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Optimal Measurement Feedback Control of Finite-time Continuous Linear Systems

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Abstract: We consider optimal control of linear continuous time systems subject to constrained inputs based on output measurements that are subject to set-membership uncertainties. The objective is to steer the system in a fixed finite-time to a given polyhedral set while minimizing a linear terminal cost function. To achieve this objective a min-max optimal control strategy is proposed, taking into account that at future time instants new measurement information is available for feedback. The proposed strategy coincides with the (computationally intractable) exact min-max dynamic programming solution if all future measurement times are considered. Limiting the number of instants at which the new measurement information is considered we achieve a compromise between the computational efforts for feedback construction and the performance of the closed-loop. *Copyright ©2008 IFAC.*

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1. INTRODUCTION

Optimal control has received a significant revival over the recent years. This is mainly driven by the need for control mechanisms that can handle state and input constraints in a structured way. Furthermore, there have been substantial advancements with respect to the available computational power and efficient numerical on-line solution methods for optimal control problems. Also nowadays optimal control based control methods such as model predictive control have been widely applied and significant theoretical achievements with respect to these methods have been made, see e.g. Mayne et al. (2000); Fontes (2003); Camacho and Bordons (2007).

An inherent problem of most optimal control strategies is that for the prediction the full knowledge of the state is required. In practice, however, not all states are directly measured. Various researchers have addressed the question of output feedback optimal control, see e.g. Moitie et al. (2002); Kurzhanski (2003); Findeisen and Allgöwer (2004); Mayne et al. (2006); Goulart and Kerrigan (2007). Often, however, these results are limited to discrete time systems.

In this note we consider the finite-time control problem for continuous time linear systems subject to constrained inputs and limited state information. The objective is to steer the system from a known set of possible initial states to a polyhedral set in finite-time while minimizing a linear terminal cost. The exact initial state of the system is assumed to be not known, rather only a bounded set estimate is available. Additional output measurements which are subject to unknown, but bounded measurement errors are available at discrete sampling times for feedback. They can be utilized to improve the estimate of the current system state.

Different solution approaches for this problem are possible. In the simplest case one can calculate an open-loop optimal control trajectory, minimizing the worst-case performance of all possible initial state trajectories. This leads, however, in general to a very conservative behavior, since one control input has to steer all possible initial states to the terminal set. On the other hand one can consider that at every future measurement instant new state estimates become available for control. While this would lead to optimal performance, since the availability of new state information is completely taken into account, the numerical solution is often computationally intense and requires approximations, see e.g. Moitie et al. (2002); Kurzhanski (2003).

In this note we propose an intermediate, suboptimal but computationally tractable approach. Basically we propose to close the loop in the prediction of the optimal control problem only at a limited number of future measurement instants. This leads to a drastically reduced computational demand. The resulting optimal control problem is then resolved at the next sampling time, taking new measurements into account. Based on this idea we outline a finite

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dimensional approximation and an efficient numerical solution approach, to allow for real time application of the proposed method.

The overall paper is structured as follows. In Section 2 we outline the mathematical problem formulation and the control objective. Section 3 reviews results related to open-loop optimal control, which are needed in the following sections. Section 4 presents the idea of once-closed optimal feedback control. In Section 4.2 we outline a finite-dimensional formulation for the new strategy. Section 4.3 discusses the existence of optimal solutions, while in Section 5 an algorithm for the construction of once-closed optimal control is presented. Section 6 provides some final conclusion.

2. PROBLEM FORMULATION

We consider linear time-invariant systems of the form

$$\dot{x} = Ax + Bu, \quad x(0) \in X_0, \tag{1}$$

where $x \in \mathbb{R}^n$ denotes the system state. The input u is constrained to the set U, i.e. $u \in U \subseteq \mathbb{R}^r$, where U is a hyper-rectangle of the form $U = \{u \in \mathbb{R}^r : |u_j| \leq 1, j = \overline{1,r}\}$. The set X_0, defined by the hyper-rectangle $X_0 = \{x \in \mathbb{R}^n : x_{\min} \leq x \leq x_{\max}\}$, is the set of possible initial states, i.e. $x(0)$ is not exactly known and not available for control purposes.

Control objective: The control objective is to steer system (1) using sampled-data control, in a fixed finite-time $T \geq 0$ to a polytope $S = \{x \in \mathbb{R}^n : h_i^T x \leq g_i, i = \overline{1,m}\}$, i.e. $x(T) \in S$, while satisfying the input constraints and minimizing a linear terminal penalty $h_0^T x(T)$. Sampled-data control refers here to the fact that the applied input only changes at fixed sampling instants and that the input is assumed to be constant in between. Sampling instants are in the following denoted by s, where $s \in \pi = \{k\delta, k = \overline{0, N-1}\}$. Here δ denotes the sampling time which is, for simplicity assumed to be constant. It is defined in terms of the discretization $N \in \mathbb{N}$ of the finite control time T: $\delta = T/N$. Thus, the input u in (1) is given by:

$$u(t) \equiv u(s), \quad t \in [s, s+\delta[,$$

where $u(s)$ depends on the current state of the system or estimates of it.

It is assumed that at the sampling instants s, where $s \in \bar{\pi} = \{k\delta, k = \overline{1,N}\}$, measurements $y(s) \in \mathbb{R}^q$ are available. However, these are subject to bounded measurement errors $\xi(s) \in \Xi$

$$y(s) = Cx(s) + \xi(s),$$

where the measurement error set Ξ is assumed to be a hyper-rectangle: $\Xi = \{\xi \in \mathbb{R}^q : \xi_{\min} \leq \xi \leq \xi_{\max}\}$.

2.1 Notation

In the following the set of all nonempty convex compact subsets of $\mathbb{R}^n$ is denoted by $\mathcal{X}$, its elements are denoted by X.

The set of all sampled-data controls defined in a segment $[t', t'']$ with values in U is shortly denoted by $\mathcal{U}_{[t',t'']}$.

In the following $x(t''; t', z, u(\cdot))$ denotes the solution of (1) at time t'' if the initial state at time t' is z and the control is $u(\cdot) \in \mathcal{U}_{[t',t'']}$; also note that $x_0(t''; t', u(\cdot)) = x(t''; t', 0, u(\cdot))$. Furthermore, let $\bar{\pi}_{[t',t'']} =]t', t''] \cap \bar{\pi}$. A set of all possible measurement trajectories $y(\cdot) = \{y(s), s \in \bar{\pi}_{[t',t'']}\}$ of the autonomous system (1) and an initial state at time t' from X, is denoted by $\mathcal{Y}_{[t',t'']}(X)$.

Since we have to distinguish between the variables used in the optimal control problem for predictions and the real system/plant variables, we will denote the latter by a superscript p. Thus, u^p and y^p denote the input and the measurement trajectory which realize in a particular control process. Moreover, $X^p(t)$ are all states $x(t)$ consistent with the initial condition $x_0 \in X_0$ and the measurements $y^p(s), s \in \bar{\pi}_{[0,t]}$.

2.2 Approaches to Optimal Output-feedback Control

So far we did not go into details in which sense the final terminal penalty term is minimized, how the availability of future measurements is considered in the prediction, and if and how often the input is recalculated based on new measurements $y(s)$. In principle different possibilities for "optimally" controlling (1) exist:

One could only calculate one "optimal" input at the first sampling instant $s = 0$ based solely on the information of $x(0) \in X_0$ and then apply this input open-loop to the system until the final time T. In this case one would search for an input $u^0(\cdot; 0, X_0) \in \mathcal{U}_{[0,T]}$ that drives all possible realizations of the initial state $x(0) \in X_0$ to the terminal set S, i.e. that $x(T; 0, x_0, u^0(\cdot; 0, X_0)) \in S \quad \forall x_0 \in X_0$. Furthermore, the input $u^0(\cdot; 0, X_0)$ should be such that it minimizes $h_0^T x(T)$ under the worst-case realization of $x_0 \in X_0$. This approach is in general referred to as *open-loop (worst-case) optimal control.* The main disadvantage of this approach is that new measurement information arising at the sampling instants s via $y(s)$ is not taken into account, which leads to a very conservative control performance. Furthermore, the approach as formulated is very conservative, since one open-loop input $u^0(\cdot; 0, X_0)$ is supposed to steer all possible realizations of $x_0 \in X_0$ to the terminal set. This might even lead to infeasibility, if the set X_0 is large.

One approach to reduce this conservatism is to repeatedly solve the open-loop optimal control problem at every sampling instant $\tau \in \pi$ subject to a shrinking horizon of length $T - \tau$ and a new set of possible states $X^p(\tau)$, or its estimate $\hat{X}^p(\tau)$. In this case the input applied to the plant is given by $u^p(t) \equiv u^0(\tau; \tau, X^p(\tau))$, $t \in [\tau, \tau+\delta[$, $\tau \in \pi$. While this does not counteract the fact that the initial optimal control problem is not feasible if the set X_0 is large, it can lead to significant performance improvements, since the set-based state estimate might improve from time step to time step. This control strategy for a finite-time problem is usually referred to as *open-loop (worst-case) optimal feedback control.*

One way to overcome the feasibility problem and the conservatism due to the assumption that one input has to cope with all possible initial states is to fuse repeated open-loop control with an output-feedback closed-loop control. In the frame of predictive control of continuous time systems without finite-time this has for example been proposed in Findeisen and Allgöwer (2004).

While this approach leads to reduced conservatism in general, it still does not take into account, that at future time instants new output measurements, and thus a new (optimal) set-based state estimate will be available. In principle one can formulate this as a dynamic programming problem, which is, however, in general intractable. In the following we will refer to the optimal dynamic programming solution as the full solution or shortly "the optimal solution" with respect to the control objective. Possible solutions to overcome this problem are the use of fixed observer dynamics, e.g. possibly non optimal observers, and fixed local controller dynamics. For discrete time linear systems this has been for example considered in Mayne et al. (2006).

In the frame of this work we consider an intermediate approach between the dynamic programming solution and the open-loop optimal feedback. We propose that in the optimal control formulation the loop is only closed at a finite number of time instants, e.g. not at all sampling instants. In the case of only one closing we will refer to this as once-closed control, e.g. in the optimal control problem it is assumed that at one single future time instant new measurement information is available. This leads to a trade-off between the computational demand and the conservatism. In case of robust state feedback this approach has been addressed in Balashevich et al. (2004); Kostyukova and Kostina (2006).

In the following section we review some results related to open-loop optimal control subject to uncertain initial states. Section 4 introduces the concept of once-closed optimal feedback control and discusses the existence of solution as well as finite-dimensional formulations. The construction of the once-closed solution is discussed in Section 5.

3. OPEN-LOOP OPTIMAL CONTROL

In the following we review some results about open-loop optimal control subject to uncertain initial states which are needed in the later sections. Further details can be found in Gabasov et al. (2007).

The general open-loop optimal control problem is denoted by $\mathcal{P}^0(\tau, X)$, where the arguments (τ, X) stress that the control process starts at the current instant $\tau \in \pi$ from an unknown state $x(\tau)$ which lays in an arbitrary initial set $X \in \mathcal{X}$. The solution of $\mathcal{P}^0(\tau, X)$ is an input $u^0(\cdot; \tau, X) \in \mathcal{U}_{[\tau,T]}$ steering every realization of the initial state $x(\tau) \in X$ into the terminal set S. Furthermore, it minimizes the worst-case cost $\max_{x\in X} h_0^T x(T; \tau, x, u(\cdot))$.

In Gabasov et al. (2007) it is shown that the optimal open-loop control $u^0(\cdot; \tau, X)$ can be obtained by the solution of the following optimal control problem

$$\min_{u(\cdot)\in\mathcal{U}_{[\tau,T]}} h_0^T x(T), \tag{2}$$

subject to

$$\dot{x} = Ax + Bu,\ x(\tau) = 0,$$
$$h_i^T x(T) \le g_i - \hat{\chi}_i(\tau, X),\ i = \overline{1, m}.$$

The resulting cost of problem $\mathcal{P}^0(\tau, X)$ is given by

$$J^0(\tau, X) = \hat{\chi}_0(\tau, X) + h_0^T x_0(T; \tau, u^0(\cdot)).$$

Note that if problem (2) is not feasible we use the convention that $J^0(\tau, X) = +\infty$.

Above the cost and the new terminal constraints are defined in terms of $\hat{\chi}_i(\tau, X)$, $i = \overline{0, m}$, which correspond to the worst-case realizations of the uncertain initial condition of the following problem:

$$\hat{\chi}_i(\tau, X) = \max \psi_i^T(\tau)x,\ x \in X.$$

Here $\psi_i(t) \in \mathbb{R}^n$, $t \in [0, T]$, denotes the solution of the adjoint equation $\dot{\psi}_i = -A^T\psi_i$ with $\psi_i(T) = h_i$.

Problem (2) is significantly easier to solve in comparison the original min-max formulation of $\mathcal{P}^0(\tau, X)$ (which is not stated here due to limited space), since all $\hat{\chi}_i$ are independent of the input. The vector $\hat{\chi}(\tau, X) = (\hat{\chi}_i(\tau, X), i = \overline{0, m})$ defines a polyhedral set-valued estimate for X of the form $\hat{X}(\tau, X) = \{x \in \mathbb{R}^n : \Psi(\tau)x \le \hat{\chi}(\tau, X)\}$, where $\Psi(\tau) = (\psi_i(\tau), i = \overline{0, m})$. Note that $u^0(\cdot; \tau, X) = u^0(\cdot; \tau, \hat{X}(\tau, X))$. Hence, the estimate $\hat{\chi}$ contains all "state information" that is sufficient for solving problem (2) and consequently $\mathcal{P}^0(\tau, X)$. For this reason we call $\hat{\chi}(\tau, X)$ a sufficient estimate of the set of possible states X.

So basically instead of the set-valued (infinite-dimensional) "state" $X \in \mathcal{X}$ we can use the not set-valued (finite-dimensional) estimate $\hat{\chi} \in \mathbb{R}^{m+1}$. This reformulation is also the core idea for the finite-dimensional formulation of our approach, as presented in Section 4.2.

3.1 Obtaining State Estimates Efficiently

At the sampling times $\tau \in \pi$ one has to solve problem $\mathcal{P}^0(\tau, X^p(\tau))$ for the current set of possible states $X^p(\tau)$, depending on the actual input $u^p(t)$, $t \in [0, \tau]$, and measurements $y^p(s)$, $s \in \bar{\pi}_{[0,\tau]}$. As outlined, the vector $\hat{\chi}^p(\tau) = \hat{\chi}(\tau, X^p(\tau))$ contains all informations required to solve (2). So if one could obtain the estimate $\hat{\chi}^p(\tau)$ in an efficient way, it would not be necessary to calculate $X^p(\tau)$.

As shown in Gabasov et al. (2007), this goal can be achieved on the base of "purified measurements" $y_0^p(s) = y^p(s) - Cx_0(s; 0, u^p(\cdot))$, $s \in \bar{\pi}_{[0,\tau]}$, by the solution of $m+1$ optimization problems

$$\hat{\gamma}_i(\tau; X_0, y_0^p(\cdot)) = \max_z\ \psi_i^T(\tau)\hat{x}(\tau),\ \dot{\hat{x}} = A\hat{x},\ \ \hat{x}(0) = z,$$
$$y_0^p(s) - C\hat{x}(s) \in \Xi,\ s \in \bar{\pi}_{[0,\tau]},\ z \in X_0. \tag{3}$$

Then $\hat{\chi}^p(\tau) = \Psi(\tau)x_0(\tau; 0, u^p(\cdot)) + \hat{\gamma}(\tau; X_0, y_0^p(\cdot))$.

4. ONCE-CLOSED OPTIMAL FEEDBACK

The open-loop optimal feedback does not take into account that at future time instants new measurements become available that can be used for feedback. This leads to rather conservative results. On the other extreme, the optimal solution by dynamic programming considers that at every future sampling instant all new measurement information is used for control. This does in general lead to a problem which is computationally not tractable. In this section we consider an intermediate case and formulate an optimal control problem that takes into account that at a small number of fixed future instants, called closing instants, new measurements are used for feedback. This allows to balance the computational efforts for the optimal feedback construction with respect to the performance of the overall closed-loop.

For simplicity of presentation we only describe the case of one closing instant, denoted by t^1. The generalization for several closing instants $\pi_{cl} = \{t^1, t^2, ..., t^\kappa\}$ is merely technical. Note that the scheme results in the optimal solution as obtained by dynamic programming if $\pi_{cl} = \bar{\pi}$.

4.1 Closure Sets and t^1-closed Controls

Consider the current sampling instant $\tau \in \pi$, an arbitrary initial set $X \in \mathcal{X}$ and a fixed closing instant $t^1 \in \pi \setminus \{0\}$.

We will distinguish between the cases that τ is bigger then the closing instant and that it is smaller. For this we define the vectors of sampling instants $\pi^1 = [0, t^1[\cap \pi$ and $\pi^0 = [t^1, T] \cap \pi$.
Case $\tau \in \pi^0$:
In this case the optimal input is given by the optimal open-loop control $u^0(\cdot; \tau, X)$ as defined by the solution of $\mathcal{P}^0(\tau, X)$, see Section 3.
Case $\tau \in \pi^1$:
We need in the following the closure set $\mathcal{X}^1$ for the time t^1:

$$\mathcal{X}^1 = \{X^1 \in \mathcal{X} : J^0(t^1, X^1) < +\infty\}.$$

Basically $\mathcal{X}^1$ consists of all sets $X^1 \in \mathcal{X}$ such that there exists an optimal open-loop control $u^0(\cdot; t^1, X^1)$ of problem $\mathcal{P}^0(t^1, X^1)$. Note that $\mathcal{X}^1 \neq \emptyset$.

We furthermore need a set that bounds the worst-case value for the open-loop optimal cost of problem $\mathcal{P}^0(t^1, X^1)$. To this end we define a so called α-closure set at time t^1:

$$\mathcal{X}^1_\alpha = \{X^1 \in \mathcal{X}^1 : \ J^0(t^1, X^1) \leq \alpha\},$$

where $\alpha \in [\alpha_{\min}, \alpha_{\max}]$, $\alpha_{\min} = \min_{x \in S} h_0^T x$; $\alpha_{\max} = \max_{x \in S} h_0^T x$. This implies that for every $X^1 \in \mathcal{X}^1_\alpha$ the optimal open-loop control guarantees the cost to be at most α.

Remark 1. The set $\mathcal{X}^1$ can be equivalently defined as consisting of all compact sets $X^1 \in \mathcal{X}$ such that there exists an admissible, not necessarily optimal, open-loop control of problem $\mathcal{P}^0(t^1, X^1)$. For the sets $X^1 \in \mathcal{X}^1_\alpha$ this control must guarantee the cost to be at most α. These equivalent definitions are easier to use for constructing the closure sets, since not necessarily the optimal open-loop control must be obtained.

Now let $u(\cdot) \in \mathcal{U}_{[\tau, t^1]}$ be a given, fixed input and $\mathcal{X}^1(u(\cdot); \tau, X)$ denote the collection of all sets of possible states X^1, that can be obtained at time t^1 if the process starts at the instant τ from $x(\tau) \in X$ under the input $u(\cdot)$.

Suppose that for some $\alpha \in [\alpha_{\min}, \alpha_{\max}]$ there exists an input $u^1_\alpha(\cdot; \tau, X) \in \mathcal{U}_{[\tau, t^1]}$, such that the following inclusion holds

$$\mathcal{X}^1(u^1_\alpha(\cdot); \tau, X) \subseteq \mathcal{X}^1_\alpha. \tag{4}$$

Construct the following control strategy: use $u^1_\alpha(\cdot; \tau, X)$ on the interval $[\tau, t^1]$, then at time t^1 switch to the optimal open-loop control $u^0(\cdot; t^1, X(t^1))$, where $X(t^1) \in \mathcal{X}^1(u^1_\alpha(\cdot); \tau, X)$ is the set of possible states which realized at t^1. Due to (4), this control strategy guarantees that the cost related to the optimal control problem is at most α.

The function $u^1_\alpha(\cdot; \tau, X)$ and $\{u^0(\cdot; t^1, X^1), X^1 \in \mathcal{X}^1(u^1_\alpha(\cdot); \tau, X)\}$ define a feasible control strategy. We will refer to this as an admissible t^1-closed or once-closed control. Consequently the optimal t^1-closed control is defined by the minimum achievable α: $\alpha^0(\tau, X) = \min \alpha$.

The optimal control problem related to the optimal t^1-closed control is denoted by $\mathcal{P}^1(\tau, X)$. Its optimal cost (if it exists) is given by:

$$J^1(\tau, X) = \alpha^0(\tau, X).$$

If no t^1-closed solution exists $J^1(\tau, X) = +\infty$.

Summarizing, the corresponding optimal input on π^1 is given by

$$u^1(\cdot; \tau, X) = u^1_{\alpha^0}(\cdot; \tau, X),$$

and the overall applied once-closed optimal feedback becomes

$$u^p(t) \equiv \begin{cases} u^1(\tau; \tau, X^p(\tau)), \ \tau \in \pi^1, \\ u^0(\tau; \tau, X^p(\tau)), \ \tau \in \pi^0, \end{cases} \tag{5}$$
$$t \in [\tau, \tau + \delta[, \ \tau \in \pi.$$

Given this feedback we can derive :
Proposition 1. Assume that a solution to $\mathcal{P}^1(0, X_0)$ exists. Then $\mathcal{P}^1(\tau, X^p(\tau))$ is feasible for all $\tau \in \pi^1$, $\mathcal{P}^0(\tau, X^p(\tau))$ is feasible for all $\tau \in \pi^0$, and $J^1(\tau, X^p(\tau))$, $\tau \in \pi^1$, $J^0(\tau, X^p(\tau))$, $\tau \in \pi^0$, are non-increasing.

This proposition implies that if the optimal once-closed control exists for $t = 0$ then it exists for all times and thus (5) can indeed be implemented for feedback control. Note that the proof is trivial due to Bellman's principle of optimality.

The definitions of the closure sets and the t^1-closed controls introduced in this section are set-based. This makes the task of deriving conditions for existence, as well as the development of suitable algorithms a demanding, non-trivial task. To avoid this we will replace the sets X^1 with their estimates $\hat{X}^1$, characterized by the sufficient estimates $\hat{\chi}^1 = \hat{\chi}(t^1, X^1)$ (see Section 3). This allows us to derive finite-dimensional counterparts for the closure sets, and the admissible and optimal t^1-closed controls. On the basis of these finite-dimensional counterparts it is possible to derive simple conditions for the existence of the optimal t^1-closed control and to derive an algorithm for solving the resulting optimal control problem $\mathcal{P}^1(\tau, X)$.

4.2 Finite-dimensional Formulation

In this section we derive finite-dimensional, not set-valued counterparts for the closure sets, and the admissible and the optimal t^1-closed controls.

To this end we consider the optimal control problem $\mathcal{P}^0(t^1, \hat{\chi}^1)$, which is equivalent to $\mathcal{P}^0(t^1, X^1)$, using the concept of sufficient estimates as introduced in Section 3:

$$\min_{u(\cdot) \in \mathcal{U}_{[t^1, T]}} h_0^T x(T),$$

subject to the constraints

$$\dot{x} = Ax + Bu, \ x(t^1) = 0,$$
$$h_i^T x(T) \leq g_i - \hat{\chi}^1_i, \ i = \overline{1, m}.$$

The resulting optimal open-loop control is denoted by $u^0(\cdot; t^1, \hat{\chi}^1)$ and the optimal cost of problem $\mathcal{P}^0(t^1, \hat{\chi}^1)$ is given by

$$J^0(t^1, \hat{\chi}^1) = \hat{\chi}^1_0 + h_0^T x_0(T; \tau, u^0(\cdot)).$$

Similar to the definition of the closure sets in the previous section we define their corresponding counterparts on the basis of open-loop solutions of problem $\mathcal{P}^0(t^1, \hat{\chi}^1)$:

$$\hat{X}^1 = \{\hat{\chi}^1 \in \mathbb{R}^{m+1} : J^0(t^1, \hat{\chi}^1) < +\infty\},$$
$$\hat{X}^1_\alpha = \{\hat{\chi}^1 \in \mathbb{R}^{m+1} : J^0(t^1, \hat{\chi}^1) \leq \alpha\}.$$

Remark 2. The sets $\hat{X}^1$, $\hat{X}^1_\alpha$ may include elements $\hat{\chi}^1$ that define empty sets X^1. These estimates, however, will not be encountered in the real processes.

To derive a counterpart of inclusion (4) the elements X^1 of the set $\mathcal{X}^1(u(\cdot);\tau,X)$ are also characterized by their sufficient estimates $\hat{\chi}^1$. To achieve this we introduce the set of all possible estimates of the form (3) that can be obtained at time t^1:

$$\hat{\Gamma}^1(\tau,X)=\{\hat{\gamma}^1\in\mathbb{R}^{m+1}: \hat{\gamma}^1=\hat{\gamma}(t^1;X,y(\cdot)),\ y(\cdot)\in\mathcal{Y}_{[\tau,t^1]}(X)\}.$$

This allows to define the finite-dimensional counterpart of the set $\mathcal{X}^1(u(\cdot);\tau,X)$:

$$\hat{X}^1(u(\cdot);\tau,X)=\{\hat{\chi}^1\in\mathbb{R}^{m+1}: \hat{\chi}^1=\Psi(t^1)x_0(t^1;\tau,u(\cdot))+\hat{\gamma}^1,\ \hat{\gamma}^1\in\hat{\Gamma}^1(\tau,X)\}.$$

Thus the control objective on the time interval $[\tau,t^1]$ is now reformulated as follows: For a given $\alpha\in[\alpha_{\min},\alpha_{\max}]$ find a control $u^1_\alpha(\cdot;\tau,X)\in\mathcal{U}_{[\tau,t^1]}$, such that

$$\hat{X}^1(u^1_\alpha(\cdot);\tau,X)\subseteq\hat{X}^1_\alpha \tag{6}$$

or, equivalently,

$$\Psi(t^1)x_0(t^1;\tau,u^1_\alpha(\cdot))+\hat{\gamma}^1\in\hat{X}^1_\alpha \text{ for all } \hat{\gamma}^1\in\hat{\Gamma}^1(\tau,X).$$

If now for a given α there exists a control $u^1_\alpha(\cdot;\tau,X)$, satisfying (6), then the admissible t^1-closed control takes the form $\{u^1_\alpha(\cdot;\tau,X);\ u^0(\cdot;t^1,\hat{\chi}^1),\hat{\chi}^1\in\hat{X}^1(u^1_\alpha(\cdot);\tau,X)\}$.

Similar to the set-based formulation the minimal value $\alpha^0(\tau,X)$, for which inclusion (6) holds, defines the optimal value of the problem and thus the t^1-closed solution. However, this is now based on a finite-dimensional reformulation.

4.3 Existence of Optimal Once-closed Controls

In this section we derive conditions for the existence of optimal t^1-closed controls. Basically this is based on establishing a connection between the t^1-closed solution of $\mathcal{P}^1(\tau,X)$, $\tau\in\pi^1$, and the solution of the following auxiliary optimal control problem

$$\rho(\alpha)=\min_{u(\cdot)}\rho, \tag{7}$$

subject to

$$\dot{x}=Ax+Bu,\quad x(\tau)=0,$$
$$\Psi(t^1)x(t^1)+\hat{\gamma}^1\in\hat{X}^1_\alpha,\ \ \hat{\gamma}^1\in\hat{\Gamma}^1(\tau,X),$$
$$|u_j(t)|\le\rho,\ j=\overline{1,r},\quad t\in[\tau,t^1[.$$

The resulting optimal open-loop input is denoted by $u^0_\rho(t;\tau,X)$, $t\in[\tau,t^1]$.

In the following we assume:

Assumption 1. Problem (7) is feasible for $\alpha=\alpha_{\max}$ and $\rho(\alpha_{\max})<1$.

Basically this assumption implies that problem $\mathcal{P}^1(\tau,X)$ has an admissible t^1-closed control.

Proposition 2. Under Assumption 1 problem $\mathcal{P}^1(\tau,X)$, $\tau\in\pi^1$, has the t^1-closed solution and the optimal value α^0 of its cost function is given by:

(i) $\rho(\alpha^0)=1$, or
(ii) $\alpha^0=\alpha'$ if $\rho(\alpha')<1$ and problem (7) is infeasible for all $\alpha<\alpha'$.

Furthermore, $u^1(\cdot;\tau,X)=u^0_{\rho(\alpha^0)}(\cdot;\tau,X)$.

Proof 1. Consider two cases: problem (7) is infeasible for $\alpha<\alpha_{\max}$, and problem (7) has a solution for all $\alpha\in[\alpha',\alpha_{\max}]$. In the former case the assertion of the theorem follows immediately.

In the second case if $\rho(\alpha)>1$ it is impossible to find an input $u^1_\alpha(\cdot)\in\mathcal{U}_{[\tau,t^1]}$ such that inclusion (6) holds. If $\rho(\alpha)\le1$ then, obviously, one can take $u^1_\alpha(\cdot)=u^0_\rho(\cdot)$. Therefore, the minimal cost is defined by the optimization problem

$$\alpha^0=\min\alpha,\ \rho(\alpha)\le1,\ \alpha\in[\alpha',\alpha_{\max}]. \tag{8}$$

Here the function $\rho:[\alpha',\alpha_{\max}]\to\mathbb{R}$ is a non-increasing function of α. Moreover, problem (7) can be reduced to a parametric linear program depending on the parameter α. Its cost then is a convex function of α. As a result problem (8) has a unique solution as stated in the proposition. □

5. CONSTRUCTION OF OPTIMAL ONCE-CLOSED CONTROLS

In this section we describe an algorithm for constructing the t^1-closed solution of problem $\mathcal{P}^1(\tau,X)$, $\tau\in\pi^1$. To shorten notations we drop the pair of parameters (τ,X) at the according functions.

Assume that $\alpha\in[\alpha_{\min},\alpha_{\max}]$ is fixed. Since the α-closure set $\hat{X}^1_\alpha$ has a very complex structure, we use an outer polytopic approximation $\tilde{X}^1_\alpha\supset\hat{X}^1_\alpha$ with faces corresponding to a given system of vectors $p_k\in\mathbb{R}^{m+1}$, $\|p_k\|=1$, $k=\overline{1,K}$. Accordingly, instead of $u^1_\alpha(\cdot)$ we will search for a function $\tilde{u}^1_\alpha(\cdot)$ satisfying the approximate inclusion

$$\hat{X}^1(\tilde{u}^1_\alpha(\cdot))\subseteq\tilde{X}^1_\alpha. \tag{9}$$

Certainly the quality of the solution depends on the choice of the vectors p_k, $k=\overline{1,K}$. This issue was investigated in Balashevich et al. (2004), hence, it is not discussed here.

In what follows it is assumed that in the neighborhood of the set $\hat{X}^1(\tilde{u}^1_\alpha(\cdot))$ the closure set $\hat{X}^1_\alpha$ is sufficiently well approximated by $\tilde{X}^1_\alpha$.

On the basis of Remark 1 one can conclude that the approximating polytope for the closure set $\hat{X}^1_\alpha$ has the form

$$\tilde{X}^1_\alpha=\{\hat{\chi}^1\in\mathbb{R}^{m+1}:p_k^T\hat{\chi}^1\le f_k(\alpha),\ k=\overline{1,K}\},$$

where

$$f_k(\alpha)=\max_{\hat{\chi}^1,u(\cdot)\in\mathcal{U}_{[t^1,T]}} p_k^T\hat{\chi}^1, \tag{10}$$

subject to

$$\dot{x}=Ax+Bu,\quad x(t^1)=0,$$
$$\hat{\chi}^1_0+h_0^Tx(T)\le\alpha,\quad \hat{\chi}^1_i+h_i^Tx(T)\le g_i,\ i=\overline{1,m},$$

where the decision variables are both the control $u(\cdot)$ and the vector $\hat{\chi}^1\in\mathbb{R}^{m+1}$. Note, that $p_k\ge0$, $k=\overline{1,K}$, otherwise the cost function of (10) is unbounded.

To construct the function $\tilde{u}^1_\alpha(\cdot)$, we change condition (6) in problem (7) considering the approximate inclusion (9). Since $\tilde{X}^1_\alpha$ is a polytope, inclusion (9) can be represented by K inequalities

$$p_k^T\Psi(t^1)x(t^1)\le f_k(\alpha)-\mu_k,\ k=\overline{1,K},$$

where

$$\mu_k=\max_{\hat{\gamma}^1\in\hat{\Gamma}^1}p_k^T\hat{\gamma}^1. \tag{11}$$

The latter is a special optimization problem of the form

$$\mu_k=\max_{y(\cdot);z_i,i=\overline{0,m}}\sum_{i=0}^m p_{ki}\psi_i^T(t^1)\hat{x}_i(t^1),$$
$$\dot{\hat{x}}_i=A\hat{x}_i,\ \hat{x}_i(\tau)=z_i,\quad y(s)-C\hat{x}_i(s)\in\Xi,\ s\in\bar{\pi}_{[\tau,t^1]},$$

$$z_i \in X,\ i = \overline{0,m},\quad y(\cdot) \in \mathcal{Y}_{[\tau,t^1]}(X).$$

The solution of this problem, denoted by $y_k(\cdot)$, gives a predicted worst-case (with respect to the projection of the set Γ^1 on a given vector p_k) measurement trajectory.

Thus, in order to find $\tilde{u}^1_\alpha(\cdot)$ one has to solve the following optimal control problem

$$\tilde{\rho}(\alpha) = \min_{u(\cdot)} \rho, \tag{12}$$

subject to

$$\dot{x} = Ax + Bu,\quad x(\tau) = 0,$$
$$p_k^T \Psi(t^1)x(t^1) \le f_k(\alpha) - \mu_k,\ k = \overline{1,K},$$
$$|u_j(t)| \le \rho,\ j = \overline{1,r},\ t \in [\tau, t^1].$$

On the basis of Proposition 2 and the approximations introduced we propose the following algorithm for constructing the suboptimal input $\tilde{u}^1(\cdot)$ of $\mathcal{P}(\tau, X)$ is proposed.

Algorithm for computing suboptimal t^1-closed controls:

Given the system of vectors p_k, $k = \overline{1,K}$, iterate the following steps:

(1) Choose $\alpha \in [\alpha_{\min}, \alpha_{\max}]$ and small parameters ε, $\varepsilon_\alpha \ge 0$.
(2) Solve K optimal control problems (10).
(3) If $\tilde{X}^1_\alpha = \emptyset$ increase α and return to step 2.
(4) Solve K optimization problems (11).
(5) Find the solution $\tilde{\rho}(\alpha)$, $\tilde{u}^0_\rho(\cdot)$ of problem (12).
(6) If (12) is infeasible increase α and return to step 2.
(7) If $\alpha < \alpha_{\max}$ and $\tilde{\rho}(\alpha) > 1$, increase α. If $\tilde{\rho}(\alpha) < 1 - \varepsilon$ and $\alpha > \alpha_{\min}$, decrease α. Return to step 2.
(8) For (i) $1 - \varepsilon \le \tilde{\rho}(\alpha) \le 1$, or (ii) $\tilde{\rho}(\alpha) \le 1$ and $\tilde{X}^1_{\alpha-\varepsilon_\alpha} = \emptyset$, set $\tilde{\alpha}^0 := \alpha$, $\tilde{u}^1(\cdot) := \tilde{u}^0_{\tilde{\rho}(\alpha)}(\cdot)$.

Remark 3. In Balashevich et al. (2004) the algorithm is expanded by an additional loop, which enhances the accuracy of the approximation $\tilde{X}^1_\alpha$ in the neighborhood of the set $\hat{X}^1(\tilde{u}^1_\alpha(\cdot))$, and relaxes it outside this neighborhood.

6. CONCLUSIONS

This paper presents a finite-time optimal measurement feedback control strategy for continuous time systems. The objective is to steer the system in finite-time to a given polyhedral terminal set while minimizing a linear terminal cost. For feedback only a set-based estimate of the initial state and sampled-data output measurements that are subject to set-membership uncertainties are available. The proposed scheme overcomes the conservatism often related to optimal control strategies that do not take into account that new state information arrives at future time instants. Specifically a min-max optimal control strategy is proposed, taking into account that at future time instants new measurement information is available. In the case that all future measurements are considered in the prediction, the scheme coincides with the full dynamic programming solution of the measurement feedback problem. Besides the conceptual formulation we present a finite-dimensional formulation of the resulting min-max problem based on so called sufficient estimates. This formulation allows to derive existence conditions for the resulting optimal control problem. Furthermore, we sketch a computational algorithm that allows to calculate the feedback control efficiently.

Summarizing, by limiting the number of instants at which the new measurement information is considered in the prediction, we achieve a compromise between the computational effort required to calculate the feedback, the conservatism, and the performance of the closed-loop.

Future work will investigate the application of the derived method to example systems, as well as the expansion to other system classes such as time-varying linear systems, and special classes of nonlinear systems.

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