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H_2 Preview Control: A Geometric Approach in the Discrete-time Domain

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Abstract: H_2 preview control in the discrete-time domain is approached in a strict geometric perspective. The original formulation in the frequency domain is recast in the time domain. Then, it is shown how the problem in the time-domain can be reduced to the combination of elementary subproblems. This approach requires a structural analysis of the properties of the singular Hamiltonian system associated to the H_2 control problem.

1. INTRODUCTION

Preview control encompasses a wide variety of methodologies aimed at solving tracking and/or rejection problems where the signals to be tracked and/or rejected are a-priori-known. Preview control can be framed either in the exact context or in the optimal context. In the former case, the control objective is to guarantee that external signals be perfectly tracked by outputs (or be completely decoupled from outputs). In the latter case, the control target is to achieve the minimal, according to some suitably chosen criterion, tracking error (or the minimal effect of the external signals on the outputs). The equivalence between tracking and rejection problems via an appropriate redefinition of the to-be-controlled variables is well-known.

The problem of achieving a right inverse of a dynamical system has received a great deal of attention from the control community since the late sixties and the problem of devising an internally stable inverse in the presence of unstable zeros of the original system, in particular, has attracted a lot of research effort and generated a large number of interesting works ever since those years. The preview of the signals to be tracked or rejected has represented the means to overwhelm the intrinsic limitation introduced by unstables zeros (see e.g. Qiu and Davison [1993] and the references therein).

The exact problem has been completely solved by means of steering along zeros techniques both in the polynomial context and in the more congenial geometric context. As to geometric solutions, necessary and sufficient conditions for perfect tracking and localization of previewed external signals, along with *ad hoc* computational algorithms, were developed in Marro et al. [2002a], Marro and Zattoni [2006], Marro et al. [2006]. Meanwhile, the problem of achieving optimal tracking and/or rejection was also investigated and preview was shown to be an effective means to obtain better performance even with that milder design requirement (see e.g. Chen et al. [2001], Hoover et al. [2004], Marro and Zattoni [2005], Moelja and Meinsma [2006]).

Most of the papers on this subject available in the literature refer to continuous-time system. Conversely, as to the discrete-time case, only few contributions can be found. Polynomial methods were developed for instance in Grimble [1991]. Algebraic methods, based on the properties on the Moore-Penrose inverse, according to a procedure developed in Marro et al. [2003], were discussed in Marro et al. [2002c]. Nonetheless, a structural approach, representing a valid discrete-time counterpart of that illustrated for continuous-time systems in Marro and Zattoni [2005] is still lacking. The main reason is that the discussion presented in Marro and Zattoni [2005] exploits a geometric analysis of the properties of the Hamiltonian system associated to the H_2 optimal control problem which cannot be trivially transferred to the discrete-time domain. The aim of this work is to develop a complete, geometric approach to H_2 preview control based on the structural properties of the singular Hamiltonian system associated to the H_2 optimal control problem holding on the quite general assumptions ensuring solvability of the relative algebraic Riccati equation.

Notation. The symbols $\mathbb{R}$, $\mathbb{C}$, $\mathbb{C}^{\odot}$, $\mathbb{C}^{\circ}$ are used for the sets of real numbers, complex numbers, complex numbers inside the open unit disc, complex numbers on the unit circle, respectively. Sets, vector spaces, and subspaces are denoted by capital script letters like $\mathcal{X}$. Matrices and linear maps are denoted by capital letters like A. The spectrum, the image, and the kernel of A are denoted by $\sigma(A)$, $\operatorname{im} A$, and $\ker A$, respectively. The symbols $\operatorname{tr}(A)$, A^{-1}, $A^{\dagger}$, and $A^{\top}$ are used for the trace, the inverse, the Moore-Penrose inverse, and the transpose of A, respectively. The symbols I and O are used for an identity matrix and a zero matrix of appropriate dimensions.

2. H_2 OPTIMAL TRACKING WITH PREVIEW IN DISCRETE-TIME SYSTEMS

In this section, the H_2 optimal tracking problem with preview is stated in terms of an equivalent problem of H_2 optimal rejection with preview.

Let us consider the discrete, time-invariant, linear system

$$\begin{aligned} \bar{x}_{t+1} &= \bar{A}\bar{x}_t + \bar{B}u_t, \\ y_t &= \bar{C}\bar{x}_t + \bar{D}u_t, \end{aligned}$$

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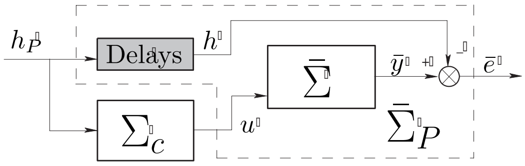

Fig. 1. Block diagram for H_2 optimal tracking with preview

with $\bar{x}_0 = 0$, $t \in [0, \infty)$, where $\bar{x} \in \mathbb{R}^{\bar{n}}$, $u \in \mathbb{R}^p$, and $y \in \mathbb{R}^q$ (with $q \geq p$) respectively denote the state, the control input, and the controlled output.

Let us consider a reference signal $h \in \mathbb{R}^q$ and let us assume that h be known with a preview time $N-1$, where $N \geq \mu$ and μ denotes the system reachability index.

The problem of the synthesis of a feedforward dynamic unit that ensures that the controlled output y tracks, in the H_2 optimal sense, the reference signal h, by taking advantage of the available preview of the latter, is reduced to an H_2 optimal rejection problem through the following, elementary manipulations (also depicted by the block diagram of Fig. 1).

As is well known, the original H_2 optimal tracking problem is reduced to an H_2 optimal rejection problem by introducing the output variable $\bar{e}_t = y_t - h_t$ and considering a new plant described by

$$\begin{aligned} \bar{x}_{t+1} &= \bar{A}\bar{x}_t + \bar{B}u_t, \\ \bar{e}_t &= \bar{C}\bar{x}_t + \bar{D}u_t - h_t. \end{aligned}$$

However, in order to get rid of the feedthrough term from the to-be-rejected input h to the controlled output $\bar{e}$, a new output $e \in \mathbb{R}^q$ is defined by inserting a cascaded unit delay in the signal flow of the original $\bar{e}$. Hence, with $z \in \mathbb{R}^q$, such that $z_{t+1} = \bar{e}_t$ and $e_t = z_t$, the new system equations are

$$\begin{aligned} \bar{x}_{t+1} &= \bar{A}\bar{x}_t + \bar{B}u_t, \\ z_{t+1} &= \bar{C}\bar{x}_t + \bar{D}u_t - h_t, \\ e_t &= z_t. \end{aligned}$$

Finally, let $x = \begin{bmatrix} \bar{x}^\top & z^\top \end{bmatrix}^\top$. Then, the equations above can also be written in standard, compact form as follows

$$x_{t+1} = Ax_t + Bu_t + Hh_t, \tag{1}$$

$$e_t = Cx_t, \tag{2}$$

where

$$A = \begin{bmatrix} \bar{A} & O \\ \bar{C} & O \end{bmatrix}, \quad B = \begin{bmatrix} \bar{B} \\ \bar{D} \end{bmatrix}, \quad H = \begin{bmatrix} O \\ -I \end{bmatrix}, \quad C = [\, O \;\; I \,].$$

In the next section, the H_2 optimal rejection problem will formally be stated for system (1), (2), with the addition of the feedthrough term from the control input u to the controlled output e, for the sake of generality.

3. H_2 OPTIMAL REJECTION WITH PREVIEW IN DISCRETE-TIME SYSTEMS: PROBLEM FORMULATION

The problem of H_2 optimal rejection of signals known with preview has been considered in the recent literature and solved with different techniques.

In Marro et al. [2002c], the problem is tackled in the time domain and, by virtue of superposition, is reduced to that of rejecting a unit pulse signal known a certain amount of time ahead of its impact on the system. The problem is then decomposed into three subproblems, a finite-horizon optimal control problem defined in the time interval corresponding to the preaction time, an infinite-horizon problem defined in the time interval corresponding to the postaction time and the problem of optimally connecting the previously mentioned subproblems by taking into account the effect of the unit pulse. The solution of the finite-horizon problem is achieved by pseudoinversion of suitably constructed matrices, whose dimensions depends on the length of the preview time interval. The dimensionality constraint implicit in pseudoinversion is avoided by resorting to a multilevel algorithm detailed in Marro et al. [2003]. The procedure is devised on the assumption of controllability of the original system.

In Zattoni [2008], the H_2 optimal rejection problem with preview is solved in the frequency domain by means of spectral factorization via Riccati equation. Namely, the transfer function matrix of the feedforward dynamic unit is derived by evaluating a suitable spectral factor connected to the transfer function matrix from the control input to the output of the to-be-controlled system. The preview of the to-be-rejected signal is taken into account by inserting a cascade of delays in the external input flow. The insertion of the delays do not affect the computational burden intrinsic in the evaluation of the spectral factor. However, it affects the factorization aimed at selecting the causal and stable part of a polynomial matrix including the spectral factor so as to define the feedforward compensator.

In this section, the H_2 optimal rejection problem with preview is tackled in the time domain and divided into three subproblems as in Marro et al. [2002c]. However, the finite-horizon optimal control problem defined in the preaction time interval is given an analytic solution, based on the study of the structural and geometric properties of the associated singular Hamiltonian system reviewed in the next Section 4. The methodology presented herein applies on the standard assumptions that guarantee the existence and uniqueness of the stabilizing solution of the discrete algebraic Riccati equation associated to the problem and, consequently, the existence and uniqueness of the solution of the associated discrete Lyapunov equation. Moreover, since it consists of a feedforward solution, it is assumed that the to-be-controlled system be stable or prestabilized, which, however, does not cause actual loss of generality with respect to the assumptions normally introduced when dealing with control problems (see e.g. Zattoni [2007] for further details on prestabilization and feedforward dynamic schemes). As to numerical reliability of possible implementations in the finite arithmetic environment of a digital computer, it is worth stressing that the sole critical aspects are those connected with the solution of the discrete algebraic Riccati equation and, although with a minor degree of criticity, those connected with the solution of the discrete Lyapunov equation. On this regard, it should be acknowledged that several studies have been carried out, also in very recent years, and different options are available in the most common Matlab© toolbox as

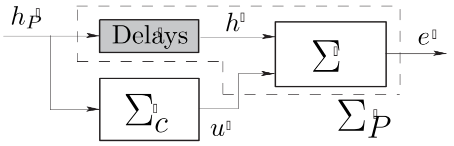

Fig. 2. Block diagram for H_2 optimal rejection with preview

well as in *ad hoc* packages like, e.g., the SLICOT© package developed by Benner and Van Dooren [2003].

Hence, the H_2 optimal rejection problem with preview for discrete-time systems is formally stated in the remainder of this section. Its solution will be outlined in Section 5, following the review on the structural properties of the singular Hamiltonian system and its exploitation in the solution of finite-horizon optimal control problems with fixed final state presented in Section 4. The treatment that follows, with the remarkable differences and peculiarities pointed out in the next sections, can be viewed as the discrete-time counterpart of that discussed in more detail for continuous-time, stabilizable systems in Marro and Zattoni [2005].

Let us consider the discrete, time-invariant, linear system

$$\begin{aligned} x_{t+1} &= A x_t + B u_t + H h_t, \\ e_t &= C x_t + D u_t, \end{aligned}$$

with $x_0 = 0$, $t \in [0, \infty)$, where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$, $h \in \mathbb{R}^s$, and $e \in \mathbb{R}^q$ (with $q \geq p$) respectively denote the state, the control input, the to-be-rejected input, and the controlled output. Let us assume:

$\mathcal{A}1$. (A, B) stabilizable;
$\mathcal{A}2$. (A, B, C, D) left invertible;
$\mathcal{A}3$. $\mathcal{Z}(A, B, C, D) \cap \mathbb{C}^{\circ} = \emptyset$, where $\mathcal{Z}(A, B, C, D)$ denotes the set of the invariant zeros of (A, B, C, D).

Moreover, let us assume that $h \in \mathbb{R}^s$ be known with a preview time $N-1$, where $N \geq \mu$ and μ denotes the system reachability index.

The problem of minimizing the effect of the input signal h reduces to a causal problem if a cascade of $N-1$ unit delays is inserted in the input h signal flow and included in a new plant Σ_P as is shown in Fig. 2. This implies that the input signal h_P of Σ_P be such that $h_{P,t} = h_{t+N-1}$. Let g denote the unit pulse response matrix of the compensated system, from the to-be-rejected input h_P to the output e.

Then, the *H_2-optimal rejection problem with preview* (henceforth, abbreviated as H_2-ORPP) is the problem of finding a feedforward linear dynamic compensator Σ_C such that

$$\|g\|_{\ell_2}^2 = \mathrm{tr}\left[\sum_{t=0}^{\infty} g_t g_t^\top\right]$$

be bounded and minimal.

Let g_j, with $j = 1, \dots, s$, denote the response of the compensated system, with zero initial state, to the input $h_{Pj} = \underline{e}_j \delta$, where $\underline{e}_j$ and δ respectively are the j-th vector of the main basis of $\mathbb{R}^s$ and the unit pulse signal. Then, the solution of the H_2-ORPP is derived from the minimization of $\|g_j\|_{\ell_2}^2$ for any $j = 1, \dots, s$, on the basis of the observation that the H_2-ORPP can be reduced to a compound optimal control problem (according to what was first stated in Marro et al. [2002c]).

Let the to-be-rejected input $h_{Pj} = \underline{e}_j \delta$ be applied to system Σ_P, with zero initial state. The problem of finding the control law u_j minimizing $\|e\|_{\ell_2}^2 = \|g_j\|_{\ell_2}^2$ is a compound optimal control problem which refers to the quadruple (A, B, C, D) and consists of

(i) the finite-horizon LQ control problem defined in $[0, N)$, with zero initial state, parameterized final state x_{Lj}, and cost functional
$$C_L(x_{Lj}) = \sum_{t=0}^{N-1} e_t^\top e_t;$$
(ii) the infinite-horizon LQ control problem defined in $[N, \infty)$, with parameterized initial state
$$x_{Rj} = x_{Lj} + H_j,$$
where H_j is the j-th column of the to-be-rejected input matrix H, and cost functional
$$C_R(x_{Lj}) = \sum_{t=N}^{\infty} e_t^\top e_t;$$
(iii) the problem of finding the intermediate state x_{Lj} minimizing the global cost functional
$$C(x_{Lj}) = C_L(x_{Lj}) + C_R(x_{Lj}).$$

4. A REVIEW OF NON-RECURSIVE SOLUTIONS TO FINITE-HORIZON OPTIMAL CONTROL PROBLEMS WITH ASSIGNED TERMINAL STATE

In this section, the main ideas and results at the basis of a structural, non-recursive solution to the discrete-time, finite-horizon, optimal control problem with assigned final state are briefly reviewed, since they are functional to the solution of the H_2 preview control problem developed in the next Section 5.

The study of the geometric properties of the Hamiltonian system associated to the optimal control problem is crucial in the derivation of the abovementioned results. In fact, in the specific case of discrete-time, stabilizable systems, it leads to the characterization of a pair of structural invariant subspaces of the singular Hamiltonian system, and then to the analytic solution of the finite-horizon problem through the expression of all the admissible trajectories of the singular Hamiltonian system and the consequent selection of the particular trajectory by setting the boundary conditions.

The analysis of the geometric and structural properties of Hamiltonian systems as a means for finding the solutions of infinite-horizon optimal control problems is well-settled in the literature. In fact, the early studies in this context are due to Van Dooren [1981] and Arnold III and Laub [1984]. More recent is the generalization of the Riccati theory via Popov function approach developed by Ionescu et al. [1999]. Also worth mentioning in this context is the straightforward, strictly geometric approach to the solution of cheap and singular, discrete-time, infinite-horizon, optimal control problems presented by Marro et al. [2002b].

As to the investigation of the structural properties of Hamiltonian systems finalized to the solution of finite-horizon, optimal control problems, the issue was completely disentangled in Marro and Zattoni [2005] as far as continuous-time, stabilizable systems are concerned. In fact, in Appendix of the abovementioned paper, it was shown how to derive the invariant subspaces of the linear transformation defined by the Hamiltonian matrix, by performing a similarity transformation (aimed at isolating the controllable part of the system) and computing the maximal and minimal symmetric solutions of the algebraic Riccati equation restricted to the sole controllable subsystem. Incidentally, this procedure, compared with the conceptually equivalent one which exploits the maximal solution of the full order algebraic Riccati equation and the solution of a Lyapunov equation — suitably associated according to the results first published in Molinari [1977] — has the crucial advantage, from the computational point of view, of involving a Riccati equation of reduced dimensions. As a matter of fact, the solution of the Riccati equation is the sole numerically critical point of the entire procedure.

As to the possibility of transferring to the discrete-time case the achievements — just mentioned — regarding the continuous-time, it is convenient to point out a remarkable difference: i.e., in the discrete-time case, the Hamiltonian system is intrinsically singular (or descriptor or generalized). In fact, it can be reduced to a regular system only at the cost of introducing some assumptions which, in the discrete-time case, turn out to be severely restrictive. These are invertibility of the control weighting matrix (while singular weighting matrices are not uncommon in discrete-time, linear quadratic problems) and, at least in the simplest case where the cross weighting matrix is zero, invertibility of the dynamic matrix of the original system (which cannot be guaranteed by pole placement under the sole hypothesis of stabilizability).

In the light of the above considerations, a structural, non-recursive solution to finite-horizon, optimal control problems addressing discrete-time, stabilizable systems was first devised in Marro and Zattoni [2007b], with focus on the case where the final state is weighted by a generic quadratic function. Lately, the technique was modified to handle the case where the final state is fixed (see Marro and Zattoni [2007a]). In the latter work, in particular, the technique was encompassed in a multi-level procedure to deal with output regulation problems stated for sets of linear systems subject to a-priori-known switches. In order to guarantee that the present paper be self-contained and legible, a summary of that technique is reported below.

Let us consider the discrete, time-invariant, linear system

$$x_{t+1} = A x_t + B u_t, \tag{3}$$

$$e_t = C x_t + D u_t, \tag{4}$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^p$, output $e \in \mathbb{R}^q$, and assume that $\mathcal{A}1$, $\mathcal{A}2$, and $\mathcal{A}3$ hold true. Let the initial state x_o and the final state x_f be assigned and compatible: i.e., let x_f be reachable from x_o within the considered time interval.

The *discrete-time, finite-horizon, linear quadratic optimal control problem with fixed final state* (henceforth abbreviated as FFS-FHLQP) is the problem of finding a control sequence u_t, with $t \in [0, T)$, driving the state from

$$x_0 = x_o \quad \text{to} \quad x_T = x_f, \tag{5}$$

while minimizing the cost functional

$$C = \sum_{t=0}^{T-1} e_t^\top e_t = \sum_{t=0}^{T-1} \left[x_t^\top Q x_t + 2 x_t^\top S u_t + u_t^\top R u_t \right], \tag{6}$$

with $C^\top C = Q$, $C^\top D = S$, $D^\top D = R$.

As is well-known, the Lagrange multiplier approach leads to a two-point boundary value problem defined by the state equations, the costate equations, the stationarity condition and the boundary conditions. In particular, the difference equations of the two-point boundary-value problem can be written as the state-space generalized system

$$\begin{bmatrix} I & O & O \\ O & -A^\top & O \\ O & -B^\top & O \end{bmatrix} \begin{bmatrix} x_{t+1} \\ p_{t+1} \\ u_{t+1} \end{bmatrix} = \begin{bmatrix} A & O & B \\ Q & -I & S \\ S^\top & O & R \end{bmatrix} \begin{bmatrix} x_t \\ p_t \\ u_t \end{bmatrix}, \tag{7}$$

also called the *singular Hamiltonian system*. The matrix on the left-hand side of (7) will be denoted by M, that on the right-hand side will be denoted by N. The matrix pencil $\lambda M - N$ is assumed to have non-vanishing determinant, i.e. $\det(\lambda M - N) \not\equiv 0$.

The stabilizing solution X of the discrete algebraic Riccati equation

$$\begin{aligned} X &= -(A^\top X B + S)(R + B^\top X B)^{-1}(B^\top X A + S^\top) \\ &\quad + A^\top X A + Q, \\ 0 &< R + B^\top X B, \end{aligned}$$

exists and is unique due to assumptions $\mathcal{A}1$–$\mathcal{A}3$. X is also positive semidefinite and is the largest real symmetric solution of the discrete algebraic Riccati equation. Let

$$\begin{aligned} K &= (R + B^\top X B)^{-1}(B^\top X A + S^\top), \\ A_F &= A - BK. \end{aligned}$$

The solution Y of the discrete Lyapunov equation

$$A_F Y A_F^\top - Y + B(R + B^\top X B)^{-1} B^\top = 0,$$

exists and is unique due to condition $\sigma(A_F) \subset \mathbb{C}^\odot$. Let

$$\bar{K} = (R + B^\top X B)^{-1}(B^\top - B^\top X A Y A_F^\top - S^\top Y A_F^\top).$$

Now, all the elements required to characterize the respective deflating subspaces of the matrix pencils $\lambda M - N$ and $\lambda N - M$, associated to the singular Hamiltonian system, have been introduced. Therefore, we can state that the subspace

$$\mathcal{V}_1 = \operatorname{im} V_1 = \operatorname{im} \begin{bmatrix} I \\ X \\ -K \end{bmatrix},$$

is a deflating subspace of the matrix pencil $\lambda M - N$ and that the spectrum of the pencil restricted to the subspace $\mathcal{V}_1$, denoted by $(\lambda M - N)|_{\mathcal{V}_1}$, is equivalent to $\lambda I - A_F$. Moreover, we can assert that the subspace

$$\mathcal{V}_2 = \operatorname{im} V_2 = \operatorname{im} \begin{bmatrix} Y A_F^\top \\ (XY - I) A_F^\top \\ \bar{K} \end{bmatrix},$$

is a deflating subspace of the matrix pencil $\lambda N - M$ and that the spectrum of the pencil restricted to the subspace $\mathcal{V}_2$, denoted by $(\lambda N - M)|_{\mathcal{V}_2}$, is equivalent to $\lambda I - A_F^\top$.

The introduction of the structural invariant subspaces $\mathcal{V}_1$ and $\mathcal{V}_2$ allows us to characterize the general form of the admissible trajectory for the singular Hamiltonian system by stating that a trajectory $\xi_t = \left[x_t^\top \; p_t^\top \; u_t^\top\right]^\top$, with $t \in [0,\, T)$, is admissible for the singular Hamiltonian system (7) if and only if it is of the form

$$\xi_t = V_1 A_F^t \alpha + V_2 (A_F^\top)^{T-t-1} \beta, \quad \text{with } t \in [0,\, T),$$

where and $\alpha, \beta \in \mathbb{R}^n$ are parameters.

The state and costate trajectories, in particular, can be written as

$$\begin{bmatrix} x_t \\ p_t \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix} A_F^t \alpha + \begin{bmatrix} Y \\ XY - I \end{bmatrix} (A_F^\top)^{T-t} \beta,$$

with $t \in [0,\, T]$. Therefore, the trajectories of the singular Hamiltonian system solving the original, two-point boundary-value problem are selected by imposing the boundary conditions. Let $[x_o^\top \; x_f^\top]^\top \in \operatorname{im}\Phi$, where

$$\Phi = \begin{bmatrix} I & Y(A_F^\top)^T \\ A_F^T & Y \end{bmatrix}.$$

A trajectory $\xi_t = \left[x_t^\top \; p_t^\top \; u_t^\top\right]^\top$, with $t \in [0,\, T)$, of the singular Hamiltonian system (7), satisfying the boundary conditions (5) is determined by

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \Phi^\dagger \begin{bmatrix} x_o \\ x_f \end{bmatrix}. \tag{8}$$

Since x_o and x_f are compatible, $\left[x_o^\top \; x_f^\top\right]^\top \in \operatorname{im}\Phi$ and the two-point boundary-value problem is solvable. Hence, let $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}^{n\times n}$ be such that

$$\begin{bmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{bmatrix} = \Phi^\dagger,$$

where $\Phi^\dagger$ is assumed to be partitioned according to (8). Then, $\alpha, \beta \in \mathbb{R}^n$ can be expressed as

$$\alpha = \alpha_1 x_o + \alpha_2 x_f, \qquad \beta = \beta_1 x_o + \beta_2 x_f.$$

Consequently, the state trajectories, the control input sequences, and the optimal value of the cost functional solving the finite-horizon optimal control problem can be expressed as functions of the initial state x_o and the final state x_f.

An optimal state trajectory x_t, with $t \in [0,\, T]$, an optimal control law u_t, with $t \in [0,\, T)$, and the optimal cost for the finite-horizon optimal control problem defined by (3)–(4) with boundary conditions (5) and cost functional (6) respectively are

$$x_t = X_{o_t} x_o + X_{f_t} x_f, \quad t \in [0,\, T), \tag{9}$$

$$u_t = U_{o_t} x_o + U_{f_t} x_f, \quad t \in [0,\, T], \tag{10}$$

$$C^o = \left[\, x_o^\top \;\; x_f^\top \,\right] \begin{bmatrix} C_{11} & C_{12} \\ C_{12}^\top & C_{22} \end{bmatrix} \begin{bmatrix} x_o \\ x_f \end{bmatrix}, \tag{11}$$

where

$$X_{o_t} = A_F^t \alpha_1 + Y (A_F^\top)^{T-t} \beta_1, \quad t \in [0,\, T), \tag{12}$$

$$X_{f_t} = A_F^t \alpha_2 + Y (A_F^\top)^{T-t} \beta_2, \quad t \in [0,\, T), \tag{13}$$

$$U_{o_t} = -K A_F^t \alpha_1 + \bar{K} (A_F^\top)^{T-t-1} \beta_1, \quad t \in [0,\, T], \tag{14}$$

$$U_{f_t} = -K A_F^t \alpha_2 + \bar{K} (A_F^\top)^{T-t-1} \beta_2, \quad t \in [0,\, T], \tag{15}$$

$$C_{11} = X\alpha_1 + (XY - I)(A_F^\top)^T \beta_1, \tag{16}$$

$$C_{12} = \frac{1}{2}((X\alpha_2 + (XY - I)(A_F^\top)^T \beta_2) \tag{17}$$

$$-(XA_F^T \alpha_1 + (XY - I)\beta_1)^\top), \tag{18}$$

$$C_{22} = -(XA_F^T \alpha_2 + (XY - I)\beta_2). \tag{19}$$

5. H_2 OPTIMAL REJECTION WITH PREVIEW IN DISCRETE-TIME SYSTEMS: PROBLEM SOLUTION

In this section, the respective solutions to subproblems (i), (ii), (iii) of Section 3 are considered in order. Then, the feedforward control scheme is synthetically described.

The solution of subproblem (i) is obtained in the light of the results summarized in Section 4. In particular, the optimal control sequence is

$$u_{j,t} = U_{f,t}\, x_{Lj}, \quad t \in [0,\, N),$$

and the optimal value of the cost functional is

$$C_L(x_{Lj}) = x_{Lj}^\top\, C_{22}\, x_{Lj},$$

where $U_{f,t}$ is given by (15) with $T = N$ and C_{22} is given by (19).

As to the solution of subproblem (ii), basic results of linear quadratic optimal control theory give the control sequence

$$u_{j,t} = -\,K\, x_{j,t}, \quad t \in [N,\, \infty),$$

and the optimal value of the cost functional is $C_R(x_{Rj}) = x_{Rj}^\top X x_{Rj}$. Hence,

$$C_R(x_{Lj}) = x_{Lj}^\top\, X\, x_{Lj} + 2\, H_j^\top\, X x_{Lj} + H_j^\top\, X H_j$$

follows from $x_{Rj} = x_{Lj} + H_j$.

As to the solution of subproblem (iii), the cost functional $J(x_{Lj})$ is minimal with

$$x_{Lj} = R\,\eta \text{ and } \quad \eta = -\,(R^\top (C_{22} + X) R)^\dagger R^\top X H_j,$$

where R denotes a basis matrix of the reachable subspace of (A, B). This result can easily be derived by imposing

$$\nabla J(\eta) = 2\eta^\top R^\top (C_{22} + X) R + 2\, H_j^\top\, X R = 0.$$

Then, let us focus on the synthesis of the feedforward control scheme. With a slight abuse of notation, let the matrix input $H_P = I\,\delta$ be applied to the extended plant Σ_P, assumed in the zero initial state. Then, the expressions of optimal control sequences and intermediate states hold in a modified form where states and controls respectively are $n \times s$ and $p \times s$ matrices, provided that x_{Lj}, $x_{j,t}$, and H_j are respectively replaced by $X_a = [x_{Lj}]_{j=1,\dots,s}$, $X_t = [x_{j,t}]_{j=1,\dots,s}$, and H.

Hence, the structure of the feedforward compensator Σ_c which ensues from the generalization of the above procedure is shown in Fig. 3. The control input is $u_t = v_t + w_t$,

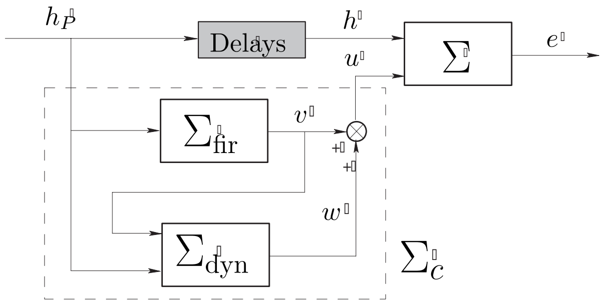

Fig. 3. Block diagram for H_2 optimal rejection with preview: structure of the feedforward compensator.

with $t \in [0, \infty)$, where v_t is the output of an FIR system Σ_{fir} whose unit pulse response matrix is

$$V_t = \begin{cases} U_{f,t}\, X_a, & \text{if } t \in [0, N), \\ O, & \text{otherwise}, \end{cases} \tag{20}$$

and w_t is the output of a standard dynamic unit Σ_{dyn} having the structure of the LQR regulator: i.e., ruled by

$$\tilde{x}_{t+1} = A_F \tilde{x}_t + B v_t + H h_{P,t-N+1},$$
$$w_t = -K\,\tilde{x}_t,$$

with $\tilde{x}_0 = 0$, $t \in [0, \infty)$.

Again, refer to the layout shown in Fig. 3. The FIR system performs its action on a system which is subject to the forcing input $w_t = -K\,\tilde{x}_t$ from the time $t = 0$ (not $t = N$ as was considered in Section 3). Nevertheless, the FIR system unit pulse response has the expression (20), due to the fact that the discrete algebraic Riccati equation associated to the quadruple $(A_F, B, C - DK, D)$ matches the discrete algebraic Riccati equation associated to the original quadruple (A, B, C, D). Hence, the superimposed feedback is zero.

6. CONCLUSION

H_2 optimal tracking and H_2 optimal rejection of previewed signals have been considered and solved in a unified framework where H_2 optimal tracking is reduced to H_2 optimal rejection. The solution has been derived by exploiting basic results of linear quadratic optimal control theory and a geometric/structural approach to the finite-horizon linear quadratic optimal control problem with assigned final state. The synthesis of the feedforward compensator, including finite impulse response systems has also benn illustrated.

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