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# Stability Analysis - Multiconvexity Approach

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**Abstract:** A new stability condition in terms of LMIs is studied in this paper, continuous- and discrete-time fuzzy systems treated in a unified manner. Based on a premise-dependent Lyapunov function and multiconvexity, we release the conservatism that commonly exists in the common $P$ approach.

## 1. INTRODUCTION

Recently a large number of literature on fuzzy control are TS model-based control where, mostly, the common $P$ approach searching for a single Lyapunov function remains active Wang et al. [1996], Tanaka et al. [1998], Kim and Lee [2000], Blanco et al. [2000], Tuan et al. [2001]. Another category emphasizes parameter-dependent functions with multiple $P_j$ matrices as a candidate of Lyapunov function Johansson et al. [1999], Kiriakidis [2001], Chadli et al. [2000], Tanaka et al. [2001a,b], Feng and Ma [2001], Cao et al. [1996]. To remove the time-derivative dependence, progress has been made recently in obtaining less conservative results using non-quadratic approach (multiple Lyapunov functions) Morere et al. [1999], Guerra and Perruquetti [2001], Guerra and Vermeiren [2001], Tanaka et al. [2003] where a fuzzy Lyapunov candidate is used for a discrete- and continuous-time T-S fuzzy model and the resulting stability condition is shown to be more relaxed than the condition derived from the common $P$ approach. In this paper, multiconvexity property from Apkarian and Tuan [2000] is combined with premise-dependent Lyapunov function to derive sufficient conditions for stability test of TS fuzzy systems.

The paper is organized as follows. Section II rehearses some useful results which forms the foundation for later developments. Section III derives the stability conditions for continuous- and discrete-time systems via a premise-dependent Lyapunov in conjunction with multiconvexity property. Two examples are illustrated in Section IV and conclusion is drawn in Section V.

## 2. PRELIMINARIES

To begin with, we introduce the following definitions and corollaries which serve as the entry point to this paper.

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A polytope $\Pi$ in $R^n$ is defined as the compact set

$$\Pi := \{\sum_{i=1}^{r} \mu_i v_i : \sum_{i=1}^{r} \mu_i = 1, \mu_i \geq 0, v_i \in R^n\} = co\ V \quad (1)$$

which constitutes the convex hull of the set $V = \{v_1, \cdots, v_r\}$. We denote the set of vertices of $\Pi$ as $vert\,\Pi := V$.

The following corollary is a useful tool permitting us to convert maximization of a function over a polytope $\Pi$ into exploring maximum of a function over $vert\ \Pi$. The corollary below clarifies this fact Apkarian and Tuan [2000]:

*Corollary 1.* (Multiconvexity). Consider a polytope $\Pi$ and the directions $d_1, \cdots, d_q$ determined by the edges of $\Pi$. $f$ has a maximum over $\Pi$ in $vert\,\Pi$ if the following is satisfied

$$\frac{\partial^2 f(v + \lambda d_l)}{\partial \lambda^2} \geq 0 \quad \forall v \in \Pi, l = 1, \cdots, q \quad (2)$$

where $v + \lambda d_l$ is a direction vector paralleling the edges of $\Pi$.

To find applications of the Corollary 1 to Lyapunov theory, it is instructive to consider the case in which $f$ is a quadratic function, $f(\mu) = \mu^T Q \mu + c^T \mu + a$. In particular $f(\mu)$ will be the time derivative function of a Lyapunov candidate function.

## 3. STABILITY ANALYSIS

In this section, we derive a stability condition for an open-loop fuzzy system that is displayed below:

$$\delta x = \sum_{i=1}^{r} \mu_i A_i x = A_\mu x \quad (3)$$

where $A_i$ is a system matrix of each rule $i$ and $\mu_i \geq 0$ is the firing strength of rule $i$. $\delta$ is a derivative operator for continuous-time systems, ($\delta x = \dot{x}(t)$) and a delay operator for discrete-time systems, ($\delta x = x(k+1)$).

*Theorem 2.* (Continuous Stability). The open loop system (3) is stable if there exist symmetric, positive definite matrices $X_j$ and upper bounds $|\dot{\mu}_j| \leq \phi_j$ satisfying the following LMIs:

$$\sum_{\rho=1}^{r} \phi_\rho X_\rho + X_j A_j^T + A_j X_j < 0, \quad 1 \leq j \leq r \tag{4}$$

$$X_i A_i^T + A_i X_i + X_j A_j^T + A_j X_j - (X_j A_i^T + A_i X_j + X_i A_j^T + A_j X_i) \geq 0, \quad 1 \leq i < j \leq r \tag{5}$$

**Proof:**
Consider a quadratic function $V(x(t))=x^T(t)X_\mu^{-1}x(t)$, where $X_\mu = \sum_{j=1}^{r} \mu_j X_j$ and $X_j$'s are symmetric, positive definite matrices such that for all $t$ and $\dot{X}_\mu = \sum_{\rho=1}^{r} \dot{\mu}_\rho X_\rho$, the time derivative of $V(x(t))$ along the state trajectories is

$$\dot{V}(x) = \dot{x}^T X_\mu^{-1} x + x^T X_\mu^{-1} \dot{x} + x^T \frac{dX_\mu^{-1}}{dt} x$$
$$= x^T (A_\mu^T X_\mu^{-1} + X_\mu^{-1} A_\mu + \frac{dX_\mu^{-1}}{dt}) x.$$

Based on Lyapunov theory, a sufficient condition is

$$A_\mu^T X_\mu^{-1} + X_\mu^{-1} A_\mu + \frac{dX_\mu^{-1}}{dt} < 0.$$

Pre- and post-multiplying the inequality above by $X_\mu$ yields

$$X_\mu A_\mu^T + A_\mu X_\mu + X_\mu \frac{dX_\mu^{-1}}{dt} X_\mu < 0.$$

Since

$$\frac{dX_\mu^{-1}}{dt} = -X_\mu^{-1} \dot{X}_\mu X_\mu^{-1}$$

we have

$$X_\mu A_\mu^T + A_\mu X_\mu - \dot{X}_\mu < 0$$

yielding

$$X_\mu A_\mu^T + A_\mu X_\mu - \sum_{\rho=1}^{r} \dot{\mu}_\rho X_\rho < 0 \tag{6}$$

By the virtue of the bounded $\dot{\mu}_\rho$ assumption, an upper bound expression is given below:

$$LHS(6) \leq \sum_{\rho=1}^{r} \phi_\rho X_\rho + \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i \mu_j M_{ij}$$
$$= \sum_{\rho=1}^{r} \phi_\rho X_\rho + \mu^T M \mu$$
$$= f(\mu) < 0 \tag{7}$$

where

$$|\dot{\mu}_\rho| \leq \phi_\rho, \quad \phi_\rho \geq 0 \text{ and } \mu = [\mu_1 \cdots \mu_r]'$$

$$M = \begin{bmatrix} M_{11} & \cdots & M_{1r} \\ \vdots & \ddots & \vdots \\ M_{r1} & \cdots & M_{rr} \end{bmatrix}, M_{ij} = (X_j A_i^T + A_i X_j)$$

and $M_{ij}$ are real symmetric, matrix-valued and linear functions of decision variables (multiple Lyapunov matrices) $X_i$. Note that the problem arisen with (7) involves infinitely many LMIs associated with each value of the parameter $\mu$ and is known to be intractable Apkarian and Tuan [2000]. By enforcing some constraints of geometric structure on the functional dependence in $\mu$, it is possible to reduce the problem to a feasibility problem of solving a finite number of LMIs. To this end, we note that the parameter vector $\mu = [\mu_1 \cdots \mu_r]'$, known as the firing strengths, evolves in the simplex defined below

$$\Gamma := \{\mu : \sum_{i=1}^{r} \mu_i = 1, \ \mu_i \geq 0\}.$$

Recalling (1), we have the polytope $\Gamma$ shown in Figure 1: To ease the proof, we assume $r = 3$ so that a geometrical

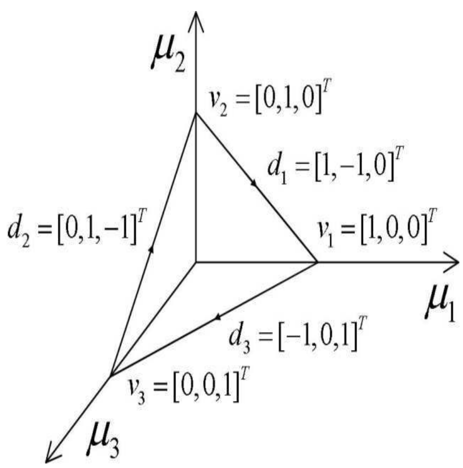


Fig. 1. Firing strength in the three-dimension space

structure becomes tangible and the vertices of $\Gamma$ can be found to be $v_1 = [1,0,0]'$, $v_2 = [0,1,0]'$, $v_3 = [0,0,1]'$ (see Figure 1) and the sufficient condition (7) for $r = 3$ becomes

$$f(\mu) = \sum_{\rho=1}^{3} \phi_\rho X_\rho + \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix}' \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix} < 0 \tag{8}$$

Equation (8) being a quadratic function of $\mu$, Corollary 1 says that $f(\mu)$ is negative whenever it is multiconvex along lines paralleling the edges of $\Gamma$ and furthermore $f(\mu)$ is negative over *vert* $\Gamma$. The remaining of the proof follows in two phases: (A) showing the second derivative condition (2) is satisfied so that the negativeness of $\dot{V}$ is assured by (B) checking the vertexes of $\Gamma$.

(A) To check the multiconvexity along the edges of $\Gamma$.

The directions $d_l, l = 1, \cdots, q$ is determined by vectors with all but two zero coordinates, the nonzero coordinates having opposite signs.

Along the direction $d_1 := [1, -1, 0]'$ of Figure 1, we get

$$f(\mu + \lambda d_1) = \sum_{\rho=1}^{r} \phi_\rho X_\rho +$$
$$\begin{bmatrix} \mu_1 + \lambda \\ \mu_2 - \lambda \\ \mu_3 \end{bmatrix}' \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \begin{bmatrix} \mu_1 + \lambda \\ \mu_2 - \lambda \\ \mu_3 \end{bmatrix}$$

which yields

$$\frac{\partial^2 f}{\partial \lambda^2} = M_{11} + M_{22} - M_{12} - M_{21}.$$

Along the direction $d_2 := [0, 1, -1]'$, we get

$$f(\mu + \lambda d_2) = \sum_{\rho=1}^{r} \phi_\rho X_\rho + [\,\mu_1 \;\; \mu_2 + \lambda \;\; \mu_3 - \lambda\,] \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 + \lambda \\ \mu_3 - \lambda \end{bmatrix}$$

which yields

$$\frac{\partial^2 f}{\partial \lambda^2} = M_{22} + M_{33} - M_{23} - M_{32}.$$

Similarly, along the direction $d_3 := [-1, 0, 1]'$ we get

$$\frac{\partial^2 f}{\partial \lambda^2} = M_{11} + M_{33} - M_{13} - M_{31}.$$

For $r = 3$, the multiconvexity is assured if

$$M_{11} + M_{22} - M_{12} - M_{21} \geq 0$$
$$M_{22} + M_{33} - M_{23} - M_{32} \geq 0$$
$$M_{11} + M_{33} - M_{13} - M_{31} \geq 0$$

are satisfied. Arguing in the same fashion as $r = 3$ case, we have the following results for the general case

$$\begin{aligned} &M_{ii} + M_{jj} - M_{ij} - M_{ji} \\ &= X_i A_i' + A_i X_i + X_j A_j' + A_j X_j - (X_j A_i' + A_i X_j + \\ &\quad X_i A_j' + A_j X_i) \geq 0, \quad 1 \leq i < j \leq r. \end{aligned}$$

This proves inequality (5). What follows is to

(B) check the vertices

At the vertex $[1, 0, 0]'$

$$\begin{aligned} f(v_1) &= \sum_{\rho=1}^{r} \phi_\rho X_\rho + [\,1\;0\;0\,] \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ &= \sum_{\rho=1}^{r} \phi_\rho X_\rho + M_{11}. \end{aligned}$$

At the vertex $[0, 1, 0]'$

$$\begin{aligned} f(v_2) &= \sum_{\rho=1}^{r} \phi_\rho X_\rho + [\,0\;1\;0\,] \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ &= \sum_{\rho=1}^{r} \phi_\rho X_\rho + M_{22}. \end{aligned}$$

Similarly, at the vertex $[0, 0, 1]'$ we have

$$f(v_3) = \sum_{\rho=1}^{r} \phi_\rho X_\rho + M_{33}.$$

For $r = 3$, to ensure negativeness, we need

$$\sum_{\rho=1}^{r} \phi_\rho X_\rho + M_{11} < 0$$
$$\sum_{\rho=1}^{r} \phi_\rho X_\rho + M_{22} < 0$$
$$\sum_{\rho=1}^{r} \phi_\rho X_\rho + M_{33} < 0.$$

Paralleling the argument for $r = 3$, we arrive at the following form for the general case.

$$\begin{aligned} &\sum_{\rho=1}^{r} \phi_\rho X_\rho + M_{jj} \\ &= \sum_{\rho=1}^{r} \phi_\rho X_\rho + X_j A_j' + A_j X_j < 0, \quad 1 \leq j \leq r. \end{aligned}$$

This proves inequality (4).

**Remark 1:** The assumption of boundedness in the rate of state-dependent firing strength $\mu$ is removed by using an idea of piecewise differential quadratic (PDQ) Lyapunov function and linear systems with jump Ma and Feng [2003].

**Remark 2:** By strengthening the condition in (4), one can slightly relax the multiconvexity requirement in (5). As an example, the feasibility problem to inequality (7) can be equivalently recast into the following problem: There exist matrices $Z_{ij}$ such that the following inequality is feasible

$$\sum_{\rho=1}^{r} \phi_\rho X_\rho + \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i \mu_j M_{ij} < -\sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i \mu_j Z_{ij}$$

where $\forall \mu \in \Gamma$ and

$$Z = \begin{bmatrix} Z_{11} & Z_{12} & \cdots & Z_{1r} \\ Z_{21} & Z_{22} & \cdots & Z_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ Z_{r1} & Z_{r2} & \cdots & Z_{rr} \end{bmatrix} \geq 0.$$

**Proof:** Similar lines to those in Apkarian and Tuan [2000].

Arguing as in Theorem 1, the associated solvability conditions are easily obtained as ($1 \leq j \leq r$, $1 \leq i < j \leq r$)

$$\sum_{\rho=1}^{r} \phi_\rho X_\rho + X_j A_j' + A_j X_j < -Z_{jj} \tag{9}$$

$$\begin{aligned} &X_i A_i' + A_i X_i + X_j A_j' + A_j X_j - (X_j A_i' + A_i X_j \\ &+ X_i A_j' + A_j X_i) \geq -(Z_{ii} + Z_{jj} - Z_{ij} - Z_{ji}) \end{aligned} \tag{10}$$

$$\begin{bmatrix} Z_{11} & Z_{12} & \cdots & Z_{1r} \\ Z_{21} & Z_{22} & \cdots & Z_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ Z_{r1} & Z_{r2} & \cdots & Z_{rr} \end{bmatrix} \geq 0. \tag{11}$$

Analogously, we will derive the stability condition for the discrete-time open-loop system (3), demonstrating that the multiconvexity property can be applied as well, constituting a unified treatment for multiconvexity stability analysis

*Theorem 3.* (Discrete Stability). The open loop system (3) is stable if there exist symmetric, positive definite matrices $X_j$ satisfying the following LMIs ($1 \le i < j \le r$):

$$\begin{bmatrix} -X_j & X_j A_j' \\ A_j X_j & -X_j \end{bmatrix} < 0,\ 1 \le j \le r \tag{12}$$

$$\begin{bmatrix} 0 & * \\ A_i X_i + A_j X_j - (A_i X_j + A_j X_i) & 0 \end{bmatrix} \ge 0 \tag{13}$$

**Proof:** Consider a quadratic function $V(x(k)) = x'(k) X_\mu^{-1} x(k)$ where $X_\mu = \sum_{j=1}^r \mu_j X_j$. The time difference of $V(x(k))$ is displayed below:

$$\begin{aligned} \Delta V &= V(x(k+1)) - V(x(k)) \\ &= x'(k+1) X_\mu^{-1} x(k+1) - x'(k) X_\mu^{-1} x(k) \\ &= x'(k) A_\mu' X_\mu^{-1} A_\mu x(k) - x'(k) X_\mu^{-1} x(k) \\ &= x'(A_\mu' X_\mu^{-1} A_\mu - X_\mu^{-1}) x < 0. \end{aligned}$$

A sufficient condition is

$$A_\mu' X_\mu^{-1} A_\mu - X_\mu^{-1} < 0$$

yielding

$$X_\mu A_\mu' X_\mu^{-1} A_\mu X_\mu - X_\mu < 0.$$

Schur complement gives

$$\begin{bmatrix} -X_\mu & X_\mu A_\mu' \\ A_\mu X_\mu & -X_\mu \end{bmatrix} < 0$$

Rewriting the matrix inequality yields

$$\begin{aligned} 0 &> \begin{bmatrix} -\sum_{j=1}^r \mu_j X_j & * \\ \sum_{i=1}^r \sum_{j=1}^r \mu_i \mu_j A_i X_j & -\sum_{j=1}^r \mu_j X_j \end{bmatrix} \\ &= -\sum_{j=1}^r \mu_j \begin{bmatrix} X_j & 0 \\ 0 & X_j \end{bmatrix} + \sum_{i=1}^r \sum_{j=1}^r \mu_i \mu_j \begin{bmatrix} 0 & X_j A_i' \\ A_i X_j & 0 \end{bmatrix} \\ &= -\mu' \hat{M} + \mu' \tilde{M} \mu \\ &= f(\mu) \end{aligned} \tag{14}$$

where $\mu = [\mu_1 \cdots \mu_r]'$ and

$$\hat{M} = \begin{bmatrix} M_1 \\ \vdots \\ M_r \end{bmatrix}, \quad M_j = \begin{bmatrix} X_j & 0 \\ 0 & X_j \end{bmatrix}$$

$$\tilde{M} = \begin{bmatrix} M_{11} & \cdots & M_{1r} \\ \vdots & \ddots & \vdots \\ M_{r1} & \cdots & M_{rr} \end{bmatrix} \quad M_{ij} = \begin{bmatrix} 0 & X_j A_i' \\ A_i X_j & 0 \end{bmatrix}$$

Notice that the matrices $M_j$ and $M_{ij}$ are real symmetric, matrix-valued and linear functions of decision variables (multiple Lyapunov matrices) $X_i$.

For $r = 3$, we have (14) displayed below.

$$f(\mu) = -\begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix}' \begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix} + \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix}' \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix}$$

Arguing in the same fashion as in the proof of continuous case, we

(A) check multiconvexity condition: Along the direction $d_1 := [1, -1, 0]'$ (Figure 1), we get

$$f(\mu + \lambda d_1) = -\begin{bmatrix} \mu_1 + \lambda \\ \mu_2 - \lambda \\ \mu_3 \end{bmatrix}' \begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix} + \begin{bmatrix} \mu_1 + \lambda \\ \mu_2 - \lambda \\ \mu_3 \end{bmatrix}' \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \begin{bmatrix} \mu_1 + \lambda \\ \mu_2 - \lambda \\ \mu_3 \end{bmatrix}$$

Then

$$\frac{\partial^2 f}{\partial \lambda^2} = M_{11} + M_{22} - M_{12} - M_{21}.$$

Similar to the first direction just shown, along the direction $d_2 := [0, 1, -1]'$ we get

$$\frac{\partial^2 f}{\partial \lambda^2} = M_{22} + M_{33} - M_{23} - M_{32}.$$

and along the direction $d_3 := [-1, 0, 1]'$, we get

$$\frac{\partial^2 f}{\partial \lambda^2} = M_{11} + M_{33} - M_{13} - M_{31}.$$

For $r = 3$, the multiconvexity condition is satisfied if

$$\begin{aligned} M_{11} + M_{22} - M_{12} - M_{21} &\ge 0 \\ M_{22} + M_{33} - M_{23} - M_{32} &\ge 0 \\ M_{11} + M_{33} - M_{13} - M_{31} &\ge 0. \end{aligned}$$

Arguing in the same fashion as $r = 3$ case, we have the following results for the general case ($1 \le i < j \le r$)

$$\begin{aligned} & M_{ii} + M_{jj} - M_{ij} - M_{ji} \\ &= \begin{bmatrix} 0 & * \\ A_i X_i + A_j X_j - (A_i X_j + A_j X_i) & 0 \end{bmatrix} \ge 0 \end{aligned}$$

This proves inequality (13).

(B) Check vertexes condition:

At the vertex $[1, 0, 0]'$

$$\begin{aligned} f(v_1) &= -\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}' \begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}' \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ &= -M_1 + M_{11}. \end{aligned}$$

Similar to vertex $[1, 0, 0]'$, we arrive at the following expression for the vertex $[0, 1, 0]'$

$$f(v_2) = -M_3 + M_{33}.$$

For $r = 3$, to ensure negativeness, we need

$$\begin{aligned} -M_1 + M_{11} &< 0 \\ -M_2 + M_{22} &< 0 \\ -M_3 + M_{33} &< 0. \end{aligned}$$

Paralleling the argument for $r = 3$, we arrive at the following form for $r > 3$

$$
\begin{aligned}
&-M_j + M_{jj}\\
&= \begin{bmatrix} -X_j & 0 \\ 0 & -X_j \end{bmatrix} + \begin{bmatrix} 0 & X_j A_j' \\ A_j X_j & 0 \end{bmatrix} < 0\\
&= \begin{bmatrix} -X_j & X_j A_j' \\ A_j X_j & -X_j \end{bmatrix} < 0,\ 1 \le j \le r.
\end{aligned}
$$

This proves inequality (12).

**Remark 3:** Likewise, by strengthening the condition in (12), one can slightly relax the multiconvexity requirement in (13). (See Remark 1)

$$
-\sum_{j=1}^{r} \mu_j M_j + \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i \mu_j M_{ij} < -\sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i \mu_j \bar{Z}_{ij}
$$

where $\forall \mu \in \Gamma$ *and*

$$
Z = \begin{bmatrix} \bar{Z}_{11} & \bar{Z}_{12} & \cdots & \bar{Z}_{1r} \\ \bar{Z}_{21} & \bar{Z}_{22} & \cdots & \bar{Z}_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{Z}_{r1} & \bar{Z}_{r2} & \cdots & \bar{Z}_{rr} \end{bmatrix} \ge 0, \quad \bar{Z}_{ij} = \begin{bmatrix} Z_{ij1} & Z_{ij2} \\ Z_{ij2} & Z_{ij3} \end{bmatrix}
$$

**Proof:** Similar lines to those in Apkarian and Tuan [2000].

Arguing as in Theorem 2, the associated feasibility conditions are easily obtained as ($1 \le j \le r$, $1 \le i < j \le r$)

$$
\begin{bmatrix} -X_j + Z_{jj1} & * \\ A_j X_j + Z_{jj2} & -X_j + Z_{jj3} \end{bmatrix} < 0 \tag{15}
$$

$$
\begin{bmatrix} Z_{ii1} + Z_{jj1} - Z_{ij1} - Z_{ji1} & \star \\ \begin{pmatrix} A_i X_i + Z_{ii2} + A_j X_j + Z_{jj2} \\ -(A_i X_j + Z_{ij2} + A_j X_i + Z_{ji2}) \end{pmatrix} & Z_{ii3} + Z_{jj3} - Z_{ij3} - Z_{ji3} \end{bmatrix} \ge 0 \tag{16}
$$

$$
\begin{bmatrix} Z_{111} & Z_{112} & \cdots & Z_{1r2} \\ Z_{112} & Z_{113} & \cdots & Z_{1r3} \\ \vdots & \vdots & \ddots & \vdots \\ Z_{r12} & Z_{r13} & \cdots & Z_{rr3} \end{bmatrix} \ge 0. \tag{17}
$$

## 4. EXAMPLES

In order to appreciate the efficiency of the proposed method, we consider examples where the T-S fuzzy models are borrowed from existing papers.

### *4.1 Continuous fuzzy systems*

A continuous fuzzy system, borrowed from Tanaka et al. [2003], composed of the following two rules

$R_1$ : IF $x_1(t)$ is $M_1$, THEN $\dot{x}(t) = A_1 x(t)$

$R_2$ : IF $x_1(t)$ is $M_2$, THEN $\dot{x}(t) = A_2 x(t)$

The fuzzy sets are described by the following two triangular membership functions:

$$
\mu_1(x(t)) = \frac{1 + sin(x_1(t))}{2}, \quad \mu_2(x(t)) = \frac{1 - sin(x_1(t))}{2}
$$

and

$$
A_1 = \begin{bmatrix} -5 & -4 \\ -1 & -2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -2 & -4 \\ 20 & -2 \end{bmatrix}
$$

the global T-S fuzzy model is:

$$
\dot{x}(t) = (\mu_1(x(t))A_1 + \mu_2(x(t))A_2)x(t)
$$

With $\phi_1 = 0.85$, $\phi_2 = 0.85$ and solving (9)-(11) and the following matrices are obtained:

$$
X_1 = \begin{bmatrix} 15.9551 & -13.0451 \\ -13.0451 & 19.0928 \end{bmatrix}, X_2 = \begin{bmatrix} 6.3777 & 0.9237 \\ 0.9237 & 27.7212 \end{bmatrix}
$$

$$
Z_{11} = \begin{bmatrix} 18.1033 & 5.6567 \\ 5.6567 & 5.2445 \end{bmatrix}, Z_{12} = \begin{bmatrix} -52.5093 & -154.8967 \\ -154.8967 & 588.9001 \end{bmatrix}
$$

$$
Z_{21} = \begin{bmatrix} 0.1002 - 0.0334 \\ -0.0334 - 0.5345 \end{bmatrix} \times 10^{-16}
$$

$$
Z_{22} = \begin{bmatrix} 6.9587 & -1.3361 \\ -1.3361 & 17.0730 \end{bmatrix}.
$$

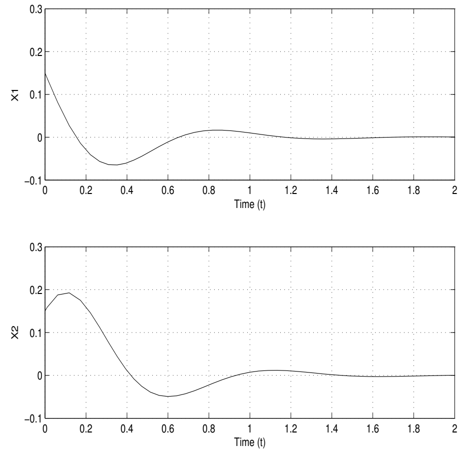


Fig. 2. States trajectory of (1) with initials values x(0)=[0.15 0.15].

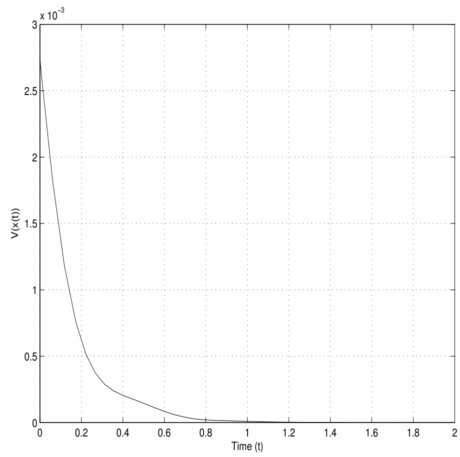


Fig. 3. Time transient of fuzzy Lyapunov function.

### *4.2 Discrete fuzzy systems*

Consider a discrete-time fuzzy system borrowed from Feng [2004] in which the rule base listed below:

$R_l$ : IF $x_l(t)$ is $\mu_l$, THEN $\dot{x}(t) = A_l x(t)$, for $l = 1, ...., 7$

The system matrices are given as

$$
A_1 = \begin{bmatrix} 1.0000 & 0.5000 \\ -0.3000 & 0.8000 \end{bmatrix} \quad A_2 = \begin{bmatrix} 1.0000 & 0.4875 \\ -0.2750 & 0.8000 \end{bmatrix}
$$

$$A_3 = \begin{bmatrix} 1.0000 & 0.4750 \\ -0.2500 & 0.8000 \end{bmatrix} \quad A_4 = \begin{bmatrix} 1.0000 & 0.4500 \\ -0.2000 & 0.8000 \end{bmatrix}$$
$$A_5 = \begin{bmatrix} 1.0000 & 0.4250 \\ -0.1500 & 0.8000 \end{bmatrix} \quad A_6 = \begin{bmatrix} 1.0000 & 0.4125 \\ -0.1250 & 0.8000 \end{bmatrix}$$
$$A_7 = \begin{bmatrix} 1.0000 & 0.4000 \\ -0.1000 & 0.8000 \end{bmatrix}.$$

By using the matlab LMI toolbox, one can easily verify that for the common $P$ solution, there exists no positive definite matrix for the fuzzy system to guarantee its stability. In other words, the fuzzy does not admit a global quadratic Lyapunov function. By solving (15)-(17) and the following matrices are obtained:

$$X_1 = \begin{bmatrix} 12.6862 & -2.6269 \\ -2.6269 & 7.5386 \end{bmatrix}, X_2 = \begin{bmatrix} 12.6589 & -2.6219 \\ -2.6219 & 7.5225 \end{bmatrix}$$
$$X_3 = \begin{bmatrix} 14.6175 & -3.0920 \\ -3.0920 & 7.7062 \end{bmatrix}, X_4 = \begin{bmatrix} 15.6588 & -3.3812 \\ -3.3812 & 7.2376 \end{bmatrix}$$
$$X_5 = \begin{bmatrix} 16.4814 & -3.6419 \\ -3.6419 & 6.5599 \end{bmatrix}, X_6 = \begin{bmatrix} 16.8279 & -3.7749 \\ -3.7749 & 6.1639 \end{bmatrix}$$
$$X_7 = \begin{bmatrix} 9.0780 & -1.3718 \\ -1.3718 & 8.1157 \end{bmatrix}$$

indicating a stable system.

## 5. CONCLUSION

In this paper, two stability conditions based on multiconvexity are developed for both continuous- and discrete-time T-S fuzzy systems. The proposed approach utilizes a premise-dependent Lyapunov function to prove Lyapunov stability of the underlying fuzzy systems, leading to a non-common $P$ method that releases the conservatism of the common $P$ scheme. It is shown and demonstrated via examples that the stability can be determined by solving a set of LMIs.

## REFERENCES

P. Apkarian and H.D. Tuan. Parameterized LMIs in control theory. *SJAM J. Control Optim.*, 38(4):1241–1264, 2000.

Y. Blanco, W. Perruquetti, and P. Borne. Relaxed stability conditions for Takagi-Sugeno's fuzzy models. In *Proc. 9th IEEE Conf. Fuzzy Syst.*, volume 2, pages 539–543, San Antonio, TX., 2000.

S.G. Cao, N.W. Rees, and G. Feng. $H^\infty$ control of nonlinear continuous-time systems based on dynamical fuzzy models. *Int'l. Journal on Systems Science*, 27(9): 821–830, 1996.

M. Chadli, D. Maquin, and J. Ragot. Relaxed stability conditions for Takagi-Sugeno fuzzy systems. In *Proc. 2000 IEEE International Conference on Systems, Man, and Cybernetics*, volume 5, pages 3514–3519, 2000.

G. Feng. Stability analysis of discrete-time fuzzy dynamic systems based on piecewise Lyapunov functions. *IEEE Trans. Fuzzy Systems*, 12(1):22–28, February 2004.

G. Feng and J. Ma. Quadratic stabilization of uncertain discrete-time fuzzy dynamic system. *IEEE Trans. Circuits and Syst. I: Fundamental Theory and Applications*, 48(11):1137–1344, November 2001.

T.M. Guerra and W. Perruquetti. Non quadratic stabilization of discrete Takagi Sugeno fuzzy models. In *FUZZ-IEEE2001 Conference*, volume 1, pages 1271–1274, Melbourne, AU, 2001.

T.M. Guerra and L. Vermeiren. Conditions for non quadratic stabilization of discrete fuzzy models. In *2001 IFAC Conference*, 2001.

M. Johansson, A. Rantzer, and K.-E. Arzen. Piecewise quadratic stability of fuzzy systems. *IEEE Trans. Fuzzy Systems*, 7(6):713–722, December 1999.

E. Kim and H. Lee. New approaches to relaxed quadratic stability condition of fuzzy control systems. *IEEE Trans. Fuzzy Systems*, 8(5):523–534, October 2000.

K. Kiriakidis. Robust stabilization of the Takagi-Sugeno fuzzy model via bilinear matrix inequalities. *IEEE Trans. Fuzzy Systems*, 9(2):269–277, April 2001.

J. Ma and G. Feng. An approach to $H_\infty$ control of fuzzy dynamic systems. *Fuzzy Set and Systems*, (136):367–386, February 2003.

Y. Morere, T.M. Guerra, L. Vermeiren, and A. Kamoun. Non quadratic stability and stabilization of discrete fuzzy models. In *IEEE-ACIDCA Conference*, 1999. reprint from authors.

K. Tanaka, T. Ikeda, and H.O. Wang. Fuzzy regulators and fuzzy observers: relaxed stability conditions and LMI-based designs. *IEEE Trans. Fuzzy Systems*, 6(2):250–265, May 1998.

K. Tanaka, T. Hori, and H.W. Wang. A fuzzy Lyapunov approach to fuzzy control system design. In *American Control Conference*, volume 6, pages 4790 –4795, Arlington, VA, 2001a.

K. Tanaka, T. Hori, and H.W. Wang. A dual design problem via multiple Lyapunov functions . In *Proc. 10th IEEE Conf. Fuzzy Syst.*, volume 1, pages 388 –391, Melbourne, Australia, 2001b.

K. Tanaka, T. Hori, and H.O. Wang. A multiple Lyapunov Function Approach to Stabilization of Fuzzy Control Systems. *IEEE Trans. Fuzzy Systems*, 11(4):582–589, August 2003.

H.D. Tuan, P. Apkarian, T. Narikiyo, and Y. Yamamoto. Parameterized linear matrix inequality techniques in fuzzy control system design. *IEEE Trans. Fuzzy Systems*, 9(2):324–332, April 2001.

H.O. Wang, K. Tanaka, and M.F. Griffin. An approach to fuzzy control of nonlinear systems: stability and design issues. *IEEE Trans. Fuzzy Systems*, 4(1):14–23, February 1996.