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Repeated Poles in Feedback over a Class of Signal-to-Noise Ratio Constrained Channels

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Abstract: In the present paper we obtain a closed form expression for the squared $H_2^{\perp}$ norm of a partial fraction expansion with repeated unstable poles. We also obtain a closed form expression for the squared H_2 norm of a partial fraction expansion with repeated stable poles. As an application we use the $H_2^{\perp}$ result to extend the closed form solution of the discrete-time linear time invariant (LTI) signal-to-noise ratio (SNR) constrained problem to the case of repeated unstable poles in the plant model.

1. INTRODUCTION

Stabilisability in the area of Control over Networks has been a growing topic of increased interest in recent years; see for example Antsaklis et al. [2004], Nair et al. [2007] and references therein. The most general results in the area call for information theoretic arguments to obtain necessary and sufficient lower bounds on the channel transmission data rate [Nair and Evans, 2004, Nair et al., 2004, Freudenberg et al., 2006, Nair et al., 2007, Charalambous and Farhadi, 2007]. For linear plant models in [Nair and Evans, 2004, Theorem 2.1] and [Freudenberg et al., 2006, Proposition III.1] it is proved that if the unstable plant is to be stabilised, then the transmission data rate has to satisfy a lower bound that depends on the open loop unstable eigenvalues of the plant. The result is extended to nonlinear plant models in [Nair et al., 2004, Theorem 1].

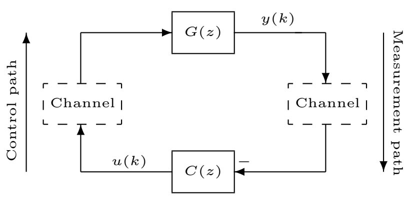

Fig. 1. General problem setting.

Another line of research introduced a framework to study stabilisability of a feedback loop over channels that have a signal to noise ratio (SNR) constraint [Braslavsky et al., 2007, Rojas et al., 2006a,b], and work in Bassam and Voulgaris [2005], Rantzer [2006]. A distinctive characteristic of the SNR approach is that it is a linear formulation. Braslavsky et al. [2007] obtained the infimal SNR required to stabilise an unstable linear time invariant (LTI) plant over a memoryless additive white Gaussian noise (AWGN) channel, whilst in Rojas et al. [2006a,b] the infimal SNR is computed for additive coloured Gaussian noise (ACGN) channels with memory, see Figure 2.

In Rojas et al. [2006a] the authors make use of a Youla parameterisation of all-stabilising LTI controllers to obtain a closed form solution for the infimal SNR (that is, the solution depends explicitly on the unstable plant poles, NMP zeros, relative degree and channel model). On the other hand, the result in Rojas et al. [2006b] make use of a linear quadratic Gaussian (LQG) optimisation to solve the discrete-time LTI SNR constrained problem in a non-closed form (that is, the solution is characterised by two Riccati equations). When discussing stabilisability, the advantage of the solution in Rojas et al. [2006b] over the solution in Rojas et al. [2006a] is that it can deal with repeated poles in the plant model, the disadvantage is that in doing so we do not obtain a closed form solution anymore.

In the present paper we address the current limitation of the closed form solution for the discrete-time LTI SNR constrained problem as presented in Rojas et al. [2006a]. To do so we start by presenting a closed form expression for the squared $H_2^{\perp}$ norm of a partial fraction expansion that contains repeated unstable poles. For completeness, we also obtain an expression for the squared H_2 norm of a partial fraction expansion with repeated stable poles. We then apply the $H_2^{\perp}$ result to the discrete-time LTI SNR constrained problem lifting the single pole assumption originally considered in Rojas et al. [2006a].

The present paper is organised as follow: in Section 2 we present the main $H_2^{\perp}$ norm technical result and the H_2 counterpart. In Section 3 we apply the $H_2^{\perp}$ result developed in the previous section to extend the closed form discrete-time LTI SNR constrained solution developed in Rojas et al. [2006a] to the case of repeated poles. Finally, in Section 4 we give our conclusions and final remarks for the present work.

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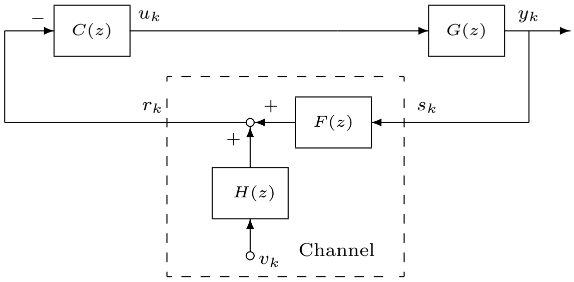

Fig. 2. LTI discrete-time control system with feedback over an additive coloured Gaussian noise (ACGN) channel with memory.

Terminology: let $\mathbb{C}$ denote the complex plane. Let $\mathbb{D}^-$, $\bar{\mathbb{D}}^-$, $\mathbb{D}^+$ and $\bar{\mathbb{D}}^+$ denote respectively the open unit-disk, closed unit-disk, open and closed unit disk complements in the complex plane $\mathbb{C}$, with $\partial\mathbb{D}$ the unit-disk itself. Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}^+$ the set of positive real numbers, $\mathbb{R}_o^+$ the set of non-negative real numbers and $\mathbb{R}^-$ the set of real negative numbers. Let $\mathbb{Z}^+$ denote the set of positive integers. A discrete-time signal is denoted by $x(k)$, $k = 0, 1, 2, \cdots$, and its $\mathcal{Z}$-transform by $X(z)$, $z \in \mathbb{C}$. The expectation operator is denoted by $\mathcal{E}$. A rational transfer function of a discrete-time system is minimum phase if all its zeros lie in $\bar{\mathbb{D}}^-$, and is non minimum phase if it has zeros in $\mathbb{D}^+$. Given $P(z)$, the transfer function of a discrete-time system, we say that $P(z) \in L_2$ if $P(z)$ is proper and bounded in $\mathbb{C}$; $P(z) \in H_2$ if $P(z)$ is proper, bounded in $\mathbb{D}^-$ and stable (i.e, all its poles lie in $\mathbb{D}^-$) and $P(z) \in H_2^\perp$ if $P(z)$ is proper, bounded in $\mathbb{D}^+$ and unstable (i.e, all its poles lie in $\mathbb{D}^+$). The L_2 norm of $P(z)$, denoted by $\|P\|_{L_2}$, is given by $\|P\|_{L_2}^2 = (1/2\pi)\int_{-\pi}^{\pi}|P(e^{\mathbf{j}\theta})|^2 d\theta$, where $\mathbf{j} = \sqrt{-1}$. Since H_2 and $H_2^\perp$ are subspaces of L_2 we have that, when appropriate, the $\|P\|_{H_2}$ and the $\|P\|_{H_2^\perp}$ have the same definition as the $\|P\|_{L_2}$. The H_∞ norm of $P(z)$ is given by $\|P\|_{H_\infty} = \sup_{\theta\in[-\pi,\pi]}\left|P(e^{\mathbf{j}\,\theta})\right|$. If a is in $\mathbb{C}$, $\bar{a}$ represents its complex conjugate and $a^H = \bar{a}^T$ the hermitian (i.e. the transposed complex conjugate of a). By general convention we have $0! = 1$ and $d^0(f(z))/dz^0 = f(z)$. LHS and RHS denote respectively the left and right hand side of an equation.

2. TECHNICAL RESULT

In the present section we develop a technical result that express in closed form the squared $H_2^\perp$ norm of a partial fraction expansion with repeated unstable poles. We introduce first a fairly simple proposition for a partial fraction expansion characterised by one pole with arbitrary multiplicity.

Proposition 1. Consider $z_i \in \mathbb{D}^-$, $n_i \in \mathbb{Z}^+$ and $f(z)$ a transfer function such that $f(z_i) \neq \infty$. Then a partial fraction expansion of $f(z)/(z-z_i)^{n_i}$ is given by

$$\frac{f(z)}{(z-z_i)^{n_i}} = \frac{r_{i,1}}{(z-z_i)} + \cdots + \frac{r_{i,n_i}}{(z-z_i)^{n_i}}, \qquad (1)$$

where

$$r_{i,l} = \frac{1}{(n_i-l)!}\frac{d^{n_i-l}}{dz^{n_i-l}}\left(f(z)\right)\Big|_{z=z_i}, \qquad (2)$$

are the residues.

Proof. The above proposition can be found for example in [Oppenheim and Schafer, 1975, pp. 56–57]. The proof is based on a repeated use of a L'Hôpital's argument. □

We address now the main technical result in the present section which refers to obtain a closed form expression for the squared $H_2^\perp$ norm of a partial fraction expansion expression.

Theorem 2. Assume $\rho_i \in \mathbb{D}^+$ to be unstable poles each with multiplicity $n_i \in \mathbb{Z}^+$, for $i = 1, \cdots, m$. Assume also that each related residue $r_{i,l} \in \mathbb{C}$, for $i = 1, \cdots, m$ and $l = 1, \cdots, n_i$ is known. Then

$$\left\|\sum_{i=1}^{m}\sum_{l=1}^{n_i}\frac{r_{i,l}}{(z-\rho_i)^l}\right\|_{H_2^\perp}^2 = \sum_{i=1}^{m}\sum_{l=1}^{n_i}\frac{r_{i,l}}{(l-1)!}\frac{d^{l-1}}{dz^{l-1}}\left(\sum_{j=1}^{m}\sum_{p=1}^{n_j}\frac{\bar{r}_{j,p}(-z)^{p-1}}{(z\bar{\rho}_j-1)^p}\right)\Bigg|_{z=\rho_i}, \qquad (3)$$

is the closed form expression for the squared $H_2^\perp$ norm of the partial fraction expansion $\sum_{i=1}^{m}\sum_{l=1}^{n_i}\frac{r_{i,l}}{(z-\rho_i)^l}$.

Proof. Start from the LHS of (3)

$$\begin{aligned}
&\left\|\sum_{i=1}^{m}\sum_{l=1}^{n_i}\frac{r_{i,l}}{(z-\rho_i)^l}\right\|_{H_2^\perp}^2 = \\
&\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{i=1}^{m}\sum_{l=1}^{n_i}\frac{r_{i,l}}{(e^{\mathbf{j}\theta}-\rho_i)^l}\right)^H\left(\sum_{i=1}^{m}\sum_{l=1}^{n_i}\frac{r_{i,l}}{(e^{\mathbf{j}\theta}-\rho_i)^l}\right)d\theta = \\
&\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{i=1}^{m}\sum_{l=1}^{n_i}\frac{\bar{r}_{i,l}}{(e^{-\mathbf{j}\theta}-\bar{\rho}_i)^l}\right)\left(\sum_{i=1}^{m}\sum_{l=1}^{n_i}\frac{r_{i,l}}{(e^{\mathbf{j}\theta}-\rho_i)^l}\right)d\theta = \\
&-\frac{1}{2\pi\mathbf{j}}\oint_{\partial\mathbb{D}}\left(\sum_{i=1}^{m}\sum_{l=1}^{n_i}\frac{\bar{r}_{i,l}z^l}{(1-z\bar{\rho}_i)^l}\right)\left(\sum_{i=1}^{m}\sum_{l=1}^{n_i}\frac{r_{i,l}}{(z-\rho_i)^l}\right)\frac{dz}{z} = \\
&\frac{1}{2\pi\mathbf{j}}\oint_{\partial\mathbb{D}}\left(\sum_{i=1}^{m}\sum_{l=1}^{n_i}\frac{\bar{r}_{i,l}(-z)^{l-1}}{(z\bar{\rho}_i-1)^l}\right)\left(\sum_{i=1}^{m}\sum_{l=1}^{n_i}\frac{r_{i,l}}{(z-\rho_i)^l}\right)dz.
\end{aligned}$$

The minus sign in the contour integral is due to the fact that in order to enclose the complement of the unit disk to obtain the $H_2^\perp$ norm, we have to maintain such region to the left throughout our contour evaluation by means of a clockwise motion. This in term implies that the original integral between $-\pi$ and π needs to be evaluated between π and $-\pi$, that is $\int_{-\pi}^{\pi}(\cdot) = -\int_{\pi}^{-\pi}(\cdot)$ and thus the introduction of a minus sign. For a better understanding consider expanding the notation in the last line of the equation above into

$$\begin{aligned}
&\left\|\sum_{i=1}^{m}\sum_{l=1}^{n_i}\frac{r_{i,l}}{(z-\rho_i)^l}\right\|_{H_2^\perp}^2 = \\
&\frac{1}{2\pi\mathbf{j}}\oint_{\partial\mathbb{D}}\frac{\bar{r}_{1,1}}{z\bar{\rho}_1-1}\frac{r_{1,1}}{z-\rho_1}dz + \cdots + \frac{1}{2\pi\mathbf{j}}\oint_{\partial\mathbb{D}}\frac{\bar{r}_{1,n_1}(-z)^{n_1-1}}{(z\bar{\rho}_1-1)^{n_1}}\frac{r_{1,n_1}}{(z-\rho_1)^{n_1}}dz + \\
&\cdots \\
&\frac{1}{2\pi\mathbf{j}}\oint_{\partial\mathbb{D}}\frac{\bar{r}_{m,1}}{z\bar{\rho}_m-1}\frac{r_{m,1}}{z-\rho_m}dz + \cdots + \frac{1}{2\pi\mathbf{j}}\oint_{\partial\mathbb{D}}\frac{\bar{r}_{m,n_m}(-z)^{n_m-1}}{(z\bar{\rho}_m-1)^{n_m}}\frac{r_{m,n_m}}{(z-\rho_m)^{n_m}}dz.
\end{aligned}$$

By application of the Residue Theorem (see for example [Churchill and Brown, 1990, pp. 169–172]) and Proposition 1 (notice that Proposition 1 is also valid if $z_i \in \mathbb{D}^-$ is replaced by $\rho_i \in \mathbb{D}^+$) on each contour integral we have

$$\begin{aligned}
&\left\|\sum_{i=1}^{m}\sum_{l=1}^{n_i}\frac{r_{i,l}}{(z-\rho_i)^l}\right\|_{H_2^\perp}^2 = \\
&\frac{r_{1,1}\bar{r}_{1,1}}{\rho_1\bar{\rho}_1-1} + \cdots + \frac{r_{1,n_1}}{(n_1-1)!}\frac{d^{n_1-1}}{dz^{n_1-1}}\left(\frac{\bar{r}_{1,n_1}(-z)^{n_1-1}}{(z\bar{\rho}_1-1)^{n_1}}\right)\Bigg|_{z=\rho_1} + \\
&\cdots \\
&\frac{r_{m,1}\bar{r}_{m,1}}{\rho_m\bar{\rho}_m-1} + \cdots + \frac{r_{m,n_m}}{(n_m-1)!}\frac{d^{n_m-1}}{dz^{n_m-1}}\left(\frac{\bar{r}_{m,n_m}(-z)^{n_m-1}}{(z\bar{\rho}_m-1)^{n_m}}\right)\Bigg|_{z=\rho_m},
\end{aligned}$$

which can be seen to be the c_{-1} coefficients of each integrand Laurent series expansion (see for example [Seron et al., 1997, pp. 315-316]). We finish by introducing a compact notation

$$\left\|\sum_{i=1}^m \sum_{l=1}^{n_i} \frac{r_{i,l}}{(z-\rho_i)^l}\right\|_{H_2^\perp}^2 = \sum_{i=1}^m \sum_{l=1}^{n_i} \frac{r_{i,l}}{(l-1)!} \frac{d^{l-1}}{dz^{l-1}} \left(\sum_{j=1}^m \sum_{p=1}^{n_j} \frac{\bar{r}_{j,p}(-z)^{p-1}}{(z\bar{\rho}_j-1)^p}\right)\Bigg|_{z=\rho_i},$$

concluding the proof. □

Example 3. In the present example we make use of the result from Theorem 2 to obtain the closed form expression for the squared $H_2^\perp$ norm when we have a pole ρ_1 with double multiplicity

$$\left\|\frac{r_1}{z-\rho_1}+\frac{r_2}{(z-\rho_1)^2}\right\|_{H_2^\perp}^2 = r_1\left(\frac{\bar{r}_1}{\rho_1\bar{\rho}_1-1}+\frac{-\bar{r}_2\rho_1}{(\rho_1\bar{\rho}_1-1)^2}\right)+\frac{r_2}{1!}\frac{d}{dz}\left(\frac{\bar{r}_1}{z\bar{\rho}_1-1}+\frac{-\bar{r}_2 z}{(z\bar{\rho}_1-1)^2}\right)\Bigg|_{z=\rho_1} = \frac{r_1\bar{r}_1}{\rho_1\bar{\rho}_1-1}-\frac{r_1\bar{r}_2\rho_1}{(\rho_1\bar{\rho}_1-1)^2}-\frac{\bar{r}_1 r_2\bar{\rho}_1}{(\rho_1\bar{\rho}_1-1)^2}+\frac{\bar{r}_2 r_2(\rho_1\bar{\rho}_1+1)}{(\rho_1\bar{\rho}_1-1)^3},$$

where we have considered here $r_1 = r_{1,1}$ and $r_2 = r_{1,2}$ to simplify the notation.

Remark 4. If $\rho_i \in \mathbb{R}$ and $r_{i,l} \in \mathbb{R}$ for all i and l, the squared $H_2^\perp$ result of Theorem 2 can be equivalently expressed in terms of a squared H_2 norm. If we consider a generic transfer function $\Psi(z)$ with real coefficients to be in $H_2^\perp$, it is well known that $\|\Psi(z)\|_{H_2^\perp}^2 = \|\Psi(z^{-1})\|_{H_2}^2$ (see for example [Zhou et al., 1996, p. 114] for the continuous-time counterpart of this argument). If we define now $\Psi(z)$ to be the partial fraction expansion in $H_2^\perp$ from Theorem 2, then

$$\left\|\sum_{i=1}^m \sum_{l=1}^{n_i} \frac{r_{i,l}}{(z-\rho_i)^l}\right\|_{H_2^\perp}^2 = \left\|\sum_{i=1}^m \sum_{l=1}^{n_i} \frac{r_{i,l}z^l}{(1-\rho_i z)^l}\right\|_{H_2}^2 \quad (4)$$

Notice, however, the squared H_2 expression in (4) does not represent a partial fraction expansion.

The result in Theorem 2 provides a closed form expression for the squared $H_2^\perp$ norm of a partial fraction expansion with repeated unstable poles. In a similar way, a closed form expression can be obtained for a partial fraction expansion that is defined in H_2. Notice that such result can not be inferred directly from Theorem 2, since $\Psi(z^{-1})$ does not represent a partial fraction expansion in H_2, as noted in Remark 4. We include the squared H_2 norm closed form result next for completeness.

Theorem 5. Assume $z_i \in \mathbb{D}^-$ to be stable poles each with multiplicity $n_i \in \mathbb{Z}^+$, for $i = 1, \cdots, m$. Assume also that each related residue $r_{i,l} \in \mathbb{C}$, for $i = 1, \cdots, m$ and $l = 1, \cdots, n_i$ is known. Then

$$\left\|\sum_{i=1}^m \sum_{l=1}^{n_i} \frac{r_{i,l}}{(z-z_i)^l}\right\|_{H_2}^2 = \sum_{i=1}^m \sum_{l=1}^{n_i} \frac{r_{i,l}}{(l-1)!} \frac{d^{l-1}}{dz^{l-1}} \left(\sum_{j=1}^m \sum_{p=1}^{n_j} \frac{\bar{r}_{j,p}z^{p-1}}{(1-z\bar{z}_j)^p}\right)\Bigg|_{z=z_i}, \quad (5)$$

is the closed form expression for the squared H_2 norm of the partial fraction expansion $\sum_{i=1}^m \sum_{l=1}^{n_i} \frac{r_{i,l}}{(z-z_i)^l}$.

Proof. The proof of this theorem follows the same steps of Theorem 2, with some minor differences. Start from the LHS of (5)

$$\left\|\sum_{i=1}^m \sum_{l=1}^{n_i} \frac{r_{i,l}}{(z-z_i)^l}\right\|_{H_2}^2 =$$
$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{i=1}^m \sum_{l=1}^{n_i} \frac{r_{i,l}}{(e^{\mathbf{j}\theta}-z_i)^l}\right)^H\left(\sum_{i=1}^m \sum_{l=1}^{n_i} \frac{r_{i,l}}{(e^{\mathbf{j}\theta}-z_i)^l}\right)d\theta=$$
$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{i=1}^m \sum_{l=1}^{n_i} \frac{\bar{r}_{i,l}}{(e^{-\mathbf{j}\theta}-\bar{z}_i)^l}\right)\left(\sum_{i=1}^m \sum_{l=1}^{n_i} \frac{r_{i,l}}{(e^{\mathbf{j}\theta}-z_i)^l}\right)d\theta=$$
$$\frac{1}{2\pi\mathbf{j}}\oint_{\partial\mathbb{D}}\left(\sum_{i=1}^m \sum_{l=1}^{n_i} \frac{\bar{r}_{i,l}z^l}{(1-z\bar{z}_i)^l}\right)\left(\sum_{i=1}^m \sum_{l=1}^{n_i} \frac{r_{i,l}}{(z-z_i)^l}\right)\frac{dz}{z}=$$
$$\frac{1}{2\pi\mathbf{j}}\oint_{\partial\mathbb{D}}\left(\sum_{i=1}^m \sum_{l=1}^{n_i} \frac{\bar{r}_{i,l}z^{l-1}}{(1-z\bar{z}_i)^l}\right)\left(\sum_{i=1}^m \sum_{l=1}^{n_i} \frac{r_{i,l}}{(z-z_i)^l}\right)dz.$$

For a better understanding consider expanding the notation in the last line above

$$\left\|\sum_{i=1}^m \sum_{l=1}^{n_i} \frac{r_{i,l}}{(z-z_i)^l}\right\|_{H_2}^2 =$$
$$\frac{1}{2\pi\mathbf{j}}\oint_{\partial\mathbb{D}}\frac{\bar{r}_{1,1}}{1-z\bar{z}_1}\frac{r_{1,1}}{z-z_1}dz+\cdots+\frac{1}{2\pi\mathbf{j}}\oint_{\partial\mathbb{D}}\frac{\bar{r}_{1,n_1}z^{n_1-1}}{(1-z\bar{z}_1)^{n_1}}\frac{r_{1,n_1}}{(z-z_1)^{n_1}}dz+$$
$$\cdots$$
$$\frac{1}{2\pi\mathbf{j}}\oint_{\partial\mathbb{D}}\frac{\bar{r}_{m,1}}{1-z\bar{z}_m}\frac{r_{m,1}}{z-z_m}dz+\cdots+\frac{1}{2\pi\mathbf{j}}\oint_{\partial\mathbb{D}}\frac{\bar{r}_{m,n_m}z^{n_m-1}}{(1-z\bar{z}_m)^{n_m}}\frac{r_{m,n_m}}{(z-z_m)^{n_m}}dz.$$

By application of the Residue Theorem (see for example [Churchill and Brown, 1990, pp. 169–172]) and Proposition 1 on each contour integral we have

$$\left\|\sum_{i=1}^m \sum_{l=1}^{n_i} \frac{r_{i,l}}{(z-z_i)^l}\right\|_{H_2}^2 =$$
$$\frac{r_{1,1}\bar{r}_{1,1}}{1-z_1\bar{z}_1}+\cdots+\frac{r_{1,n_1}}{(n_1-1)!}\frac{d^{n_1-1}}{dz^{n_1-1}}\left(\frac{\bar{r}_{1,n_1}z^{n_1-1}}{(1-z\bar{z}_1)^{n_1}}\right)\Bigg|_{z=z_1}+$$
$$\cdots$$
$$\frac{r_{m,1}\bar{r}_{m,1}}{1-z_m\bar{z}_m}+\cdots+\frac{r_{m,n_m}}{(n_m-1)!}\frac{d^{n_m-1}}{dz^{n_m-1}}\left(\frac{\bar{r}_{m,n_m}z^{n_m-1}}{(1-z\bar{z}_m)^{n_m}}\right)\Bigg|_{z=z_m},$$

which can be seen to be the c_{-1} coefficients of each integrand Laurent series expansion ([pp. 315-316]Seron et al. [1997]). We finish by introducing a compact notation

$$\left\|\sum_{i=1}^m \sum_{l=1}^{n_i} \frac{r_{i,l}}{(z-z_i)^l}\right\|_{H_2}^2 = \sum_{i=1}^m \sum_{l=1}^{n_i} \frac{r_{i,l}}{(l-1)!} \frac{d^{l-1}}{dz^{l-1}} \left(\sum_{j=1}^m \sum_{p=1}^{n_j} \frac{\bar{r}_{j,p}z^{p-1}}{(1-z\bar{z}_j)^p}\right)\Bigg|_{z=z_i},$$

concluding the proof. □

Example 6. In the present example we make use of the result from Theorem 5 to obtain the closed form expression for the squared H_2 norm in the case of a pole z_1 with double multiplicity

$$\left\|\frac{r_1}{z-z_1}+\frac{r_2}{(z-z_1)^2}\right\|_{H_2}^2 = r_1\left(\frac{\bar{r}_1}{1-z_1\bar{z}_1}+\frac{\bar{r}_2 z_1}{(1-z_1\bar{z}_1)^2}\right)+\frac{r_2}{1!}\frac{d}{dz}\left(\frac{\bar{r}_1}{1-z\bar{z}_1}+\frac{\bar{r}_2 z}{(1-z\bar{z}_1)^2}\right)\Bigg|_{z=z_1} = \frac{r_1\bar{r}_1}{1-z_1\bar{z}_1}+\frac{r_1\bar{r}_2 z_1}{(1-z_1\bar{z}_1)^2}+\frac{\bar{r}_1 r_2\bar{z}_1}{(1-z_1\bar{z}_1)^2}+\frac{\bar{r}_2 r_2(1+z_1\bar{z}_1)}{(1-z_1\bar{z}_1)^3},$$

where we have considered $r_1 = r_{1,1}$ and $r_2 = r_{1,2}$ to simplify the notation. Notice that the case of complex conjugate poles is not represented by the above result, since in that case each pole would have multiplicity one and therefore the correct expression would be

$$\left\|\frac{r_1}{z-z_1}+\frac{r_2}{z-\bar{z}_1}\right\|_{H_2}^2 = \frac{r_1\bar{r}_1}{1-z_1\bar{z}_1}+\frac{r_1\bar{r}_2}{1-z_1^2}+\frac{\bar{r}_1 r_2}{1-\bar{z}_1^2}+\frac{\bar{r}_2 r_2}{1-z_1\bar{z}_1}.$$

The previous example shows the use of Theorem 5 for the simple case of one pole with double multiplicity. We clarify that Theorem 2 and Theorem 5 can deal with poles in the complex plane, not just in the real line, although in general we focus on the case of $z_i \in \mathbb{R}$, $|z_i| < 1$, $i = 1, \cdots, m$ or $\rho_i \in \mathbb{R}$,$|\rho_i| > 1$ $i = 1, \cdots, m$.

The results in the present section are technical. In the next section we propose their application, in particular of

Proposition 1 and Theorem 2, to the discrete-time LTI SNR constrained problem defined in Rojas et al. [2006a].

3. APPLICATION TO THE DISCRETE-TIME LTI SNR CONSTRAINED PROBLEM

We start the present section by listing the general assumptions for the LTI filters in Figure 2:

Plant model: through the present work, if not stated otherwise, it is assumed that the plant model $G(z)$ has m unstable poles, $|\rho_i| > 1, \forall i = 1, \cdots, m$, each with multiplicity n_i; q NMP zeros, $|a_j| > 1, \forall j = 1, \cdots, q$ (no NMP zeros match any of the m unstable poles), and overall relative degree $n_g \geq 1$.
Channel model: the channel model $F(z)$ is a stable, biproper transfer function with f NMP zeros, $|w_j| > 1, \forall j = 1, \cdots, f$ (no NMP zeros match any of the m unstable poles).
Channel additive noise process: the channel additive noise process is labelled $n(k)$ and it is a zero-mean i.i.d. Gaussian white noise process with variance σ^2.
Noise model: the system $H(z)$ colouring the channel additive white noise $n(k)$ is assumed to be a stable, biproper and minimum phase transfer function.

We assume that $C(z)$ is such that the closed-loop system is stable in the sense that, for any distribution of initial conditions, the distribution of all signals in the loop will converge exponentially rapidly to a stationary distribution. The channel input power, defined by $\|s\|_{Pow} \triangleq \lim_{k\to\infty} \mathcal{E}\left\{y^2(k)\right\}$ is required to satisfy an imposed power constraint

$$\mathcal{P} > \mathcal{E}\{y^2\}, \tag{6}$$

for some predetermined power level $\mathcal{P}$, where $\mathcal{E}\left\{y^2\right\}$ stands for $\lim_{k\to\infty} \mathcal{E}\left\{y^2(k)\right\}$ and it is introduced to simplify the notation. Under reasonable stationarity assumptions [Åström, 1970, §4.4], the power in the channel input may be computed as

$$\mathcal{E}\{y^2\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} |T_{yn}(e^{j\omega})|^2 \sigma^2 d\omega,$$

where

$$T_{yn}(z) = -\frac{C(z)G(z)}{1+C(z)G(z)F(z)} H(z), \tag{7}$$

is the transfer functions that relate $y(k)$ with $n(k)$. Since the feedback control system is stable, we have

$$\mathcal{E}\{y^2\} = \|T_{yn}(z)\|_{H_2}^2 \sigma^2.$$

Thus, the power constraint (6) at the input of the channel, translates into a SNR bound defined by the squared H_2 norm of $T_{yn}(z)$

$$\frac{\mathcal{P}}{\sigma^2} > \|T_{yn}(z)\|_{H_2}^2. \tag{8}$$

Remark 7. It can be seen from (7) that the biproper assumption for $F(z)$ and $H(z)$ is without loss of generality. Indeed, if the transfer function $F(z)$ has relative degree n_f, with $n_f \geq 1$, then the case of $F(z)$ strictly proper would be equivalent to consider $F_{bip}(z) = z^{n_f}F(z)$ and $\tilde{G}(z) = \frac{G(z)}{z^{n_f}}$, since the factor z^{-n_f} would not modify the squared H_2 norm of $T_{yn}(z)$. Similarly, if the transfer function $H(z)$ has relative degree n_h, with $n_h \geq 1$, we can observe from equation (7) that this would be equivalent to $H_{bip}(z) = z^{n_h}H(z)$, since the factor z^{n_h} will not modify the squared H_2 norm of $T_{yn}(z)$.

From (8) we observe that a fundamental limitation in the SNR of the ACGN channel will be given by the infimum of $\|T_{yn}(z)\|_{H_2}^2$, which indeed is at the core of the infimal SNR problem definition that follows.

Problem 8. **(Infimal SNR for LTI Stabilisability Problem).** *Find a proper rational stabilising LTI controller $C(z)$ such that the feedback control loop is stable and the transfer function in* (7) *achieves the infimum possible constraint* (8) *imposed on the admissible channel SNR.*

The problem stated here of characterising a lower bound for the SNR required for stabilisability of a discrete-time LTI output feedback, as in Figure 2, has been previously addressed in Rojas et al. [2006a] for $n_i = 1, \forall i = 1, \cdots, m$. In the present paper by means of the technical result from the previous section we consider the more general case of $n_i \neq 1$ for some i, $i = 1, \cdots, m$.

Denote the Blaschke product containing the unstable poles of $G(z)$ (that is the poles in $\mathbb{D}^+$) by

$$B_\rho(z) = \prod_{i=1}^m \left(\frac{z-\rho_i}{1-z\bar{\rho}_i}\right)^{n_i}. \tag{9}$$

The use of such a definition for the Blaschke product when dealing with unstable poles with multiplicity greater than one can also be found, for example, in [Toker et al., 2002, §III] where it was applied to solve a unit-step tracking problem. Define

$$\beta_k \triangleq \frac{1}{k!}\frac{d^k}{dz^k} B_\rho(z)\Big|_{z=0}. \tag{10}$$

Denote, also, the Blaschke product containing the $\bar{\mathbb{D}}^+$ zeros of $G(z)$ and $F(z)$ by

$$B_{\zeta G}(z) = \prod_{j=1}^q \frac{z-a_j}{1-z\bar{a}_j},\ B_{\zeta F}(z) = \prod_{j=1}^f \frac{z-w_j}{1-z\bar{w}_j}. \tag{11}$$

In general, if it is not necessary to stress the different sources of the NMP zeros we will use $B_\zeta(z)$ as notation, with $B_\zeta(z) = B_{\zeta G}(z)B_{\zeta F}(z)$ and $\{\zeta_j | j = 1, \cdots, q+f\} = \{a_1, \cdots, a_q, w_1, \cdots, w_f\}$.

Theorem 9. Consider the discrete-time output LTI feedback represented in Figure 2 and that $G(z)$, $F(z)$ and $H(z)$ satisfy the assumptions listed in the present section, then

$$\frac{\mathcal{P}}{\sigma^2} > \sum_{i=1}^m \sum_{l=1}^{n_i} \frac{r_{i,l}}{(l-1)!}\frac{d^{l-1}}{dz^{l-1}}\left(\sum_{j=1}^m \sum_{p=1}^{n_j} \frac{\bar{r}_{j,p}(-z)^{p-1}}{(z\bar{\rho}_j-1)^p}\right)\Bigg|_{z=\rho_i} + \delta, \tag{12}$$

in which we have

$$r_{i,l} = \frac{1}{(n_i-l)!}\frac{d^{n_i-l}}{dz^{n_i-l}}\left((z-\rho_i)^{n_i} B_\rho^{-1}(z)B_\zeta^{-1}(z)\tilde{F}^{-1}(z)H(z)\right)\Big|_{z=\rho_i}, \tag{13}$$

$$\delta = \begin{cases} 0, & \text{if } \mathrm{n_g}=1 \\ \sum_{k=1}^{n_g-1}|\mu_k|^2, & \text{if } \mathrm{n_g}>1 \end{cases}, \tag{14}$$

and

$$\mu_k = \sum_{i=1}^m \sum_{l=1}^{min\{k,n_i\}} \binom{k-1}{l-1} r_{i,l}\rho_i^{k-l}. \tag{15}$$

Proof. We proceed by considering the function spaces L_2, H_2, $H_2^\perp$, and RH_∞ defined in the Introduction, with the stability region given by the open unit disk in the complex plane. Introduce a coprime factorisation $F(z)G(z) = N(z)/M(z)$, and the parameterisation of all stabilising controllers (see Doyle et al. [1992, pp. 64-65])

$$C(z) = (X(z)+M(z)Q(z))/(Y(z)-N(z)Q(z)),$$

where $X(z)$ and $Y(z)$ satisfy the Bezout identity, $N(z)X(z) +M(z)Y(z) = 1$. It follows that $T_{yn}(z) = -\left(N(z)X(z)+ N(z)M(z)Q(z)\right)F^{-1}(z)H(z)$. Further factorise $M(z) = B_\rho(z)M_0(z)$, where $B_\rho(z)$ is the Blaschke product in (9) and $N(z) = B_{\zeta G}(z)B_{\zeta F}(z)N_0(z)$, where $B_{\zeta G}(z)$ and $B_{\zeta F}(z)$ are the Blaschke products in (11) and $M_o(z)$ and $N_o(z)$ are in H_2. It follows from the Bezout identity that $B_\rho^{-1}(z)$ and $M_0(z)Y(z)$ have power series expansions at infinity of the form

$$B_\rho^{-1}(z)=\sum_{k=0}^{\infty}\beta_k z^{-k},$$
$$M_0(z)Y(z)=\sum_{k=0}^{n_g-1}\beta_k z^{-k}+\sum_{k=n_g}^{\infty}\alpha_k z^{-k}, \tag{16}$$

where β_k is defined as in (10). Since $B_\rho(z)$ is biproper, $N(z)$ and $N_0(z)$ have relative degrees n_g, and the set $\left\{z^{-k}; k = 0, \cdots, \infty\right\}$ forms an orthonormal basis for H_2. It follows that

$$\inf_{Q(z)\in RH_\infty}\|T_{yn}\|_{H_2}^2 =\inf_{Q(z)\in RH_\infty}\left\|\left(B_\rho^{-1}B_\zeta^{-1}-M_0YB_\zeta^{-1}+M_0N_0Q\right)\tilde{F}^{-1}H\right\|_{L_2}^2, \tag{17}$$
$$=\inf_{Q(z)\in RH_\infty}\Big\|B_\rho^{-1}B_\zeta^{-1}\tilde{F}^{-1}H-B_\zeta^{-1}\tilde{F}^{-1}H\sum_{k=0}^{n_g-1}\beta_k z^{-k} -B_\zeta^{-1}\tilde{F}^{-1}H\sum_{k=n_g}^{\infty}\alpha_k z^{-k}+M_0N_0Q\tilde{F}^{-1}H\Big\|_{L_2}^2,$$

where $\tilde{F}^{-1}(z) = B_{z\zeta}(z)F^{-1}(z)$. Consider also a partial fraction expansion of $B_\rho^{-1}(z)B_\zeta^{-1}(z)\tilde{F}^{-1}(z)H(z)$ which permits the decomposition

$$B_\rho^{-1}B_\zeta^{-1}\tilde{F}^{-1}H=\Gamma^\perp+\Gamma, \tag{18}$$

where

$$\Gamma^\perp=\sum_{i=1}^{m}\sum_{l=1}^{n_i}\frac{r_{i,l}}{(z-\rho_i)^l}+\sum_{j=1}^{q+f}\frac{t_j}{z-\zeta_j}, \tag{19}$$

and

$$r_{i,l}=\frac{1}{(n_i-l)!}\frac{d^{n_i-l}}{dz^{n_i-l}}\left((z-\rho_i)^{n_i}B_\rho^{-1}(z)B_\zeta^{-1}(z)\tilde{F}^{-1}(z)H(z)\right)\Big|_{z=\rho_i},$$
$$t_j=(1-|\zeta_j|^2)B_\rho^{-1}(\zeta_j)\tilde{F}^{-1}(\zeta_j)H(\zeta_j)\prod_{\substack{l=1\\l\neq j}}^{q+f}\frac{1-\zeta_j\bar{\zeta}_l}{\zeta_j-\zeta_l}. \tag{20}$$

The expression for $r_{i,l}$ comes from direct application of Proposition 1 recognising $f(z) = (z - \rho_i)^{n_i}B_\rho^{-1}(z)\,B_\zeta^{-1}(z)\,\tilde{F}^{-1}(z)\,H(z)$.

Consider now $\left(\sum_{k=0}^{n_g-1}\beta_k z^{-k}\right)B_\zeta^{-1}(z)\tilde{F}^{-1}(z)H(z)$ in (17) and use a partial fraction expansion to isolate the terms in $B_\zeta(z)$ due to NMP zeros

$$\left(\sum_{k=0}^{n_g-1}\beta_k z^{-k}\right)B_\zeta^{-1}\tilde{F}^{-1}H=\sum_{j=1}^{q+f}\left(\sum_{k=0}^{n_g-1}\frac{\beta_k}{\zeta_j^k}\right)\frac{m_j}{z-\zeta_j}+\Theta, \tag{21}$$

where

$$m_j=(1-|\zeta_j|^2)\left(\prod_{\substack{l=1\\l\neq j}}^{q+f}\frac{1-\zeta_j\bar{\zeta}_l}{\zeta_j-\zeta_l}\right)\tilde{F}^{-1}(\zeta_j)H(\zeta_j), \tag{22}$$

and $\Theta(z)$ is in H_2. Consider now the third term in (17) defined by

$$\left(\sum_{k=n_g}^{\infty}\alpha_k z^{-k}\right)B_\zeta^{-1}\tilde{F}^{-1}H=\sum_{j=1}^{q+f}\frac{q_j}{z-\zeta_j}+\Omega, \tag{23}$$

in which the RHS is obtained using again partial fraction expansion and Ω is in H_2, where

$$q_j=(1-|\zeta_j|^2)\left(\prod_{\substack{l=1\\l\neq j}}^{q+f}\frac{1-\zeta_j\bar{\zeta}_l}{\zeta_j-\zeta_l}\right)\tilde{F}^{-1}(\zeta_j)H(\zeta_j)\left(B_\rho^{-1}(\zeta_j)-\sum_{k=0}^{n_g-1}\frac{\beta_k}{\zeta_j^k}\right),$$
$$=t_j-\left(\sum_{k=0}^{n_g-1}\frac{\beta_k}{\zeta_j^k}\right)m_j. \tag{24}$$

Finally, this allow us to redefine the expression in (17) as

$$=\inf_{Q(z)\in RH_\infty}\Big\|\sum_{i=1}^{m}\sum_{l=1}^{n_i}\frac{r_{i,l}}{(z-\rho_i)^l}+\sum_{j=1}^{q+f}\frac{t_j}{z-\zeta_j}-\sum_{j=1}^{q+f}\left(\sum_{k=0}^{n_g-1}\frac{\beta_k}{\zeta_j^k}\right)\frac{m_j}{z-\zeta_j}-\sum_{j=1}^{q+f}\frac{q_j}{z-\zeta_j}- \Omega-\Theta+\Gamma+M_0N_0Q\tilde{F}^{-1}H\Big\|_{L_2}^2. \tag{25}$$

A close analysis of the zeros related residue coefficients reveals that

$$t_j-\left(\sum_{k=0}^{n_g-1}\frac{\beta_k}{\zeta_j^k}\right)m_j-q_j=0,\quad \forall j=1,\cdots,q+f, \tag{26}$$

due to the result on q_j from (24). This noticeable simplify expression (25) into

$$\inf_{Q(z)\in RH_\infty}\|T_{yn}\|_{H_2}^2= \inf_{Q(z)\in RH_\infty}\left\|\sum_{i=1}^{m}\sum_{l=1}^{n_i}\frac{r_{i,l}}{(z-\rho_i)^l}-\Omega-\Theta+\Gamma+M_0N_0Q\tilde{F}^{-1}H\right\|_{L_2}^2. \tag{27}$$

At this point we are only faced with the relative degree difference of $\Gamma(z)$ and $\Theta(z)$ (biproper, since filters $F(z)$ and $H(z)$ have been selected to be biproper too [1]), on one side, and $\Omega(z)$ and $N_o(z)M_o(z)Q(z)\tilde{F}^{-1}(z)H(z)$ of degree n_g, on the other.

The term $\Gamma(z)$ can be defined, from (18), as

$$\Gamma(z)=B_\rho^{-1}B_\zeta^{-1}\tilde{F}^{-1}H-\sum_{i=1}^{m}\sum_{l=1}^{n_i}\frac{r_{i,l}}{(z-\rho_i)^l}-\sum_{j=1}^{q+f}\frac{t_j}{z-\zeta_j}. \tag{28}$$

Similarly for $\Theta(z)$, from (21) we have

$$\Theta(z)=\left(\sum_{k=0}^{n_g-1}\beta_k z^{-k}\right)B_\zeta^{-1}\tilde{F}^{-1}H-\sum_{j=1}^{q+f}\left(\sum_{k=0}^{n_g-1}\frac{\beta_k}{\zeta_j^k}\right)\frac{m_j}{z-\zeta_j}. \tag{29}$$

At this point we are interested in the impulse response of $\Gamma(z) - \Theta(z)$ and since by definition

$$B_\rho^{-1}=\sum_{k=0}^{n_g-1}\beta_k z^{-k}, \tag{30}$$

making explicit the first $n_g - 1$ sampling times, we have that

$$\Gamma(z)-\Theta(z)=-\sum_{i=1}^{m}\sum_{l=1}^{n_i}\left(\sum_{k=l}^{n_g-1}\binom{k-1}{l-1}r_{i,l}\rho_i^{k-l}z^{-k}\right) -\sum_{j=1}^{q+f}\left(\sum_{k=1}^{n_g-1}t_j\zeta_j^{k-1}z^{-k}\right)+ \sum_{j=1}^{q+f}\left[\sum_{k=1}^{n_g-1}\left(\sum_{i=0}^{n_g-1}\frac{\beta_i}{\zeta_j^i}\right)m_j\zeta_j^{k-1}z^{-k}\right]+\Xi(z), \tag{31}$$

where from explicitly considering the impulse response of $\sum_{i=1}^{m}\sum_{l=1}^{n_i}\frac{r_{i,l}}{(z-\rho_i)^l}$, we observe that the coefficients $r_{i,l}\rho^{k-l}$ show an arrangement according to Pascal's triangle (defined by the binomial coefficient $\binom{k-1}{l-1}$). The $\Xi(z)$ term in (31) is in H_2 with relative degree n_g. From (30) we also have the fact that $B_\rho^{-1}(\zeta_j) = \sum_{k=0}^{n_g-1}\beta_k\zeta_j^{-k}$. Together with the definition for t_j and m_j, this effectively cancels out the second and third term on the RHS of (31), leaving us with

$$\Gamma(z)-\Theta(z)=-\sum_{k=1}^{n_g-1}\underbrace{\left(\sum_{i=1}^{m}\sum_{l=1}^{min\{k,n_i\}}\binom{k-1}{l-1}r_{i,l}\rho_i^{k-l}\right)}_{\mu_k}z^{-k}+\Xi(z). \tag{32}$$

The final result is then

$$=\Big\|\sum_{i=1}^{m}\sum_{l=1}^{n_i}\frac{r_{i,l}}{(z-\rho_i)^l}\Big\|_{H_2^\perp}^2+\Big\|-\sum_{k=1}^{n_g-1}\mu_k z^{-k}\Big\|_{L_2}^2+ \inf_{Q(z)\in RH_\infty}\left\|\Xi-\Omega+M_0N_0Q\tilde{F}^{-1}H\right\|_{H_2}^2. \tag{33}$$

[1] Notice that since $F(z) = B_{\zeta F}(z)\tilde{F}(z)$ then also $\tilde{F}(z)$ will be biproper.

By taking $Q = -(\Xi - \Omega)M_o^{-1}N_o^{-1}\tilde{F}H^{-1}$, the last term in (33) is zero, therefore equation (33) becomes

$$\inf_{Q(z)\in RH_\infty}\|T_{yn}\|_{H_2}^2 = \sum_{i=1}^{m}\sum_{l=1}^{n_i}\frac{r_{i,l}}{(l-1)!}\frac{d^{l-1}}{dz^{l-1}}\left(\sum_{j=1}^{m}\sum_{p=1}^{n_j}\frac{\bar{r}_{j,p}(-z)^{p-1}}{(z\bar{\rho}_j-1)^p}\right)\Bigg|_{z=\rho_i} + \sum_{k=1}^{n_g-1}|\mu_k|^2, \quad (34)$$

with

$$r_{i,l}=\frac{1}{(n_i-l)!}\frac{d^{n_i-l}}{dz^{n_i-l}}\left((z-\rho_i)^{n_i}B_\rho^{-1}(z)B_\zeta^{-1}(z)\tilde{F}^{-1}(z)H(z)\right)\Big|_{z=\rho_i},$$

$$\mu_k=\sum_{i=1}^{m}\sum_{l=1}^{min\{k,n_i\}}\binom{k-1}{l-1}r_{i,l}\rho_i^{k-l}, \quad (35)$$

which completes the proof. □

Remark 10. Notice from Theorem 9 that we regain the result of Theorem 2 in Rojas et al. [2006a] whenever $n_i = 1$, $i = 1, \cdots, m$, that is

$$\frac{\mathcal{P}}{\sigma^2} > \sum_{i=1}^{m}\sum_{j=1}^{m}\frac{r_i\bar{r}_j}{\rho_i\bar{\rho}_j-1}+\delta, \quad (36)$$

where $r_i = r_{i,1}$ in order to simplify the notation, and in which

$$r_i=(1-|\rho_i|^2)B_\zeta^{-1}(\rho_i)\tilde{F}^{-1}(\rho_i)H(\rho_i)\prod_{\substack{j=1\\j\neq i}}^{m}\frac{1-\rho_i\bar{\rho}_j}{\rho_i-\rho_j}, \quad (37)$$

$$\delta=\begin{cases}0, & \text{if } \mathrm{n_g}=1\\ \sum_{k=1}^{n_g-1}|\mu_k|^2, & \text{if } \mathrm{n_g}>1\end{cases}, \quad (38)$$

where

$$\mu_k=\sum_{i=1}^{m}r_i\rho_i^{k-1}. \quad (39)$$

Remark 11. In the application of the technical result developed in Section 2 to the proof of Theorem 9, we have not considered the case of repeated NMP zeros (either from the plant or channel model). Such choice was made to avoid increasing unnecessarily the complexity of the proof of Theorem 9. Nonetheless, we are of the opinion that such extension should be feasible.

4. CONCLUSION AND REMARKS

In the present paper we have developed closed form expressions for the squared $H_2^\perp$ and H_2 norms of a partial fraction expansion that explicitly considers repeated poles. The result for the squared $H_2^\perp$ case is then applied to the infimal discrete-time LTI SNR for stabilisability problem, where we observe that the obtained solution agrees with the earlier result in Rojas et al. [2006a] when dealing with single unstable poles. Further directions of research should consider application of the technical results developed here to other suitable problems such as the one addressed in Rojas et al. [2008].

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