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Quantizer Design for Interconnected Feedback Control Systems

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Abstract: In this paper, we consider design of interconnected $\mathcal{H}_\infty$ feedback control systems with quantized signals. We first assume that a decentralized state feedback has been designed for an interconnected continuous-time LTI system so that the closed-loop system is stable and a desired $\mathcal{H}_\infty$ disturbance attenuation level is achieved, and that the subsystems' states are quantized before they are passed to the local controller. We propose a local-state-dependent strategy for updating the quantizers' parameters, so that the overall closed-loop system is asymptotically stable and achieves the same $\mathcal{H}_\infty$ disturbance attenuation level. We then extend the result to the case of decentralized static output feedback where the measurement outputs are quantized, and propose a local-output-dependent strategy for updating the quantizers' parameters. Both the pre-designed controllers and the quantizers' parameters are constructed in a decentralized manner, depending on local information.

Keywords: interconnected continuous-time LTI system, decentralized $\mathcal{H}_\infty$ control, quantizer, quantization, matrix inequality, state feedback, output feedback.

1. INTRODUCTION

In classical feedback control theory, various signals or data in the control loop have been assumed to be passed directly without data loss, except in saturated systems. However, this is not true in many real applications. For example, in networked control systems Bushnell [2001], Ishii & Francis [2002] where all signals are transferred through network, package dropouts or data transfer rate limitations always happen. Another important aspect, which is well known in signal processing area, is signal quantization. Since quantization always exists in computer based control systems, many researchers have begun to study the analysis and design problems for control systems involving various quantization methods. Delchamps [1990] addressed the problem of stabilizing an unstable linear system by means of quantized state feedback, i.e., state feedback where the measurements of the system state are quantized. The quantizer in Delchamps [1990] takes value in a countable set. Brockett and Liberzon [2000] defined a quantizer taking value in a finite set and considered quantized feedback stabilization for linear systems. It has been shown there that if it is possible to change the sensitivity of the quantizer on the basis of available quantized measurements, then a hybrid control strategy, for both continuous- and discrete-time systems, can be designed to guarantee global asymptotic stability. While the approach in Brockett and Liberzon [2000] relies on the possibility of making discrete online adjustments of quantizer parameters, Liberzon [2003] extended the approach for more general nonlinear systems with general types of quantizers involving the states of the system, the measured outputs, and the control inputs. The idea and results in Liberzon [2003] are applied for stabilization of discrete-time LTI systems with quantized measurement outputs in Matsumoto *et al.* [2003].

Later, Zhai *et al.* [2004] considered the stabilization problem for a discrete-time LTI system via state feedback involving both quantized states and control inputs. As assumed in Liberzon [2003], the system considered in Zhai *et al.* [2004] is supposed to be stabilizable and a stabilizing state feedback has been designed without taking quantization into account. However, the system's states are quantized before they are passed to the controller, and the control inputs are quantized before they are passed to the system. This is a natural setting in networked control systems, where all informations (reference inputs, plant outputs, control inputs, etc.) are exchanged through a network among control system components (sensors, controllers, actuators, etc.). Due to the quantization effects, the desired system stability can not be guaranteed. For this reason, Zhai *et al.* [2004] defined the two quantizers with general forms as in Liberzon [2003] and then proposed a hybrid quantized state feedback strategy where the values of the quantizer parameters are updated at discrete instants of time. Further, they extended the results to $\mathcal{H}_\infty$ feedback control systems in Zhai *et al.* [2005], dealing with both state feedback and dynamic output feedback. The key point is to propose a state-dependent (or output-dependent) strategy for updating the quantizer's parameter, so that the system is asymptotically stable and achieves the same $\mathcal{H}_\infty$ disturbance attenuation level. It was also noted in Zhai *et al.* [2005] that the control strategies of updating the quantizer's parameter are dependent on time in the existing works (Brockett and Liberzon [2000], Liberzon [2003], Matsumoto *et al.* [2003], Zhai *et al.* [2004]), and such control strategies can not be applied for the case of $\mathcal{H}_\infty$ control systems since the value of the disturbance inputs is not available and thus we can not drive the state into an invariant region, as done in Liberzon [2003], Matsumoto *et al.* [2003], Zhai *et al.* [2004]. As a great contrast, the control strategy in Zhai *et al.* [2005] is state or output dependent, which is usually regarded to have more robustness.

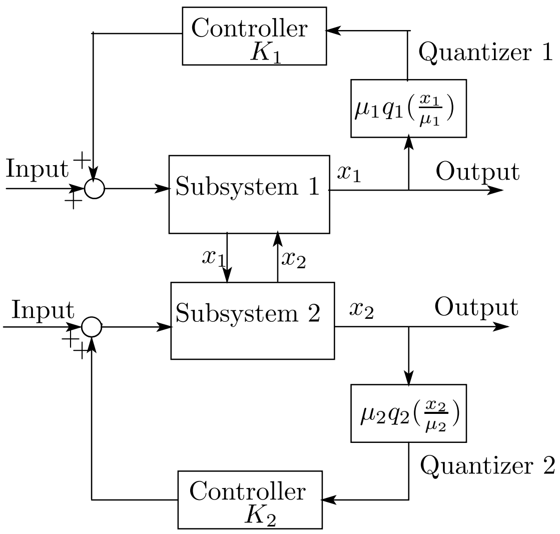

Fig.1 Interconnected Feedback System with Quantized State or Quantized Measurement Output

In this paper, we extend the discussion in Zhai *et al.* [2005] to interconnected feedback control systems, as described in Fig. 1. As also noted later, although the discussion and the result are valid for the case where there are more than three subsystems involved, we assume for notation simplicity that the number of subsystems is two. The two subsystems have their own state, control input, measurement output, disturbance input, controlled output, and they interconnect each other through their states. In decentralized control, the two controllers in Fig. 1 are designed in a decentralized manner, depending on each subsystem's local state (or measurement output).

Now, as in our previous work, we assume that for each subsystem, a local state feedback (or static output feedback) has been designed such that the overall system is stable and some $\mathcal{H}_\infty$ disturbance attenuation level is achieved. However, the subsystems' local states (or outputs) are quantized before they are passed to the controller, and due to the quantization effects, the desired system stability and $\mathcal{H}_\infty$ disturbance attenuation level can not be guaranteed. Here, we suppose that the quantizers are in a generalized form and there is a zoom parameter which can adjusted. Then, we propose to update the quantizers' parameters in a reasonable decentralized online manner, i.e., to change the parameter's value depending on each subsystem's state (or output) information. We show that under some flexible sufficient condition, there exists a decentralized control strategy for updating each quantizer's zoom parameter, such that the overall closed-loop system is asymptotically stable and the same $\mathcal{H}_\infty$ disturbance attenuation level is achieved.

The rest of this paper is organized as follows. Section 2 gives the definition and the property of generalized quantizer. Section 3 describes the control problem formulation and how to predesign the controller in the case of decentralized state feedback. Section 4 proposes a local-state-dependent strategy for updating the quantizers' parameters, so that the overall closed-loop system is asymptotically stable and achieves the same $\mathcal{H}_\infty$ disturbance attenuation level. Section 5 extends the consideration to the case of decentralized static output feedback, and obtain parallel result. Finally, Section 6 gives some concluding remarks.

2. QUANTIZER DESCRIPTION

First, we give the definition of a quantizer with general form as introduced in Liberzon [2003]. Let $z \in \Re^l$ be the variable being quantized. A *quantizer* is defined as a piecewise constant function $q : \Re^l \to \mathcal{D}$, where $\mathcal{D}$ is a finite subset of $\Re^l$. This leads to a partition of $\Re^l$ into a finite number of quantization regions of the form $\{z \in \Re^l : q(z) = i\}$, $i \in \mathcal{D}$. These quantization regions are not assumed to have any particular shapes. We assume that there exist positive real numbers M and Δ such that the following conditions hold:

(1) If
$$|z| \le M \tag{1}$$
then
$$|q(z) - z| \le \Delta\,. \tag{2}$$
(2) If
$$|z| > M$$
then
$$|q(z)| > M - \Delta\,.$$

Throughout this paper, we denote by $|\cdot|$ the standard Euclidean norm in the n-dimensional vector space $\Re^n$, and denote by $\|\cdot\|$ the corresponding induced matrix norm in $\Re^{n\times n}$. Condition 1 gives a bound on the quantization error when the quantizer does not saturate. Condition 2 provides a way to detect the possibility of saturation. We will refer to M and Δ as *the range of* q and the *quantization error*, respectively. We also assume that $q(x) = 0$ for x in some neighborhood of the origin. The example of satisfying the above requirements is given by the quantizer with rectangular quantization regions in Brockett and Liberzon [2000], Liberzon [2000].

In the control strategy to be developed below, we will use quantized measurements of the form
$$q_\mu(z) \triangleq \mu q(\frac{z}{\mu})\,, \tag{3}$$
where $\mu > 0$ is the parameter. The extreme case of $\mu = 0$ is regarded as setting the output of the quantizer as zero. The range of this quantizer is $M\mu$ and the quantization error is $\Delta\mu$. We can view μ as a "zoom" variable: increasing

μ corresponds to zooming out and essentially obtaining a new quantizer with larger range and larger quantization error, while decreasing μ corresponds to zooming in and obtaining a quantizer with smaller range but also smaller quantization error. We will update μ later depending on the system local state (or the local measurement output). In this sense, it can be considered as another state of the resultant closed-loop system.

3. PROBLEM FORMULATION

Although the discussion in this paper can be easily extended to the case where more than two subsystems are interconnected, we focus our attention on the case of two subsystems (as in Fig. 1) which are described by

$$\begin{cases} \dot{x}_1 = A_{11}x_1 + A_{12}x_2 + B_{11}w_1 + B_{21}u_1 \\ z_1 = C_1x_1 + D_1w_1 \end{cases} \tag{4}$$

and

$$\begin{cases} \dot{x}_2 = A_{21}x_1 + A_{22}x_2 + B_{12}w_2 + B_{22}u_2 \\ z_2 = C_2x_2 + D_2w_2 \end{cases} \tag{5}$$

where $x_1 \in \Re^{n_1}$ and $x_2 \in \Re^{n_2}$ are the subsystems' states, $u_1 \in \Re^{m_1}$ and $u_2 \in \Re^{m_2}$ are the control inputs, $w_1 \in \Re^{h_1}$ and $w_2 \in \Re^{h_2}$ are the disturbance inputs, $z_1 \in \Re^{p_1}$ and $z_2 \in \Re^{p_2}$ are the controlled outputs. The matrices A_{ij},B_{ij},C_i and D_i $(i, j = 1, 2)$ are constant and of appropriate dimension, and $A_{12}x_2$ in (4) and $A_{21}x_1$ in (5) are the interconnection terms between the two subsystems. We assume that the pairs (A_{11}, B_{21}) and (A_{22}, B_{22}) are stabilizable.

Suppose that for the system (4) and (5), we have designed a decentralized controller composed of two local state feedbacks

$$u_1 = K_1x_1\,, \quad u_2 = K_2x_2 \tag{6}$$

so that the closed-loop system, composed of (4), (5) and (6), is stable and the $\mathcal{H}_\infty$ norm of the transfer function from $w = [w_1^T\ w_2^T]^T$ to $z = [z_1^T\ z_2^T]^T$ is less than a specified level γ. More precisely, the closed-loop system is writen as

$$\begin{cases} \dot{x} = Ax + B_1w \\ z = Cx + Dw \end{cases} \tag{7}$$

where $x = [x_1^T\ x_2^T]^T$, and

$$\begin{aligned} A &= \begin{bmatrix} A_{11} + B_{21}K_1 & A_{12} \\ A_{21} & A_{22} + B_{22}K_2 \end{bmatrix}, \\ B_1 &= \begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix},\ C = [\,C_1\ C_2\,]\,,\ D = \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix}. \end{aligned} \tag{8}$$

Then, the hypothesis is that, without taking quantization into consideration, the gains K_1 and K_2 in (6) are designed so that A is stable and $\|D + C(sI - A)^{-1}B_1\|_\infty < \gamma$. Therefore, according to the well known Bounded Real Lemma Iwasaki *et al.* [1998], there exists a positive definite matrix P satisfying the matrix inequality

$$\begin{bmatrix} A^TP + PA & PB_1 & C^T \\ B_1^TP & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix} < 0\,. \tag{9}$$

Furthermore, as in typical decentralized controller design settings, we consider block-diagonal matrix P as $P = \text{diag}\{P_1, P_2\}$. In this case, Pre- and post-multiplying (9) by $\text{diag}\{P^{-1}, I, I\}$, and setting $P_1^{-1} = Q_1, P_2^{-1} = Q_2$, $K_1Q_1 = M_1$, $K_2Q_2 = M_2$, results in

$$\begin{bmatrix} \begin{matrix} A_{11}Q_1 + B_{21}M_1 + \\ (A_{11}Q_1 + B_{21}M_1)^T \end{matrix} & A_{12}Q_2 + Q_1A_{21}^T & Q_1C_1^T & B_{11} \\ A_{21}Q_1 + Q_2A_{12}^T & \begin{matrix} A_{22}Q_2 + B_{22}M_2 + \\ (A_{22}Q_2 + B_{22}M_2)^T \end{matrix} & Q_2C_2^T & B_{12} \\ C_1Q_1 & C_2Q_2 & -\gamma I & D \\ B_{11}^T & B_{12}^T & D^T & \gamma I \end{bmatrix} < 0 \tag{10}$$

which is a linear matrix inequality (LMI) with respect to $Q_1 > 0$, $Q_2 > 0$, M_1 and M_2. When it is feasible, the state feedback gains are obtained by $K_1 = M_1Q_1^{-1}$, $K_2 = M_2Q_2^{-1}$. This is a well known design procedure for decentralized state feedback of interconnected systems, and we also assume that the state feedbacks (6) are obtained using the above procedure.

Throughout this paper, we will let $\lambda_m(\cdot)$ and $\lambda_M(\cdot)$ denote the smallest and the largest eigenvalue of a symmetric matrix, respectively. Then, for any positive definite matrix W, the inequality

$$\lambda_m(W)\,|x|^2 \le x^TWx \le \lambda_M(W)\,|x|^2 \tag{11}$$

holds for any x.

Here, as depicted in Fig. 1, we deal with the case where only quantized local state information is available. For this reason, we modify the state feedback (6) using the quantized information of x as

$$u_1 = K_1\mu_1q_1(\frac{x_1}{\mu_1})\,, \quad u_2 = K_2\mu_2q_2(\frac{x_2}{\mu_2})\,. \tag{12}$$

For any fixed positive scalars μ_1 and μ_2, the closed-loop system composed of the systems (4), (5) and the new state feedback (12) is given by

$$\begin{cases} \dot{x} = Ax + B_1w + F(\mu, x) \\ z = Cx + Dw\,, \end{cases} \tag{13}$$

where

$$F(\mu, x) = \begin{bmatrix} F_1(\mu_1, x_1) \\ F_2(\mu_2, x_2) \end{bmatrix} \triangleq \begin{bmatrix} \mu_1B_{21}K_1\left(q_1(\frac{x_1}{\mu_1}) - \frac{x_1}{\mu_1}\right) \\ \mu_2B_{22}K_2\left(q_2(\frac{x_2}{\mu_2}) - \frac{x_2}{\mu_2}\right) \end{bmatrix}. \tag{14}$$

Now, the control problem is very natural. Due to the existence of quantization error, the stability of the closed-loop system and the desired $\mathcal{H}_\infty$ disturbance attenuation level γ is not guaranteed. For this reason, we formulate our control problem as follows:

Decentralized Quantizer Design Problem. *Design a decentralized control strategy which adjusts μ_1 depending on the local state x_1 and adjusts μ_2 depending on the local state x_2 appropriately, so that the stability of the overall closed-loop system and the same $\mathcal{H}_\infty$ disturbance attenuation level γ is achieved.*

4. MAIN RESULT

Since (9) is a matrix inequality, we can always find a block diagonal positive matrix $R = \text{diag}\{R_1,\ R_2\}$ such that

$$\begin{bmatrix} A^TP + PA + R & PB_1 & C^T \\ B_1^TP & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix} < 0 \qquad (15)$$

which is equivalent to

$$\begin{bmatrix} A^TP + PA + R + \frac{1}{\gamma}C^TC & PB_1 + \frac{1}{\gamma}C^TD \\ B_1^TP + \frac{1}{\gamma}D^TC & -\gamma I + \frac{1}{\gamma}D^TD \end{bmatrix} < 0\,. \quad (16)$$

We are in the position to state and prove the first main result in this paper.

Theorem 1. *Assume that for the two quantizers M_i is chosen large enough compared to Δ_i so that*

$$M_i > 2\Delta_i \frac{\|P_iB_{2i}K_i\|}{\lambda_m(R_i)}\,,\ i = 1, 2\,. \qquad (17)$$

Then, there exists a control strategy for updating μ_i, which is dependent on the local state x_i, such that the closed-loop system (13) *is asymptotically stable and the $\mathcal{H}_\infty$ disturbance attenuation level γ is achieved.*

Proof. Since $\frac{x_i}{\mu_i}$ $(i = 1, 2)$ is quantized before going to the state feedback, we obtain by using the properties of general quantizers in (1) and (2) that whenever $|x_i| \le M_i\mu_i$, the following holds.

$$\left| q_i(\frac{x_i}{\mu_i}) - \frac{x_i}{\mu_i} \right| \le \Delta_i \qquad (18)$$

We consider the Lyapunov function candidate

$$V(x) = x^TPx \qquad (19)$$

for the closed-loop system (13). By using the matrix inequality (16), we obtain that when $|x_i| \le M_i\mu_i$, the derivative of $V(x)$ along solutions of (13) satisfies

$$\begin{aligned}
\dot{V} &= (Ax + B_1w + F(\mu,x))^T Px \\
&\quad + x^TP(Ax + B_1w + F(\mu,x)) \\
&= \left[\, x^T\ w^T \,\right] \begin{bmatrix} A^TP + PA & PB_1 \\ B_1^TP & 0 \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \\
&\quad + x^TPF(\mu,x) + F^T(\mu,x)Px \\
&\le \left[\, x^T\ w^T \,\right] \begin{bmatrix} -R - \frac{1}{\gamma}C^TC & -\frac{1}{\gamma}C^TD \\ -\frac{1}{\gamma}D^TC & \gamma I - \frac{1}{\gamma}D^TD \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \\
&\quad + x^TPF(\mu,x) + F^T(\mu,x)Px \\
&= -\frac{1}{\gamma}z^Tz + \gamma w^Tw \\
&\quad - x_1^TR_1x_1 + x_1^TP_1F_1(\mu_1,x_1) + F_1^T(\mu_1,x_1)P_1x_1 \\
&\quad - x_2^TR_2x_2 + x_2^TP_2F_2(\mu_2,x_2) + F_2^T(\mu_2,x_2)P_2x_2 \\
&\le -\frac{1}{\gamma}z^Tz + \gamma w^Tw \\
&\quad - \lambda_m(R_1)|x_1| \left(|x_1| - 2\Delta_1 \frac{\|P_1B_{21}K_1\|}{\lambda_m(R_1)}\mu_1 \right) \\
&\quad - \lambda_m(R_2)|x_2| \left(|x_2| - 2\Delta_2 \frac{\|P_2B_{22}K_2\|}{\lambda_m(R_2)}\mu_2 \right)\,. \qquad (20)
\end{aligned}$$

According to (17), we can always find a scalar $\epsilon \in (0, 1)$ such that

$$M_i > 2\Delta_i \frac{\|P_iB_{2i}K_i\|}{\lambda_m(R_i)} \times \frac{1}{1-\epsilon}\,,\ i = 1, 2\,, \qquad (21)$$

which is equivalent to

$$\frac{1}{1-\epsilon} \times 2\Delta_i \frac{\|P_iB_{2i}K_i\|}{\lambda_m(R_i)}\mu_i < M_i\mu_i\,,\ i = 1, 2\,. \qquad (22)$$

Therefore, for any nonzero x_i, we can find a positive scalar μ_i such that

$$\frac{1}{1-\epsilon} \times 2\Delta_i \frac{\|P_iB_{2i}K_i\|}{\lambda_m(R_i)}\mu_i \le |x_i| \le M_i\mu_i\,. \qquad (23)$$

This is also true in the case of $x = 0$, where we set $\mu = 0$ as an extreme case and consider the output of the quantizer as zero.

In other words, since we can always choose μ so that (23) is satisfied, (20) holds and thus

$$\begin{aligned}
\dot{V} &\le -\frac{1}{\gamma}z^Tz + \gamma w^Tw - \epsilon\lambda_m(R_1)|x_1|^2 - \epsilon\lambda_m(R_2)|x_2|^2 \\
&\le -\frac{1}{\gamma}z^Tz + \gamma w^Tw - \epsilon\frac{\lambda_m(R)}{\lambda_M(P)}V \\
&= -\epsilon\frac{\lambda_m(R)}{\lambda_M(P)}V - \frac{1}{\gamma}\Gamma(t)\,, \qquad (24)
\end{aligned}$$

where $\Gamma(t) \stackrel{\triangle}{=} z^T(t)z(t) - \gamma^2w^T(t)w(t)$.

First, by setting $w = 0$ in (24), we see clearly that the system is asymptotically stable.

Next, since $V(t) \ge 0$, we obtain from (24) that $\dot{V} \le -\frac{1}{\gamma}\Gamma(t)$, and thus for any $t > t_0$,

$$V(t) - V(t_0) \le -\frac{1}{\gamma}\int_{t_0}^{t} \Gamma(\tau)d\tau\,. \qquad (25)$$

Using $V(t) \ge 0$ again, we obtain

$$\int_{t_0}^{t} z^T(\tau)z(\tau)d\tau \le \gamma V(t_0) + \gamma^2 \int_{t_0}^{t} w^T(\tau)w(\tau)d\tau\,, \qquad (26)$$

which implies that the $\mathcal{H}_\infty$ disturbance attenuation level γ is achieved. This completes the proof. ∎

Remark 1. In the existing references (for example, Liberzon [2003], Zhai *et al.* [2004]), the value of μ is updated in a time-controlled manner, i.e., when to change the value of μ is dependent only on time. This is not possible for the present situation because we do not know the value of $w(t)$ and thus we can not drive $x(t)$ into a specified invariant region, as done in Liberzon [2003], Zhai *et al.* [2004]. To overcome this difficulty, we have proposed a state-dependent strategy (23) for adjusting the value of μ. As also pointed out in many other references, such a state-dependent strategy is usually more robust to modelling imperfection than time-dependent one.

Remark 2. There is an important observation concerning the implementation of the quantizer proposed in this section, and it is also valid for the quantizer in the next section. We assume that the function $q_i(\cdot)$, which may be very complicated, has been designed and we implement $\mu_i q_i(\frac{x_i}{\mu_i})$ (NOT $q(\frac{x_i}{\mu_i})$ only) as a parameter-dependent quantizer. Since the variable of the function $q_i(\cdot)$ is $\frac{x_i}{\mu_i}$, the quantizer can flexibly deal with large or small state x_i by adjusting the value of μ_i, so that the condition (23) is satisfied. This is very important in $\mathcal{H}_\infty$ control problems since the state x_i may be very large temporarily due to unexpected disturbance input. In the case where only $q_i(\frac{x_i}{\mu_i})$ is viewed as a quantizer, the output of the quantizer has to be scaled by μ_i before it is passed to the controller. The function $q_i(\cdot)$ in this paper is a general concept for quantization, and thus careful consideration is required in real implementation.

Remark 3. Although the $\mathcal{H}_\infty$ disturbance attenuation level γ is fixed in this paper, the same discussion is applicable for any positive $\gamma > \gamma_{opt}$, where γ_{opt} is the optimal $\mathcal{H}_\infty$ norm that the system composed of (4) and (5) can reach via decentralized state feedback.

5. EXTENSION TO OUTPUT FEEDBACK

In the case where the state information is not available in the feedback loop and also in the quantizer, we need to pull out certain output information from the system and then consider output feedback. For this reason, we consider in this section the interconnected system described by

$$\begin{cases} \dot{x}_1 = A_{11}x_1 + A_{12}x_2 + B_{11}w_1 + B_{21}u_1 \\ z_1 = C_1x_1 + D_1w_1 \\ y_1 = E_1x_1 \end{cases} \quad (27)$$

and

$$\begin{cases} \dot{x}_2 = A_{21}x_1 + A_{22}x_2 + B_{12}w_2 + B_{22}u_2 \\ z_2 = C_2x_2 + D_2w_2 \\ y_2 = E_2x_2 \end{cases} \quad (28)$$

where $y_1 \in \Re^{q_1}$ and $y_2 \in \Re^{q_2}$ are the local measurement outputs, E_1 and E_2 are constant matrices of appropriate dimension, and all the other vectors and matrices are the same as before. We assume that the triples (A_{11}, B_{21}, E_1) and (A_{22}, B_{22}, E_2) are stabilizable and detectable.

Although the following discussion is also valid for dynamical output case, we assume for notation simplicity that, without taking quantization into consideration, a decentralized static output feedback

$$u_1 = K_1y_1\,, \quad u_2 = K_2y_2 \quad (29)$$

has been designed such that the overall closed-loop system is stable and the $\mathcal{H}_\infty$ norm of the transfer function from w to z is less than a specified level γ. Then, the matrix inequality (9) and (16) are satisfied with

$$A = \begin{bmatrix} A_{11} + B_{21}K_1E_1 & A_{12} \\ A_{21} & A_{22} + B_{22}K_2E_2 \end{bmatrix}. \quad (30)$$

Without causing confusion, we use the same notation A, which is different from the matrix in (8). As in the previous section, we suppose that the gains K_1 and K_2 are designed in a decentralized manner using a block diagonal positive definite matrix P.

Now, we deal with the case where only quantized measurements of the output y_i are available, and thus the decentralized static output feedback (29) takes the form of

$$u_1 = K_1\mu_1 q_1(\frac{y_1}{\mu_1})\,, \quad u_2 = K_2\mu_2 q_2(\frac{y_2}{\mu_2})\,. \quad (31)$$

Again, although we used the same quantizer notation q_1 and q_2 for notation simplicity, they are generally different from the previous ones since the dimensions of x_i and y_i are basically different.

The closed-loop system composed of (27), (28) and (31) is

$$\begin{cases} \dot{x} = Ax + B_1w + F(\mu, y) \\ z = Cx + Dw\,, \end{cases} \quad (32)$$

where

$$F(\mu, y) = \begin{bmatrix} F_1(\mu_1, y_1) \\ F_2(\mu_2, y_2) \end{bmatrix} \triangleq \begin{bmatrix} \mu_1B_{21}K_1\left(q_1(\frac{y_1}{\mu_1}) - \frac{y_1}{\mu_1}\right) \\ \mu_2B_{22}K_2\left(q_2(\frac{y_2}{\mu_2}) - \frac{y_2}{\mu_2}\right) \end{bmatrix}. \quad (33)$$

Also, due to the existence of quantization error, the stability and the desired $\mathcal{H}_\infty$ disturbance attenuation level γ is not guaranteed. Next, we propose a control strategy which adjusts the quantizers' parameters μ_1 and μ_2 appropriately, depending on the local measurement outputs, so that the stability and the desired $\mathcal{H}_\infty$ disturbance attenuation level γ is achieved.

Theorem 2. *Assume that for the two quantizers M_i is chosen large enough compared to Δ_i so that*

$$M_i > 2\Delta_i\frac{\|P_iB_{2i}K_i\|\|E_i\|}{\lambda_m(R_i)}\,, \quad i = 1, 2\,. \quad (34)$$

Then, there exists a control strategy for updating μ_i, which is dependent on the local measurement output y_i, such that the closed-loop system (32) is asymptotically stable and the $\mathcal{H}_\infty$ disturbance attenuation level γ is achieved.

Proof. Since $\frac{y_i}{\mu_i} = \frac{E_ix_i}{\mu_i}$ $(i = 1, 2)$ is quantized before being passed to the feedback, we obtain by using the properties of general quantizers in (1) and (2) that whenever $|y_i| \le M_i\mu_i$, the inequality

$$\left|\frac{y_i}{\mu_i} - q(\frac{y_i}{\mu_i})\right| \le \Delta_i \quad (35)$$

is true. We consider the same Lyapunov function candidate (19) for the closed-loop system (32). By making the same calculation as in the proof of Theorem 1, we obtain that when $|y_i| \le M_i\mu_i$, the derivative of $V(x)$ along solutions of (32) satisfies

$$
\begin{aligned}
\dot{V} &= (Ax + B_1 w + F(\mu, y))^T Px \\
&\quad + x^T P (Ax + B_1 w + F(\mu, y)) \\
&\le -\frac{1}{\gamma} z^T z + \gamma w^T w \\
&\quad - x_1^T R_1 x_1 + x_1^T P_1 F_1(\mu_1, y_1) + F_1^T(\mu_1, y_1) P_1 x_1 \\
&\quad - x_2^T R_2 x_2 + x_2^T P_2 F_2(\mu_2, y_2) + F_2^T(\mu_2, y_2) P_2 x_2 \\
&\le -\frac{1}{\gamma} z^T z + \gamma w^T w \\
&\quad - \lambda_m(R_1)|x_1| \left(|x_1| - 2\Delta_1 \frac{\|P_1 B_{21} K_1\|}{\lambda_m(R_1)} \mu_1 \right) \\
&\quad - \lambda_m(R_2)|x_2| \left(|x_2| - 2\Delta_2 \frac{\|P_2 B_{22} K_2\|}{\lambda_m(R_2)} \mu_2 \right) \\
&\le -\frac{1}{\gamma} z^T z + \gamma w^T w \\
&\quad - \lambda_m(R_1) \frac{|x_1|}{\|E_1\|} \left(|y_1| - 2\Delta_1 \frac{\|P_1 B_{21} K_1\| \|E_1\|}{\lambda_m(R_1)} \mu_1 \right) \\
&\quad - \lambda_m(R_2) \frac{|x_2|}{\|E_2\|} \left(|y_2| - 2\Delta_2 \frac{\|P_2 B_{22} K_2\| \|E_2\|}{\lambda_m(R_2)} \mu_2 \right) .
\end{aligned} \tag{36}
$$

According to (34), we can always find a scalar $\epsilon \in (0, 1)$ such that

$$M_i > 2\Delta_i \frac{\|P_i B_{2i} K_i\| \|E_i\|}{\lambda_m(R_i)} \times \frac{1}{1 - \epsilon} , \tag{37}$$

which is equivalent to

$$\frac{1}{1 - \epsilon} \times 2\Delta_i \frac{\|P_i B_{2i} K_i\| \|E_i\|}{\lambda_m(R_i)} \mu_i < M_i \mu_i . \tag{38}$$

Similarly as in Theorem 1, if we choose the quantizer's parameter μ_i for any y_i such that

$$\frac{1}{1 - \epsilon} \times 2\Delta_i \frac{\|P_i B_{2i} K_i\| \|E_i\|}{\lambda_m(R_i)} \mu_i \le |y_i| \le M_i \mu_i , \tag{39}$$

then (36) is true and thus

$$\dot{V} \le -\frac{1}{\gamma} \Gamma(t) - \epsilon \lambda_m(R_1) \frac{|x_1|}{\|E_1\|} |y_1| - \epsilon \lambda_m(R_2) \frac{|x_2|}{\|E_2\|} |y_2| . \tag{40}$$

The remaining proof, concerning the asymptotic stability and the $\mathcal{H}_\infty$ disturbance attenuation level, is the same as in Theorem 1, and is thus omitted. ∎

Remark 4. The difference between the control strategies (23) and (39) is that (23) is dependent on the local state x_i while (39) is dependent on the local measurement output y_i. This is natural since in the present situation we can not obtain the state information directly.

Remark 5. Both the condition (17) in Theorem 1 and the condition (34) in Theorem 2 are flexible, in the sense that we can choose the matrices P_i, R_i (and K_i) so that these conditions are satisfied. These matrices are not independent and they must satisfy the matrix inequality (16), but we still have much design freedom since it is an inequality and we can incorporate some optimization requirement when solving (9) and (16).

6. CONCLUSION

In this paper, we have studied stabilization and $\mathcal{H}_\infty$ disturbance attenuation problem for interconnected feedback control systems where the states or the measurement outputs are quantized before they go to the controller. We have proposed a local-state-dependent (or local-output-dependent) control strategy for updating the quantizers' parameters on line so that the overall closed-loop system is asymptotically stable and achieves the same $\mathcal{H}_\infty$ disturbance attenuation level as in the case where no quantization is involved.

Our next interest is the $\mathcal{H}_\infty$ disturbance attenuation problem for interconnected feedback control systems with two quantizers (quantization of both states/outputs and control inputs), as mentioned in Zhai *et al.* [2004]. Furthermore, the application of these results for design of large-scale networked control systems is an interesting and challenging problem.

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