymbol{G}_v\boldsymbol{V}^{-1}(i)\boldsymbol{G}_v^T\boldsymbol{P}(i)\big)\boldsymbol{G} \end{array} \tag{37}$$

Then, using Sherman - Morrisson - Woodbury identity

$$\begin{array}{c} (\boldsymbol{A}+\boldsymbol{B}\boldsymbol{C}\boldsymbol{B}^T)^{-1} = \\ = \boldsymbol{A}^{-1}-\boldsymbol{A}^{-1}\boldsymbol{B}(\boldsymbol{C}^{-1}+\boldsymbol{B}^T\boldsymbol{A}^{-1}\boldsymbol{B})^{-1}\boldsymbol{B}^T\boldsymbol{A}^{-1} \end{array} \tag{38}$$

and (22), one can obtain

$$\begin{array}{c} \boldsymbol{H}^{-1}(i) = (\boldsymbol{P}(i)+\boldsymbol{P}(i)\boldsymbol{G}_v\boldsymbol{V}^{-1}(i)\boldsymbol{G}_v^T\boldsymbol{P}(i))^{-1} = \\ = \boldsymbol{P}^{-1}(i)-\boldsymbol{G}_v(\boldsymbol{V}(i)+\boldsymbol{G}_v^T\boldsymbol{P}(i)\boldsymbol{G}_v)^{-1}\boldsymbol{G}_v^T = \\ = \boldsymbol{P}^{-1}(i)-\gamma^{-2}\boldsymbol{G}_v\boldsymbol{G}_v^T \end{array} \tag{39}$$

$$\boldsymbol{H}(i) = \boldsymbol{P}(i)(\boldsymbol{I}-\gamma^{-2}\boldsymbol{G}_v\boldsymbol{G}_v^T\boldsymbol{P}(i))^{-1} \tag{40}$$

respectively, and (37) can be rewritten as

$$\boldsymbol{W}(i) = \boldsymbol{G}^T\boldsymbol{H}(i)\boldsymbol{G} \tag{41}$$

It is evident, that $\boldsymbol{V}(i) > 0$ implies $\boldsymbol{H}(i) > 0$, $\boldsymbol{W}(i) > 0$. Using notations

$$\boldsymbol{U}(i) = \boldsymbol{R}+\boldsymbol{W}(i) = \boldsymbol{R}+\boldsymbol{G}^T\boldsymbol{H}(i)\boldsymbol{G} \tag{42}$$

$$\boldsymbol{K}(i) = \boldsymbol{U}^{-1}(i)\boldsymbol{K}_u^T(i)\boldsymbol{V}(i)\boldsymbol{K}_q(i) \tag{43}$$

with $\boldsymbol{u}^*(i)$ given in (24), to complete to squares for $\boldsymbol{u}(i)$ from (31), (33) gives

$$\begin{array}{c} \boldsymbol{u}^T(i)\boldsymbol{U}(i)\boldsymbol{u}(i)+\boldsymbol{q}^T(i)\boldsymbol{K}_q^T(i)\boldsymbol{V}(i)\boldsymbol{K}_u(i)\boldsymbol{u}(i)+ \\ +\boldsymbol{u}^T(i)\boldsymbol{K}_u^T(i)\boldsymbol{V}(i)\boldsymbol{K}_q(i)\boldsymbol{q}(i) = \\ = [\boldsymbol{u}(i)+\boldsymbol{K}(i)\boldsymbol{q}(i)]^T\boldsymbol{U}(i)[\boldsymbol{u}(i)+\boldsymbol{K}(i)\boldsymbol{q}(i)]- \\ -\boldsymbol{q}^T(i)\boldsymbol{K}^T(i)\boldsymbol{U}(i)\boldsymbol{K}(i)\boldsymbol{q}(i) = \\ = \|\boldsymbol{u}(i)-\boldsymbol{u}^*(i)\|^2_{\boldsymbol{U}(i)}-\boldsymbol{q}^T(i)\boldsymbol{K}^T(i)\boldsymbol{U}(i)\boldsymbol{K}(i)\boldsymbol{q}(i) \end{array} \tag{44}$$

Rearranging (29) with (40) results

$$\begin{array}{c} \underset{a}{E}\{\Delta V(\boldsymbol{q}(i))\} = \\ = \underset{a}{E}\left\{\begin{array}{c} \|\boldsymbol{u}(i)-\boldsymbol{u}^*(i)\|^2_{\boldsymbol{U}(i)}-\|\boldsymbol{v}(i)-\boldsymbol{v}^*(i)\|^2_{\boldsymbol{V}(i)}+ \\ +\boldsymbol{q}^T(i)\left\{\begin{array}{c}\boldsymbol{F}^T\boldsymbol{P}(i)\boldsymbol{F}+\boldsymbol{F}_a^T\boldsymbol{P}(i)\boldsymbol{F}_a+ \\ +\boldsymbol{C}^T\boldsymbol{C}-\boldsymbol{P}(i-1)- \\ -\boldsymbol{K}^T(i)\boldsymbol{U}(i)\boldsymbol{K}(i)+ \\ +\boldsymbol{K}_q^T(i)\boldsymbol{V}(i)\boldsymbol{K}_q(i)\end{array}\right\}\boldsymbol{q}(i)+ \\ +2\boldsymbol{q}^T(i)\boldsymbol{F}^T\boldsymbol{P}(i)\boldsymbol{G}\boldsymbol{u}(i)+ \\ +\gamma^2\boldsymbol{v}^T(i)\boldsymbol{v}(i)-\boldsymbol{z}^T(i)\boldsymbol{z}(i) \end{array}\right\} = \\ = \underset{a}{E}\left\{\begin{array}{c} \|\boldsymbol{u}(i)-\boldsymbol{u}^*(i)\|^2_{\boldsymbol{U}(i)}-\|\boldsymbol{v}(i)-\boldsymbol{v}^*(i)\|^2_{\boldsymbol{V}(i)}+ \\ +\boldsymbol{q}^T(i)\boldsymbol{T}(i)\boldsymbol{q}(i)+2\boldsymbol{q}^T(i)\boldsymbol{F}^T\boldsymbol{P}(i)\boldsymbol{G}\boldsymbol{u}(i)+ \\ +\gamma^2\boldsymbol{v}^T(i)\boldsymbol{v}(i)-\boldsymbol{z}^T(i)\boldsymbol{z}(i) \end{array}\right\} \end{array} \tag{45}$$

with $\boldsymbol{T}(i)$ given in (23). This concludes the proof.

5. OPTIMAL CONTROL

Theorem 2. For a system given in (1), (2) with performance index (5) solution to the optimal control is

$$\boldsymbol{u}(i) = -\boldsymbol{K}(i)\boldsymbol{q}(i) \tag{46}$$

$$\boldsymbol{K}(i) = -\boldsymbol{U}^{-1}(i)\boldsymbol{G}^T\boldsymbol{P}(i)\boldsymbol{F} \tag{47}$$

conditioned by $\boldsymbol{V}(i) > 0$. Here $\boldsymbol{P}(i)$ is a solution of the discrete matrix equation

$$\boldsymbol{P}(i-1) = \boldsymbol{F}^T\boldsymbol{Y}(i)\boldsymbol{F}+\boldsymbol{F}_a^T\boldsymbol{P}(i)\boldsymbol{F}_a+\boldsymbol{C}^T\boldsymbol{C} \tag{48}$$

$$\boldsymbol{Y}(i) = \boldsymbol{H}(i)+\boldsymbol{P}(i)\boldsymbol{G}\boldsymbol{U}^{-1}(i)\boldsymbol{G}^T\boldsymbol{P}(i) \tag{49}$$

and $\boldsymbol{U}(i) = \boldsymbol{U}^T(i)$, $\boldsymbol{H}(i) = \boldsymbol{H}^T(i)$, and $\boldsymbol{V}(i) = \boldsymbol{V}^T(i)$ are given in (20), (21), and (22), respectively.

Using that control for $\boldsymbol{P}(0) < \gamma^2\boldsymbol{Q}^*$ the performance index is negative and takes value

$$\begin{array}{c} J_N = \boldsymbol{q}^T(0)(\boldsymbol{P}(0)-\gamma^2\boldsymbol{Q}^*)\boldsymbol{q}(0)+ \\ +\sum_{i=0}^{N-1}\underset{a}{E}\Big\{-\|\boldsymbol{v}(i)-\boldsymbol{v}^*(i)\|^2_{\boldsymbol{V}(i)}\Big\} < 0 \end{array} \tag{50}$$

Proof. In the system under consideration mean value of Lyapunov function along a trajectory of system under control can be computed using (18), (19) as follows

$$\sum_{i=0}^{N-1} \underset{a}{E}\{\Delta V(\boldsymbol{q}(i))\} = \\ = \underset{a}{E}\left\{\boldsymbol{q}^T(N)\boldsymbol{P}(N-1)\boldsymbol{q}(N)\right\} - \boldsymbol{q}^T(0)\boldsymbol{P}(0)\boldsymbol{q}(0) \tag{51}$$

and using (45) as

$$\sum_{i=0}^{N-1} \underset{a}{E}\left\{\Delta V(\boldsymbol{q}(i))\right\} = \\ = \sum_{i=0}^{N-1} \underset{a}{E}\left\{\begin{array}{c}\|\boldsymbol{u}(i)-\boldsymbol{u}^*(i)\|^2_{\boldsymbol{U}(i)} - \|\boldsymbol{v}(i)-\boldsymbol{v}^*(i)\|^2_{\boldsymbol{V}(i)} + \\ +\boldsymbol{q}^T(i)\boldsymbol{T}(i)\boldsymbol{q}(i) + 2\boldsymbol{q}^T(i)\boldsymbol{F}^T\boldsymbol{P}(i)\boldsymbol{G}\boldsymbol{u}(i) + \\ +\gamma^2\boldsymbol{v}^T(i)\boldsymbol{v}(i) - \boldsymbol{z}^T(i)\boldsymbol{z}(i)\end{array}\right\} \tag{52}$$

Adding (52) to (5) and subtracting (51) from (5) the cost function for control law specified in equation (4), with $\boldsymbol{P}(N-1) = \boldsymbol{Q}^\bullet$, can be brought to the form

$$J_N = \boldsymbol{q}^T(0)(\boldsymbol{P}(0) - \gamma^2\boldsymbol{Q}^*)\boldsymbol{q}(0) + \\ + \sum_{i=0}^{N-1} \underset{a}{E}\left\{\begin{array}{c}\|\boldsymbol{u}(i)-\boldsymbol{u}^*(i)\|^2_{\boldsymbol{U}(i)} - \|\boldsymbol{v}(i)-\boldsymbol{v}^*(i)\|^2_{\boldsymbol{V}(i)} + \\ +\boldsymbol{q}^T(i)\boldsymbol{T}(i)\boldsymbol{q}(i) + 2\boldsymbol{q}^T(i)\boldsymbol{F}^T\boldsymbol{P}(i)\boldsymbol{G}\boldsymbol{u}(i)\end{array}\right\} \tag{53}$$

Clearly, the optimal control strategy for $\boldsymbol{u}(i)$ is given by $\boldsymbol{u}(i) = \boldsymbol{u}^*(i)$, where $\boldsymbol{P}(i)$ satisfies

$$\boldsymbol{F}^T\boldsymbol{P}(i)\boldsymbol{F} + \boldsymbol{F}_a^T\boldsymbol{P}(i)\boldsymbol{F}_a + \boldsymbol{C}^T\boldsymbol{C} + \\ +\boldsymbol{K}_q^T(i)\boldsymbol{V}(i)\boldsymbol{K}_q(i) - \boldsymbol{K}^T(i)\boldsymbol{U}(i)\boldsymbol{K}(i) - \\ -\boldsymbol{F}^T\boldsymbol{P}(i)\boldsymbol{G}\boldsymbol{K}(i) - \boldsymbol{K}^T(i)\boldsymbol{G}^T\boldsymbol{P}(i)\boldsymbol{F} - \boldsymbol{P}(i-1) = \boldsymbol{0} \tag{54}$$

Substituting (43) and completing to square for that $\boldsymbol{K}(i)$ from (54) gives

$$\boldsymbol{F}^T\boldsymbol{P}(i)\boldsymbol{G}\boldsymbol{K}(i) + \boldsymbol{K}^T(i)\boldsymbol{G}^T\boldsymbol{P}(i)\boldsymbol{F} + \boldsymbol{K}^T(i)\boldsymbol{U}(i)\boldsymbol{K}(i) = \\ = \boldsymbol{F}^T\boldsymbol{P}(i)\boldsymbol{G}\boldsymbol{U}^{-1}(i)\boldsymbol{K}_u^T(i)\boldsymbol{V}(i)\boldsymbol{K}_q(i) + \\ +\boldsymbol{K}_q^T(i)\boldsymbol{V}(i)\boldsymbol{K}_u(i)\boldsymbol{U}^{-1}(i)\boldsymbol{G}^T\boldsymbol{P}(i)\boldsymbol{F} + \\ +\boldsymbol{K}_q^T(i)\boldsymbol{V}(i)\boldsymbol{K}_u(i)\boldsymbol{U}^{-1}(i)\boldsymbol{K}_u^T(i)\boldsymbol{V}(i)\boldsymbol{K}_q(i) = \\ = \boldsymbol{S}(i) - \boldsymbol{F}^T\boldsymbol{P}(i)\boldsymbol{G}\boldsymbol{U}^{-1}(i)\boldsymbol{G}^T\boldsymbol{P}(i)\boldsymbol{F} \tag{55}$$

$$\boldsymbol{S}(i) = \|\boldsymbol{G}^T\boldsymbol{P}(i)\boldsymbol{F} + \boldsymbol{K}_u^T(i)\boldsymbol{V}(i)\boldsymbol{K}_q(i)\|_{\boldsymbol{U}^{-1}(i)} = \\ = \|\boldsymbol{G}^T\boldsymbol{P}(i)\boldsymbol{F} + \boldsymbol{U}(i)\boldsymbol{K}(i)\|_{\boldsymbol{U}^{-1}(i)} \tag{56}$$

Setting

$$\boldsymbol{K}(i) = -\boldsymbol{U}^{-1}(i)\boldsymbol{G}^T\boldsymbol{P}(i)\boldsymbol{F} \tag{57}$$

moves $\boldsymbol{S}(i)$ to zero, and then, using(55) and (34), equation (54) takes the form

$$\boldsymbol{P}(i-1) = \boldsymbol{F}^T\left\{\begin{array}{c}\boldsymbol{P}(i) + \boldsymbol{P}(i)\boldsymbol{G}_v\boldsymbol{V}^{-1}(i)\boldsymbol{G}_v^T\boldsymbol{P}(i) \\ +\boldsymbol{P}(i)\boldsymbol{G}\boldsymbol{U}^{-1}(i)\boldsymbol{G}^T\boldsymbol{P}(i)\end{array}\right\}\boldsymbol{F} + \\ +\boldsymbol{F}_a^T\boldsymbol{P}(i)\boldsymbol{F}_a + \boldsymbol{C}^T\boldsymbol{C} \tag{58}$$

or, taking into account (39), then next form

$$\boldsymbol{P}(i-1) = \boldsymbol{F}^T\boldsymbol{Y}(i)\boldsymbol{F} + \boldsymbol{F}_a^T\boldsymbol{P}(i)\boldsymbol{F}_a + \boldsymbol{C}^T\boldsymbol{C} \tag{59}$$

$$\boldsymbol{Y}(i) = \boldsymbol{H}(i) + \boldsymbol{P}(i)\boldsymbol{G}\boldsymbol{U}^{-1}(i)\boldsymbol{G}^T\boldsymbol{P}(i) \tag{60}$$

It is evident, using (57), (59), (60) the performance index (53) takes value (50). (Obtained results are any revised to those presented in Gershon and Shaked (2005).)

6. STABILIZING CONTROL

Stabilizing control can be designed using a steady-state solution $\boldsymbol{P} > 0$ of discrete algebraic inequality derived from (54), i.e. the terminal weight matrix $\boldsymbol{P} \in I\!R^{n\times n}$ needs to be calculated such, that for a steady-state solution $\boldsymbol{P} > 0$ (59) be negative definite.

Since, using (38), equality (42) can be written as

$$\boldsymbol{U}^{-1} = \boldsymbol{R}^{-1} - \boldsymbol{Q} \tag{61}$$

$$\boldsymbol{Q} = \boldsymbol{R}^{-1}\boldsymbol{G}^T(\boldsymbol{H}^{-1} + \boldsymbol{G}\boldsymbol{R}^{-1}\boldsymbol{G}^T)^{-1}\boldsymbol{G}\boldsymbol{R}^{-1} > 0 \tag{62}$$

and since for (40) be

$$\boldsymbol{H}^{-1} = \boldsymbol{P}^{-1} - \gamma^{-2}\boldsymbol{G}_v\boldsymbol{G}_v^T \tag{63}$$

an algebraic inequality derived from (59) can takes form

$$\boldsymbol{F}^T(\boldsymbol{H} + \boldsymbol{P}\boldsymbol{G}\boldsymbol{R}^{-1}\boldsymbol{G}^T\boldsymbol{P})\boldsymbol{F} + \boldsymbol{F}_a^T\boldsymbol{P}\boldsymbol{F}_a + \boldsymbol{C}^T\boldsymbol{C} - \boldsymbol{P} < 0 \tag{64}$$

Taking into account that for $\boldsymbol{U} > 0$ is $\boldsymbol{S} > 0$, as well as that

$$\boldsymbol{P}\boldsymbol{G}\boldsymbol{R}^{-1}\boldsymbol{G}^T\boldsymbol{P} < (\boldsymbol{P}^{-1} - \boldsymbol{G}\boldsymbol{R}^{-1}\boldsymbol{G}^T)^{-1} = \boldsymbol{P}_G^{-1} \tag{65}$$

desired algebraic inequalities can be reformulated as follows

$$\boldsymbol{F}^T(\boldsymbol{H} + \boldsymbol{P}_G^{-1})\boldsymbol{F} + \boldsymbol{F}_a^T\boldsymbol{P}\boldsymbol{F}_a + \boldsymbol{C}^T\boldsymbol{C} - \boldsymbol{P} < 0 \tag{66}$$

$$\boldsymbol{V} = \gamma^2\boldsymbol{I}_p - \boldsymbol{G}_v^T\boldsymbol{P}\boldsymbol{G}_v > 0 \tag{67}$$

The procedure for deriving matrix $\boldsymbol{P}$ yields, after using nontrivial transformations, a set of linear matrix inequalities (LMI).

Theorem 3. For a system given in (1), (2) with performance index (5) and given $\gamma > 0$, a necessary and sufficient condition for a stabilizing control is that there exists a matrix $\boldsymbol{X} = \boldsymbol{X}^T > 0 \in I\!R^{n\times n}$ that satisfies the following LMIs

$$\boldsymbol{Z} = \begin{bmatrix} -\boldsymbol{X} & \boldsymbol{X}\boldsymbol{F}^T & \boldsymbol{X}\boldsymbol{F}^T & \boldsymbol{X}\boldsymbol{F}_a^T & \boldsymbol{X}\boldsymbol{C}^T \\ \boldsymbol{F}\boldsymbol{X} & -\boldsymbol{X}_G & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{F}\boldsymbol{X} & \boldsymbol{0} & -\boldsymbol{X}_\delta & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{F}_a\boldsymbol{X} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{X} & \boldsymbol{0} \\ \boldsymbol{C}\boldsymbol{X} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{I}_m \end{bmatrix} < 0 \tag{68}$$

$$\boldsymbol{V} = \begin{bmatrix} -\delta\boldsymbol{I}_p & \boldsymbol{G}_v^T \\ \boldsymbol{G}_v & -\boldsymbol{X} \end{bmatrix} < 0 \tag{69}$$

$$\boldsymbol{X}_G = \boldsymbol{X} - \boldsymbol{G}\boldsymbol{R}^{-1}\boldsymbol{G}^T, \quad \boldsymbol{X}_\delta = \boldsymbol{H}^{-1} = \boldsymbol{X} - \delta^{-1}\boldsymbol{G}_v\boldsymbol{G}_v^T \tag{70}$$

and $\delta = \gamma^2 > 0$.

Then a solution to the stabilizing control is

$$\boldsymbol{u}(i) = -\boldsymbol{K}\boldsymbol{q}(i) \tag{71}$$

$$\boldsymbol{K} = -\boldsymbol{U}^{-1}\boldsymbol{G}^T\boldsymbol{X}^{-1}\boldsymbol{F} \tag{72}$$

$$\boldsymbol{U} = \boldsymbol{G}^T\boldsymbol{X}_\delta^{-1}\boldsymbol{G} + \boldsymbol{R} \tag{73}$$

Proof. Inequality (66) can be transformed into an LMI using technique based on Schur complements. Pre-multiplying (66) on the left and right hand side by matrix $\boldsymbol{P}^{-1}$ gives

$$\boldsymbol{P}^{-1}\boldsymbol{F}^T(\boldsymbol{H} + \boldsymbol{P}_G^{-1})\boldsymbol{F}\boldsymbol{P}^{-1} + \\ +\boldsymbol{P}^{-1}\boldsymbol{F}_a^T\boldsymbol{P}\boldsymbol{F}_a\boldsymbol{P}^{-1} + \boldsymbol{P}^{-1}\boldsymbol{C}^T\boldsymbol{C}\boldsymbol{P}^{-1} - \boldsymbol{P}^{-1} < 0 \tag{74}$$

Since $\boldsymbol{R}$ is positive definite, defining new LMI variable

$$\boldsymbol{X} := \boldsymbol{P}^{-1} > 0 \tag{75}$$

(74) takes this form

$$\begin{aligned}\boldsymbol{X}\boldsymbol{F}^T\boldsymbol{H}\boldsymbol{F}\boldsymbol{X} + \boldsymbol{X}\boldsymbol{F}^T(\boldsymbol{X}-\boldsymbol{G}\boldsymbol{R}^{-1}\boldsymbol{G}^T)^{-1}\boldsymbol{F}\boldsymbol{X}+ \\ +\boldsymbol{X}\boldsymbol{F}_a^T\boldsymbol{X}^{-1}\boldsymbol{F}_a\boldsymbol{X} + \boldsymbol{X}\boldsymbol{C}^T\boldsymbol{C}\boldsymbol{X} - \boldsymbol{X} < 0\end{aligned} \tag{76}$$

and (67) this form

$$\boldsymbol{V} = \gamma^2\boldsymbol{I}_p - \boldsymbol{G}_v^T\boldsymbol{X}^{-1}\boldsymbol{G}_v > 0 \tag{77}$$

The LMIs for (75), (76) and (77) can now be written as

$$-\boldsymbol{X} < 0 \tag{78}$$

$$-\boldsymbol{V} = \begin{bmatrix} -\delta\boldsymbol{I}_p & \boldsymbol{G}_v^T \\ \boldsymbol{G}_v & -\boldsymbol{X} \end{bmatrix} < 0 \tag{79}$$

$$\boldsymbol{Z} = \begin{bmatrix} -\boldsymbol{X} & \boldsymbol{X}\boldsymbol{F}^T & \boldsymbol{X}\boldsymbol{F}^T & \boldsymbol{X}\boldsymbol{F}_a^T & \boldsymbol{X}\boldsymbol{C}^T \\ \boldsymbol{F}\boldsymbol{X} & -\boldsymbol{X}_G & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{F}\boldsymbol{X} & \boldsymbol{0} & -\boldsymbol{X}_\delta & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{F}_a\boldsymbol{X} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{X} & \boldsymbol{0} \\ \boldsymbol{C}\boldsymbol{X} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{I}_m \end{bmatrix} < 0 \tag{80}$$

$$\boldsymbol{X}_G = \boldsymbol{X} - \boldsymbol{G}\boldsymbol{R}^{-1}\boldsymbol{G}^T, \quad \boldsymbol{X}_\delta = \boldsymbol{H}^{-1} = \boldsymbol{X} - \delta^{-1}\boldsymbol{G}_v\boldsymbol{G}_v^T \tag{81}$$

with $\delta = \gamma^2 > 0$, where $\boldsymbol{0}$ denotes a null matrix with consistent dimension.

Starting by applying the Schur complement to (68) leads to the condition

$$0 > [\boldsymbol{F}\boldsymbol{X}\ \boldsymbol{F}\boldsymbol{X}\ \boldsymbol{F}_a\boldsymbol{X}\ \boldsymbol{C}\boldsymbol{X}]^T \begin{bmatrix} \boldsymbol{X}_G & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{X}_\delta & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{X} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I}_p \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{F}\boldsymbol{X} \\ \boldsymbol{F}\boldsymbol{X} \\ \boldsymbol{F}_a\boldsymbol{X} \\ \boldsymbol{C}\boldsymbol{X} \end{bmatrix} - \boldsymbol{X} \tag{82}$$

For $\boldsymbol{X} > 0$, $\boldsymbol{X}_G > 0$, $\boldsymbol{X}_\delta > 0$ it follows that

$$\begin{bmatrix} \boldsymbol{X}_G & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{X}_\delta & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{X} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I}_p \end{bmatrix} > 0 \tag{83}$$

$$\left\{\begin{array}{c} \boldsymbol{X} > 0 \\ \left\{\begin{array}{c} -\boldsymbol{X} + \boldsymbol{X}\boldsymbol{F}^T\boldsymbol{X}_G^{-1}\boldsymbol{F}\boldsymbol{X}+ \\ +\boldsymbol{X}\boldsymbol{F}^T\boldsymbol{X}_\delta^{-1}\boldsymbol{F}\boldsymbol{X}+ \\ +\boldsymbol{X}\boldsymbol{F}_a^T\boldsymbol{X}^{-1}\boldsymbol{F}_a\boldsymbol{X} + \boldsymbol{X}\boldsymbol{C}^T\boldsymbol{C}\boldsymbol{X} \end{array}\right\} < 0 \end{array}\right. \tag{84}$$

respectively.

Substituting variable defined in (75) in (84) gives

$$\left\{\begin{array}{c} \boldsymbol{P}^{-1} > 0 \\ \left\{\begin{array}{c} -\boldsymbol{P}^{-1} + \boldsymbol{P}^{-1}\boldsymbol{F}^T(\boldsymbol{P}_G^{-1} + \boldsymbol{H})\boldsymbol{F}\boldsymbol{P}^{-1}+ \\ +\boldsymbol{P}^{-1}\boldsymbol{F}_a^T\boldsymbol{P}\boldsymbol{F}_a\boldsymbol{P}^{-1} + \boldsymbol{P}^{-1}\boldsymbol{C}^T\boldsymbol{C}\boldsymbol{P}^{-1} \end{array}\right\} < 0 \end{array}\right. \tag{85}$$

Then by pre-multiplying and post-multiplying both matrix inequalities in (85) by $\boldsymbol{P}$ one can obtain

$$\left\{\begin{array}{l} \boldsymbol{P} > 0 \\ -\boldsymbol{P} + \boldsymbol{F}_a^T\boldsymbol{P}\boldsymbol{F}_a + \boldsymbol{C}^T\boldsymbol{C} + \boldsymbol{F}^T(\boldsymbol{P}_G^{-1} + \boldsymbol{H})\boldsymbol{F} < 0 \end{array}\right. \tag{86}$$

Consequently, feasibility of (80) and feasibility of (86) are equivalent.

7. CONSTRAINED STABILIZING CONTROL

Theorem 4. For a system given in (1), (2) with equality constraint (7), the performance index (5), and gain matrices (14), (15) a solution to the constrained stabilizing control is given by

$$\boldsymbol{u}(i) = -\boldsymbol{M}\boldsymbol{q}(i) + \boldsymbol{N}\boldsymbol{u}^\circ(i) \tag{87}$$

Here

$$\boldsymbol{u}^\circ(i) = -\boldsymbol{K}^\circ\boldsymbol{q}(i) \tag{88}$$

$$\boldsymbol{K}^\circ = -\boldsymbol{U}^{-1}\boldsymbol{G}^{\circ T}\boldsymbol{X}^{-1}\boldsymbol{F}^\circ \tag{89}$$

$$\boldsymbol{U} = \boldsymbol{G}^{\circ T}\boldsymbol{X}_\delta^{-1}\boldsymbol{G}^\circ + \boldsymbol{R} \tag{90}$$

$$\boldsymbol{F}^\circ = \boldsymbol{F} - \boldsymbol{G}\boldsymbol{M} \tag{91}$$

$$\boldsymbol{G}^\circ = \boldsymbol{G}\boldsymbol{N} \tag{92}$$

and $\boldsymbol{X} = \boldsymbol{X}^T > 0$ is a solution of matrix inequalities

$$\boldsymbol{Z} = \begin{bmatrix} -\boldsymbol{X} & \boldsymbol{X}\boldsymbol{F}^{\circ T} & \boldsymbol{X}\boldsymbol{F}^{\circ T} & \boldsymbol{X}\boldsymbol{F}_a^T & \boldsymbol{X}\boldsymbol{C}^T \\ \boldsymbol{F}^\circ\boldsymbol{X} & -\boldsymbol{X}_G^\circ & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{F}^{\circ T}\boldsymbol{X} & \boldsymbol{0} & -\boldsymbol{X}_\delta & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{F}_a\boldsymbol{X} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{X} & \boldsymbol{0} \\ \boldsymbol{C}\boldsymbol{X} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{I}_m \end{bmatrix} < 0 \tag{93}$$

$$-\boldsymbol{V} = \begin{bmatrix} -\delta\boldsymbol{I}_p & \boldsymbol{G}_v^T \\ \boldsymbol{G}_v & -\boldsymbol{X} \end{bmatrix} < 0 \tag{94}$$

where $\delta = \gamma^2 > 0$ and

$$\boldsymbol{X}_G^\circ = \boldsymbol{X} - \boldsymbol{G}^\circ\boldsymbol{R}^{-1}\boldsymbol{G}^{\circ T}, \ \boldsymbol{X}_\delta = \boldsymbol{H}^{-1} = \boldsymbol{X} - \delta^{-1}\boldsymbol{G}_v\boldsymbol{G}_v^T \tag{95}$$

Proof. Using identity $\boldsymbol{q}(i) = \boldsymbol{q}(i)$ and control (16), the system transform can be introduced as follows

$$\begin{bmatrix} \boldsymbol{q}(i) \\ \boldsymbol{u}(i) \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ -\boldsymbol{M} & \boldsymbol{N} \end{bmatrix} \begin{bmatrix} \boldsymbol{q}(i) \\ \boldsymbol{u}^\circ(i) \end{bmatrix} = \boldsymbol{T} \begin{bmatrix} \boldsymbol{q}(i) \\ \boldsymbol{u}^\circ(i) \end{bmatrix} \tag{96}$$

$$\boldsymbol{T} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ -\boldsymbol{M} & \boldsymbol{N} \end{bmatrix} \tag{97}$$

to describe modified control law representation for matrices given in (15), (16).

Since system (1), (2) is linear in $\boldsymbol{q}(i)$ the quadratic Lyapunov function can be of the form

$$v(\boldsymbol{q}(i)) = \boldsymbol{q}^T(i)\boldsymbol{P}\boldsymbol{q}(i) \tag{98}$$

where $\boldsymbol{P} \in I\!R^{n\times n}$ is a symmetric positive definite matrix. If Lyapunov function takes form (98), its difference is

$$\Delta v(\boldsymbol{q}(i), \boldsymbol{u}(i)) = v(\boldsymbol{q}(i+1)) - v(\boldsymbol{q}(i)) \tag{99}$$

$$\Delta v(\boldsymbol{q}(i), \boldsymbol{u}(i)) = \left[\, \boldsymbol{q}^T(i)\ \boldsymbol{u}^T(i) \,\right] \boldsymbol{J}_V(i) \begin{bmatrix} \boldsymbol{q}(i) \\ \boldsymbol{u}(i) \end{bmatrix} \tag{100}$$

respectively, where

$$\begin{aligned}\boldsymbol{J}_V(i) = \\ = \begin{bmatrix} (\boldsymbol{F} + \boldsymbol{F}_a a(i))^T\boldsymbol{P}(\boldsymbol{F} + \boldsymbol{F}_a a(i)) - \boldsymbol{P} & \boldsymbol{Z} \\ \boldsymbol{Z}^T & \boldsymbol{G}^T\boldsymbol{P}\boldsymbol{G} \end{bmatrix}\end{aligned} \tag{101}$$

$$\boldsymbol{Z} = (\boldsymbol{F} + \boldsymbol{F}_a a(i))^T\boldsymbol{P}\boldsymbol{G} \tag{102}$$

Using transform (96), (97) the Lyapunov function diference given in (100), (101) can be equivalently rewritten to next form

$$\Delta v\big(\boldsymbol{q}(i), \boldsymbol{u}^\circ(i)\big) = \left[\, \boldsymbol{q}^T(i)\ \boldsymbol{u}^{\circ T}(i) \,\right] \boldsymbol{J}_V^\circ(i) \begin{bmatrix} \boldsymbol{q}(i) \\ \boldsymbol{u}^\circ(i) \end{bmatrix} \tag{103}$$

with

$$\boldsymbol{J}_V^\circ(i) = \boldsymbol{T}^T \boldsymbol{J}_V(i) \boldsymbol{T} = \begin{bmatrix} (\boldsymbol{F}^\circ + \boldsymbol{F}_a a(i))^T \boldsymbol{P} (\boldsymbol{F}^\circ + \boldsymbol{F}_a a(i)) - \boldsymbol{P} & \boldsymbol{Z}^\circ \\ \boldsymbol{Z}^{\circ T} & \boldsymbol{G}^{\circ T} \boldsymbol{P} \boldsymbol{G}^\circ \end{bmatrix} \quad (104)$$

$$\boldsymbol{Z}^\circ = (\boldsymbol{F}^\circ + \boldsymbol{F}_a a(i)) \boldsymbol{P} \boldsymbol{G}^\circ \quad (105)$$

$$\boldsymbol{F}^\circ = \boldsymbol{F} - \boldsymbol{G}\boldsymbol{M} \quad (106)$$

$$\boldsymbol{G}^\circ = \boldsymbol{G}\boldsymbol{N} \quad (107)$$

Defining algebraic inequality

$$\boldsymbol{F}^{\circ T} (\boldsymbol{P}_G^{\circ -1} + \boldsymbol{H}) \boldsymbol{F}^\circ + \boldsymbol{F}_a^T \boldsymbol{P} \boldsymbol{F}_a + \boldsymbol{C}^T \boldsymbol{C} - \boldsymbol{P} < 0 \quad (108)$$

then, by following the similar approach used for deriving (68), (69), one can obtain (93), (94) for equivalent system, defined in (91), (92).

Corollary 1. Since

$$\begin{aligned} \boldsymbol{F}^\circ - \boldsymbol{G}^\circ \boldsymbol{K}^\circ &= \boldsymbol{F} - \boldsymbol{G}\boldsymbol{M} - \boldsymbol{G}\boldsymbol{N}\boldsymbol{K}^\circ = \\ &= \boldsymbol{F} - \boldsymbol{G}\boldsymbol{M} - \boldsymbol{G}(\boldsymbol{K} - \boldsymbol{M}) = \boldsymbol{F} - \boldsymbol{G}\boldsymbol{K} \end{aligned} \quad (109)$$

one can see that the eigenvalues spectrum $\rho(\boldsymbol{F} - \boldsymbol{G}\boldsymbol{K})$ of the closed-loop system matrix $\boldsymbol{F}_C = \boldsymbol{F} - \boldsymbol{G}\boldsymbol{K}$ is the same as the eigenvalues spectrum of designed closed-loop system matrix $\boldsymbol{F}_C^\circ = \boldsymbol{F}^\circ - \boldsymbol{G}^\circ \boldsymbol{K}^\circ$. Note that the new matrix $\boldsymbol{F}_C^\circ$ is singular and we have to solve a singular control problem.

Corollary 2. Since (28) implies

$$\begin{aligned} &\boldsymbol{z}^T(i)\boldsymbol{z}(i) = \\ &= [\, \boldsymbol{q}(i) \;\; \boldsymbol{u}(i) \,]^T \begin{bmatrix} \boldsymbol{C}^T\boldsymbol{C} & \boldsymbol{C}^T\boldsymbol{D} \\ \boldsymbol{D}^T\boldsymbol{C} & \boldsymbol{D}^T\boldsymbol{D} \end{bmatrix} \begin{bmatrix} \boldsymbol{q}(i) \\ \boldsymbol{u}(i) \end{bmatrix} \leq \\ &\leq 2\boldsymbol{q}^T(i)\boldsymbol{C}^T\boldsymbol{C}\boldsymbol{q}(i) + 2\boldsymbol{u}^T(i)\boldsymbol{D}^T\boldsymbol{D}\boldsymbol{u}(i) = \\ &= 2\boldsymbol{q}^T(i)\boldsymbol{C}^T\boldsymbol{C}\boldsymbol{q}(i) + \boldsymbol{u}^T(i)\boldsymbol{R}\boldsymbol{u}(i) \end{aligned} \quad (110)$$

it is evident, that for stabilizing control can be matrix $\boldsymbol{R} > 0$ chosen independently on $\boldsymbol{C}^T\boldsymbol{C}$ as a free design parameter, to modify eigenvalue structure of closed-loop matrix $\boldsymbol{F}_C^\circ$, and consequently $\boldsymbol{F}_C$, too.

8. ILLUSTRATIVE EXAMPLE

To demonstrate properties one simple system described by the discrete-time state-space equations (1), (2), was considered, where

$$\boldsymbol{F} = \begin{bmatrix} 0.9993 & 0.0987 & 0.0042 \\ -0.0212 & 0.9612 & 0.0775 \\ -0.3875 & -0.7187 & 0.5737 \end{bmatrix}, \; \boldsymbol{G}_v = \begin{bmatrix} 0.1 \\ 0.3 \\ 0.7 \end{bmatrix}$$

$$\boldsymbol{G} = \begin{bmatrix} 0.0051 & 0.0050 \\ 0.1029 & 0.0987 \\ 0.0387 & -0.0388 \end{bmatrix}, \; \boldsymbol{C} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\boldsymbol{F}_a = \begin{bmatrix} 0 & 0 & 0.0004 \\ 0 & 0 & 0 \\ 0.0388 & 0 & 0 \end{bmatrix}, \; \boldsymbol{R} = 40 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

for sampling period $\Delta t = 0.1$ s. Assuming matrix equality constraint as

$$\boldsymbol{L} = [\, 2 \;\; -1 \;\; 1 \,]$$

there were obtained feedback gain matrix parameters

$$\boldsymbol{M} = \begin{bmatrix} -4.5975 & 4.1756 & -1.4212 \\ -10.8552 & 9.8590 & -3.3557 \end{bmatrix}$$

$$\boldsymbol{N} = \begin{bmatrix} 0.8479 & -0.3591 \\ -0.3591 & 0.1521 \end{bmatrix}$$

New design parameters were then recomputed as follows

$$\boldsymbol{F}^\circ = \begin{bmatrix} 1.0770 & 0.0281 & 0.0282 \\ 1.5233 & -0.4415 & 0.5550 \\ -0.6308 & -0.4978 & 0.4985 \end{bmatrix}$$

$$\boldsymbol{G}^\circ = \begin{bmatrix} 0.0025 & -0.0011 \\ 0.0518 & -0.0219 \\ 0.0467 & -0.0198 \end{bmatrix}$$

The LMIs (93), (94) was solved using the Self-Dual-Minimization (SeDuMi) package for MATLAB (Peaucelle et al. (2002)). Problem was solvable and yielding for given system matrices the terminal weight matrix

$$\boldsymbol{X} = \begin{bmatrix} 0.0017 & -0.0177 & -0.0312 \\ -0.0177 & 0.6122 & 0.3201 \\ -0.0312 & 0.3201 & 0.5831 \end{bmatrix}, \; (\gamma = 4.9444 \cdot 10^3)$$

for which feedback gain matrix, as well as the eigenvalues set are

$$\boldsymbol{K}^\circ = \begin{bmatrix} 50.2308 & 0.0620 & 2.6530 \\ -21.2742 & -1.0263 & -1.1236 \end{bmatrix}, \; \text{eig}\, \boldsymbol{F}_C^\circ = \begin{bmatrix} 0.7457 \\ 0.0884 \\ 0.0000 \end{bmatrix}$$

9. CONCLUDING REMARKS

The paper presents the control design principle for discrete-time linear stochastic multi-variable dynamic systems with multiplicative noise and state variable constraints in the form of matrix equalities. Presented applications can be considered as a task concerned the class of H_∞ stabilization control problems where the stabilizing solutions were new formulated. It should be emphasized that the proposed stabilizing approach, based on LMIs, offers possibility to tune eigenvalue set of the closed-loop system matrix.

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