Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

A full-block $\mathcal{S}$-procedure application to delay-dependent $\mathcal{H}_\infty$ state-feedback control of uncertain time-delay systems

C. Briat * O. Sename * J.F. Lafay **

* *GIPSA-Lab, Departement of Control Systems (former LAG), Grenoble Universités, ENSIEG - BP46, 38402 Saint Martin d'Hères - Cedex FRANCE,* { Corentin.Briat,Olivier.Sename}@gipsa-lap.inpg.fr
** *IRCCyN - Centrale de Nantes, 1 rue de la Noë - BP 92101, 44321 Nantes Cedex 3 - FRANCE,* Jean-Francois.Lafay@irccyn.ec-nantes.fr

Abstract: This paper deals about the robust stabilization of uncertain systems with time-varying state delays in the delay dependent framework. The system is represented using LFR and stability is deduced from Lyapunov-Krasovskii theorem and full-block $\mathcal{S}$-procedure. We derive sufficient conditions to the existence of a robust $\mathcal{H}_\infty$ state-feedback control law. As this sufficient condition is expressed in terms of NMI we propose a relaxation based on the cone-complementary algorithm which is known to lead to good results for such problems. We show the efficiency of our method trough an example.

Keywords: Robust controller synthesis; Robust time-delay systems; Convex optimization

1. INTRODUCTION

Since several years, constant state-delayed systems have been heavily studied since they are often responsible of instability and poor performances (see [Gouaisbaut and Peaucelle, 2006b], [Fridman, 2006b],[Niculescu, 2001] and references therein). More recently, time-varying delays, appearing for instance in communication networks, have suggested more and more interest (See [Fridman, 2006a],[He et al., 2004],[Suplin et al., 2006], [Wu, 2003],[Gu et al., 2003] and references therein).

Two kinds of stability results exist in time-delay systems (TDS): delay independent and delay dependent. The first one guarantees stability for every delay from 0 to ∞. This result is actually conservative due to the consideration of delays near $+\infty$. However, delay-independent stabilization may be useful when delayed terms matrices are small in front of non-delayed terms matrices (i.e. the effect of the delay is small). The second one guarantees the system stability over a compact set of delay (e.g $[0, h_M]$) and leads then to less conservative results. Actually, this type of result better fits the reality because the delays are always bounded from a practical point of view.

In the context of uncertain systems, an useful tool is the $\mathcal{H}_\infty$ synthesis which provides powerful robust analysis and control design tools. Nevertheless, due to the small gain condition, some conservatism is always induced. That is why scalings are used in order to obtain better results while reducing the conservatism. The scalings generalize the notion of small-gain condition while considering how the system and the uncertainties are connected (and not only their apparent norm as it is used in the classical small-gain theorem). This leads to the notion of well-posedness of feedback systems [Iwasaki and Hara, 1998] which unifies in a nice unique framework stability and robustness analysis. The scalings in well-posedness analysis are often called separators (full-block multipliers) since they separate the graph of the system and the inverse graph of the uncertainty and provide then a necessary and sufficient condition for well-posedness (and hence stability) of feedback systems. The great interest of full block multipliers come from the fact that there is no inertia constraint on a whole space but only on particular subspaces (which is not the case for scaled-small gain theorem where the scalings must be positive definite) [Wu, 2000], [Scherer, 2001], [Wu, 2001].

This papers brings a new method to design state-feedback for TDS using full-block multipliers, and includes the following contributions:

- The type of uncertainties here considered is quite large as the formulation allows to include polytopic uncertainties, matrix bounded uncertainties ...
- First we provide sufficient conditions to delay dependent asymptotic stability of uncertain time-delay system. We extend the result of [Gouaisbaut and Peaucelle, 2006a] to the case of uncertain time-delay stability written as an interconnection of the system and the uncertainty using the linear fractional transformation. The stability with $\mathcal{H}_\infty$ performance is given using the so-called full-block $\mathcal{S}$-procedure extending to the delay-dependent case the results in [Wu, 2003].
- Second we derive from this stability lemma a stabilizability lemma (or robust state-feedback existence lemma) in terms of Linear and Nonlinear Matrix Inequalities (LMI and NMI).
- As the stabilizability lemma is not tractable we propose a relaxation based on the cone-complementary

algorithm [Ghaoui et al., 1997] which is known to provide good convergence properties in practice (despite of its local convergence).
- Finally, we show the efficiency of our approach and compare it to other methods through several examples.

We consider in this paper systems of the form

$$\begin{aligned} \dot{x}(t) &= \mathscr{A}(\Delta)x(t) + \mathscr{A}_h(\Delta)x_h(t) + \mathscr{B}_u(\Delta)u(t) \\ &\quad + \mathscr{B}_1(\Delta)w_1(t) \\ z_1(t) &= \mathscr{C}_1(\Delta)x(t) + \mathscr{C}_{1h}(\Delta)x_h(t) + \mathscr{D}_{1u}(\Delta)u(t) \\ &\quad + \mathscr{D}_{11}(\Delta)w_1(t) \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$, $x_h = x(t - h(t))$, $h(t) \in \mathscr{H}$, $u \in \mathbb{R}^{n_u}$, $w_1 \in \mathbb{R}^{n_w}$, $z_1 \in \mathbb{R}^{n_z}$ are respectively the system state, the delayed state, the delay, the control input, the exogenous inputs and the controlled outputs. $\Delta \in \mathbf{\Delta}$ represents bounded multiplicative uncertainties. The sets $\mathscr{H}$ and $\mathbf{\Delta}$ are detailed further.

The paper is structured as follows, Section 2 presents useful lemmas and system description. Section 3 presents two theorems on robust stability/performance for uncertain system with time varying delays. Section 4 proposes a design method of state-feedback controller through LMIs. The notation is quite standard but let us define $Im(A_\perp)$ as the orthogonal complement of $Im(A)$ (defined as $A^T A_\perp = 0$). A is positive definite (negative definite) on a subspace $\mathscr{S}$ means that $x^T x > 0$ for all $x \in \mathscr{S}$ ($x^T A x < 0$ for all $x \in \mathscr{S}$).

2. RECALL AND DEFINITIONS

This section briefly recall the necessary background.

2.1 Useful lemmas

Lemma 2.1. Full block S-procedure Suppose $\mathscr{S}$ is a subspace of $\mathbb{R}^n$, $T \in \mathbb{R}^{l \times n}$ is a full row rank matrix, N is a

compact set of matrices of full row rank. Define the family of subspaces for each $U \in \mathbf{U}$

$$\mathscr{S}_U = \mathscr{S} \cap Ker(UT) = \{x \in \mathscr{S} : UTx = 0\}$$

Then the following conditions are equivalent:

1. For any $U \in \mathbf{U}$,
$$N < 0 \text{ on } \mathscr{S}_U \text{ and } \mathscr{S}_U \cap \mathscr{S}_0 = \{0\}$$
where $\mathscr{S}_0$ is a fixed subspace of $\mathscr{S}$ such that $dim(\mathscr{S}_0) \geq k$ and $N \geq 0$ on $\mathscr{S}_0$
2. There exists a symmetric matrix R such that for all $U \in \mathbf{U}$
$$N + T^T R T < 0 \text{ on } \mathscr{S} \text{ and } R > 0 \text{ on } Ker(U)$$

In our case $\mathscr{S}$ represents the nominal system, T specifies the interconnection between the nominal system $\mathscr{S}$ and the uncertainty set $\mathbf{U}$. Therefore $\mathscr{S}_U$ denotes the uncertain system. The lemma renders the implicit conditions based on the uncertain system data to an explicit expression through the full block multiplier R (See [Scherer, 2001] for more details).

2.2 System description

Without loss of generality let us consider system (1) rewritten using the linear fractional transformation procedure as in figure 1:

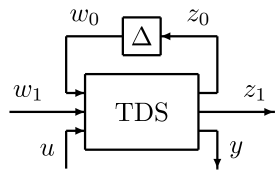

Fig. 1. Uncertain linear time delay system

$$\begin{aligned} \begin{bmatrix} \dot{x}(t) \\ z_0(t) \\ z_1(t) \end{bmatrix} &= \begin{bmatrix} A & B_0 & B_1 \\ C_0 & D_{00} & D_{01} \\ C_1 & D_{10} & D_{11} \end{bmatrix} \begin{bmatrix} x(t) \\ w_0(t) \\ w_1(t) \end{bmatrix} + \begin{bmatrix} A_h \\ C_{0h} \\ C_{1h} \end{bmatrix} x_h(t) \\ &\quad + \begin{bmatrix} B_u \\ D_{0u} \\ D_{1u} \end{bmatrix} u(t) \qquad w_0(t) = \Delta z_0(t) \end{aligned} \tag{2}$$

The delay is assumed to belong to the set

$$\mathscr{H} := \left\{ h \in \mathscr{C}^1(\mathbb{R}_+, [0, h_M]) : h_M < +\infty,\ |\dot{h}| \leq \mu < 1 \right\}$$

The time-varying uncertainties Δ belong to the following uncertainty set

$$\mathbf{\Delta} := \left\{ \bigoplus_i^s \delta_i I_{d_i} : |\delta_i| \leq k_i < +\infty,\ d_i \in \mathbb{N}_* \right\}$$

This representation not only captures the size of the uncertainties but also their structure. This block representation has been widely used in robust control. It is also possible to extend it to full uncertainty blocks.
Assuming that the uncertain system is well posed (ie. $(I - \Delta D_{00}$ nonsingular for all $\Delta \in \mathbf{\Delta}$) then it is possible to represent the uncertain system into an LFT form:

$$\begin{aligned} \begin{bmatrix} \mathscr{A}(\Delta) & \mathscr{A}_h(\Delta) & \mathscr{B}_1(\Delta) \\ \mathscr{C}_0(\Delta) & \mathscr{C}_{0h}(\Delta) & \mathscr{D}_{01}(\Delta) \\ \mathscr{C}_1(\Delta) & \mathscr{C}_{1h}(\Delta) & \mathscr{D}_{11}(\Delta) \end{bmatrix} &= \begin{bmatrix} A & A_h & B_1 \\ C_0 & C_{0h} & D_{01} \\ C_1 & C_{1h} & D_{11} \end{bmatrix} \\ &\quad + \begin{bmatrix} B_0 \\ D_{00} \\ D_{10} \end{bmatrix} (I - \Delta D_{00})^{-1} \Delta \left[C_0 \ C_{0h} \ D_{01} \right] \end{aligned}$$

The full-block S-procedure lemma will translate the stability and performance tests for uncertain systems into their equivalent formulation using a full-block multiplier. Let us introduce the full block multiplier set $\mathscr{F}$ associated with the uncertainty set $\mathbf{\Delta} \in \ltimes_{\nvdash} \times \ltimes_{\nvdash}$.

$$\mathscr{F} := \left\{ F \in \mathbb{S}^{2n_0} : \left[\Delta^T \ I_{n_0} \right] F \begin{bmatrix} \Delta \\ I_{n_0} \end{bmatrix} > 0,\ \forall \Delta \in \mathbf{\Delta} \right\}$$

Since Δ is infinite dimensional the previous constraint leads to an infinite number of constraints, which is not implementable. However, this can be relaxed under certain conditions on the uncertainty and the multiplier structure. For more details on these relaxations, the readers should refer to [Scherer, 2001, Wu, 2003, Scherer and Hol, 2006].

3. DELAY DEPENDENT STABILITY CRITERIUM

This section propose a robust delay-dependent lemma for uncertain time-delay system with rational dependence onto the parameters. Generally, only few papers consider (see for instance Wu [2003]) such a type of dependence while generally treat the case of polytopic uncertainties Gouaisbaut and Peaucelle [2006b], Suplin et al. [2006] or norm bounded uncertain matrices Xu et al. [2006]).

The robust delay-dependent stability lemma for uncertain time-delay systems with time-varying delays is presented below:

Theorem 3.1. The system (2) is asymptotically stable with a $\mathcal{L}_2$ induced norm on channel $w_1 \to z_1$ lower than $\gamma > 0$ for all $\Delta \in \boldsymbol{\Delta}$ if there exist symmetric matrices $P, Q, R > 0$ and a scaling matrix $F \in \mathscr{F}$ such that

$$\Pi + \Theta^T F \Theta < 0 \tag{4}$$

where $\Theta = \begin{bmatrix} 0 & 0 & I & 0 & 0 & 0 \\ C_0 & C_{0h} & D_{00} & D_{01} & 0 & 0 \end{bmatrix}$ and Π is defined by (3).

Proof: A sketch of a proof is developed in appendix A. □

Remark 3.1. It is worth noting that the proof embeds a rigorous application of the full-block $\mathcal{S}$-procedure lemma (as done in [Wu, 2003]). This explains the length and the weight of the proof.

Note also that in the present stability lemma, we do not introduce any slack variable as usually done in the literature. The only matrix introduced is the topological separator [Iwasaki and Hara, 1998] added by the full-block $\mathcal{S}$-procedure which is theoretically lossless. This matrix is radically different than 'slack' variables used in many delay-dependent results (see [Park, 1999, Xu and Lam, 2007]). It plays a fundamental role in the stability of interconnections ([Iwasaki and Hara, 1998, Scherer, 2001]).

4. ROBUST $\mathcal{H}_\infty$ STATE FEEDBACK DESIGN

In that section we propose a result to design a state feedback with $\mathcal{H}_\infty$ performance achievement for uncertain state-delayed systems with time-varying delays.

We consider now the closed-loop system obtained from the interconnection of system (2) and the control law $u(t) = Kx(t)$:

$$\begin{aligned} \begin{bmatrix} \dot{x}(t) \\ z_0(t) \\ z_1(t) \end{bmatrix} &= \begin{bmatrix} A_{cl} & B_0 & B_1 \\ C_{0cl} & D_{00} & D_{01} \\ C_{1cl} & 0 & D_{11} \end{bmatrix} \begin{bmatrix} x(t) \\ w_0(t) \\ w_1(t) \end{bmatrix} + \begin{bmatrix} A_h \\ C_{0h} \\ 0 \end{bmatrix} x_h(t) \\ w_0(t) &= \Delta z_0(t) \end{aligned} \tag{5}$$

where $A_{cl} = A + B_u K$, $C_{0cl} = C_0 + D_{0u}K$ and $C_{1cl} = C_1 + D_{1u}K$. Note that we consider here that the controlled output is certain (i.e. $D_{10} = 0$) and does not contain any delayed-state ($C_{1h} = 0$). This is a weak assumption since on one hand there is no need to control the delayed-state and on the other hand the controlled output is a virtual output for design purpose and hence does not involve any uncertainties.

The stabilization problem formulation is here expressed using the backward adjoint of a time-delay system [Bensoussan et al., 2006].

4.1 Adjoint system

The backward adjoint of the closed-loop system (5) is defined as [Bensoussan et al., 2006]

$$\begin{bmatrix} \dot{\tilde{x}}(t) \\ \tilde{z}_0(t) \\ \tilde{z}_{0h}(t) \\ \tilde{z}_1(t) \end{bmatrix} = \left[\begin{array}{cccc|c} A_{cl}^T & C_{0cl}^T & C_{1h}^T & C_{1cl}^T & A_h^T \\ B_0^T & D_{00}^T & 0 & 0 & 0 \\ 0 & 0 & D_{00}^T & 0 & B_0^T \\ B_1^T & D_{01}^T & 0 & D_{11}^T & 0 \end{array}\right] \left[\begin{array}{c} \tilde{x}(t) \\ \tilde{w}_0(t) \\ \tilde{w}_{0h}(t) \\ \tilde{w}_1(t) \\ \hline \tilde{x}_h(t) \end{array}\right] \tag{6}$$

$$\begin{bmatrix} \tilde{w}_0(t) \\ \tilde{w}_{0h}(t) \end{bmatrix} = \bar{\Delta} \begin{bmatrix} \tilde{z}_0(t) \\ \tilde{z}_{0h}(t) \end{bmatrix}, \ \bar{\Delta} = \Delta \oplus \Delta \tag{7}$$

Due to the fact that the matrix C_{0h} is non-zero then we obtain this particular form for the adjoint system. As the output signal $z_0(t)$ is the original system contains a delayed-signal (i.e. $x_h(t)$), then we obtain an adjoint system with a delay-input signal. In order to account for it in the stability analysis, we back-propagate delay-operator through the uncertainty. This explains why the adjoint system involves a delayed uncertainty and supplementary signals (i.e. $w_{0h}(t)$).

It is worth noting that when either the matrix acting on the delayed-state is certain or it is the only uncertain matrix, the adjoint system admits a more simple expression where the uncertainty matrix does not need to be repeated.

In all cases, the multipliers set becomes

$$\bar{\mathscr{F}} := \left\{ F \in \mathbb{R}^{4n_0} : \begin{bmatrix} \bar{\Delta}^T & I_{2n_0} \end{bmatrix} F \begin{bmatrix} \bar{\Delta} \\ I_{2n_0} \end{bmatrix} > 0, \ \forall \bar{\Delta} \in \bar{\boldsymbol{\Delta}} \right\} \tag{8}$$

4.2 State-Feedback existence lemma

We prove in this section the state-feedback existence lemma.

Lemma 4.1. There exists a robust state-feedback control law of the form $u(t) = Kx(t)$ such that the closed-loop system (5) is asymptotically stable for all $\Delta \in \boldsymbol{\Delta}$ if there exist symmetric matrices $P, Q, R > 0$, a scalar $\gamma > 0$ and $F \in \bar{\mathscr{F}}$ such that

$$\mathcal{M}_1 + \Theta_1 F \Theta_1^T < 0 \tag{9}$$

$$\mathcal{N}_2^T (\mathcal{M}_2 + \Theta_2 F \Theta_2^T) \mathcal{N}_2 < 0 \tag{10}$$

where $\mathcal{M}_1$ is defined in (11), $\mathcal{M}_2$ in (12), $\Theta_1 = \begin{bmatrix} 0 & 0 & B_0 & 0 \\ 0 & 0 & 0 & B_0 \\ I & 0 & D_{00} & 0 \\ 0 & I & 0 & D_{00} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\Theta_2 = \begin{bmatrix} 0 & 0 & B_0 & 0 \\ I & 0 & D_{00} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_0 \\ 0 & I & 0 & D_{00} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $\mathcal{N}_2 = Ker \begin{bmatrix} B_u^T & D_{0u}^T & D_{1u}^T \end{bmatrix} \oplus I$.

Proof: See appendix B. □

As the matrix inequality (9) is nonlinear due to the term $-h_M^{-1} P R^{-1} P$, this lemma cannot be easily solved. We reformulate it into the following form.

Lemma 4.2. There exists a robust state feedback control of the form $u(t) = Kx(t)$ such that the closed-loop system (5) is asymptotically stable for all $\Delta \in \boldsymbol{\Delta}$ if there exist symmetric matrices $P, Q, R, S, T > 0$, symmetric definite matrices $W, Z < 0$ and a scalar $\gamma > 0$ and $F \in \bar{\mathscr{F}}$ such that

$$\Pi = \begin{bmatrix} A^TP+PA+Q-h_M^{-1}R & \star & \star & \star & \star & \star \\ A_h^TP+h_M^{-1}R & -(1-\mu)Q-h_M^{-1}R & \star & \star & \star & \star \\ B_0^TP & 0 & 0 & \star & \star & \star \\ B_1^TP & 0 & 0 & -\gamma I_{n_w} & \star & \star \\ C_1 & 0 & D_{10} & D_{11} & -\gamma I_{n_z} & \star \\ h_MRA & h_MRA_h & h_MRB_0 & h_MRB_1 & 0 & -h_MR \end{bmatrix} \tag{3}$$

$$\mathcal{M}_1 := \begin{bmatrix} Q-h_M^{-1}R-h_M^{-1}PR^{-1}P & \star & \star & \star & \star & \star \\ h_M^{-1}R & -(1-\mu)Q-h_M^{-1}R & \star & \star & \star & \star \\ 0 & 0 & 0 & \star & \star & \star \\ 0 & 0 & 0 & 0 & \star & \star \\ 0 & 0 & 0 & 0 & -\gamma I_{n_z} & \star \\ B_1^T & 0 & D_{01}^T & 0 & D_{11}^T & -\gamma I_{n_w} \end{bmatrix} \tag{11}$$

$$\mathcal{M}_2 := \begin{bmatrix} AP+PA^T+Q-h_M^{-1}R & \star & \star & \star & \star & \star & \star \\ C_0P & 0 & \star & \star & \star & \star & \star \\ C_1P & 0 & -\gamma I_{n_z} & \star & \star & \star & \star \\ A_hP+h_M^{-1}R & 0 & 0 & -(1-\mu)Q-h_M^{-1}R & \star & \star & \star \\ C_{0h}P & 0 & 0 & 0 & 0 & \star & \star \\ B_1^T & D_{01}^T & D_{11}^T & 0 & 0 & -\gamma I_{n_w} & \star \\ h_MRA^T & h_MRC_0^T & h_MRC_1^T & h_MRA_h^T & h_MRC_{0h}^T & 0 & -h_MR \end{bmatrix} \tag{12}$$

$$\mathcal{M}_3 := \begin{bmatrix} Q-h_M^{-1}R+h_M^{-1}Z & \star & \star & \star & \star & \star \\ h_M^{-1}R & -(1-\mu)Q-h_M^{-1}R & \star & \star & \star & \star \\ 0 & 0 & 0 & \star & \star & \star \\ 0 & 0 & 0 & 0 & \star & \star \\ 0 & 0 & 0 & 0 & -\gamma I_{n_z} & \star \\ B_1^T & 0 & D_{01}^T & 0 & D_{11}^T & -\gamma I_{n_w} \end{bmatrix} \tag{13}$$

$$\mathcal{M}_3 + \Theta_1 F \Theta_1^T < 0 \tag{14}$$

$$\mathcal{N}_2^T(\mathcal{M}_2 + \Theta_2 F \Theta_2^T)\mathcal{N}_2 < 0 \tag{15}$$

$$\begin{bmatrix} W & T \\ T & -S \end{bmatrix} \leq 0 \tag{16}$$

with $ZW = I$, $RS = I$ and $PT = I$, where $\mathcal{M}_2$ is defined in (12), $\mathcal{M}_3$ in (13),

Proof: The proof is identical as in [Chen and Zheng, 2006]. □ This problem is obviously non-convex due to equalities $ZW = I$, $RS = I$ and $PT = I$ but as in [Chen and Zheng, 2006] such a problem can be approximatively solved using the cone complementary algorithm (see [Ghaoui et al., 1997]).

Algorithm 1. Cone complementary algorithm

(1) Fix $\gamma = \gamma_0$.
(2) Fix $k = 0$, γ and find $P_0, Q_0, R_0, S_0, T_0, W_0, T_0$ satisfying (14), (15), (16).
(3) Find $(P_{k+1}, T_{k+1}, R_{k+1}, S_{k+1}, W_{k+1}, Z_{k+1})$ that solves $\min_{P,Q,R,S,T,W,Z} \mathrm{Tr}(PT_k+P_kT+RS_k+R_kS+WZ_k+W_kZ)$ with (14) (15), (16) and

$$\begin{bmatrix} P & I \\ I & T \end{bmatrix} \geq 0 \quad \begin{bmatrix} R & I \\ I & S \end{bmatrix} \geq 0 \quad \begin{bmatrix} W & I \\ I & Z \end{bmatrix} \leq 0 \tag{17}$$

(4) If it is feasible
  - If the optimal value is $6n$ reduce then update γ to a smaller value and go to step 2
  - else $k \leftarrow k + 1$ and go to step 3.

  else if $\gamma \geq \gamma_{max}$ then exit else update γ to a larger value and go to step 2.

4.3 Controller Construction

We provide here methods to construct the controller from the solutions of the stabilizability lemma.

Explicit Construction This method uses a simple algorithm borrowed from [Iwasaki and Skelton, 1995].

Algorithm 2. (1) Find $\lambda > 0$ such that $\Phi := (\lambda U^TU - \mathcal{M}_2 - \Theta_2 F \Theta_2^T)^{-1} > 0$ where $U = \begin{bmatrix} B_u^T & D_{0u}^T & D_{1u}^T & 0 & 0 & 0 & 0 \end{bmatrix}$
(2) Compute $K = -\lambda U \Phi V^T (V \Phi V^T)^{-1}$ with $V = [P\ 0\ 0\ 0\ 0\ 0\ h_MR]$

This method allows to construct explicitly the controller and is parametrized by the term λ hence there exists an infinite number of stabilizing controllers satisfying $\mathcal{H}_\infty$ closed-loop performances. Note that λ can be easily found using SDP. For a full parametrization see for instance [Iwasaki and Skelton, 1994] and references therein.

Implicit Construction This part explains how to construct the controller in an implicit manner. In this case, the controller is found as a solution of a SDP and allows to add supplementary constraints.

Lemma 4.3. The controller matrix K is found while solving the following SDP

$$\begin{aligned} &\min_{K,\nu,t} \ J(K,\nu,t) \\ &\qquad \mathcal{M}_2 + \Theta_2 F \Theta_2^T + U^TKV + V^TK^TU \leq -tI \\ &\qquad \mathcal{L}(K,\nu,t) < 0 \end{aligned} \tag{18}$$

where J is a cost to minimize, ν, t are additional decision terms and $\mathcal{L}$ supplementary convex constraints.

Notice that if the constraints $\mathcal{L}$ are too strong then it may be not possible to find a feasible solution to this LMI problem even if the stabilizability problem is feasible. In this case, the constraints should be make weaker.

5. EXAMPLE

Consider now the following system

$$\begin{aligned}\dot{x}(t) &= \begin{bmatrix}-1.3 & 0.2\\ 0.2 & -1\end{bmatrix} x(t) + \begin{bmatrix}1\\0\end{bmatrix} u(t) + \begin{bmatrix}1\\1\end{bmatrix} w(t)\\ &+ \left(\begin{bmatrix}-0.6 & -0.5\\ -0.5 & -0.6\end{bmatrix} + \begin{bmatrix}1 & 0\\ 1 & -0.8\end{bmatrix}\rho + \begin{bmatrix}-0.9 & 1\\ 0.1 & -1\end{bmatrix}\rho^2\right) x_h(t)\\ z(t) &= \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} x(t) + \begin{bmatrix}0\\ 0.1\end{bmatrix} u(t)\end{aligned} \tag{19}$$

We aim compare our approach to the approach proposed in [Suplin et al., 2006] which gives very good results for polytopic systems. However this system cannot be directly expressed as a polytope due to the polynomial dependence onto the uncertain parameter $\rho \in [-1, 1]$. One way to consider it as a polytope is to make the following restrictive change of variables: $\rho_1 := \rho$ and $\rho_2 := \rho^2 \in [0, 1]$ (it would be also possible to construct a ellipsoid containing the family of matrices parametrized by ρ on). It is obvious that such an approximation is not good since we consider aberrant points (for instance $\rho_1 = 1$ and $\rho_2 = 0$ which never occurs). It is possible to reduce the size of the polytope but the epigraph of the function $f(\rho_1) = \rho_1^2$ has to remain convex which is source of conservatism. With no polytope reduction, the approach of [Suplin et al., 2006] does not lead to find a stabilizing controller while using the approach presented in this paper we obtain $\gamma = 8.765$ with a state-feedback gain $K = [35.0747\ \ -19.3227]$.

We can easily imagine that a system looses stabilizability for some values into the epigraph of $f(\cdot)$ but not onto the image set of $f(\cdot)$. In that case, the polytopic approach would lead to more conservative result than methods using the linear fractional transformation.

6. CONCLUSION

We have proposed a new method of $\mathcal{H}_\infty$ robust stability/performance analysis for uncertain system with time-varying delays through full block multipliers in the delay-dependent framework in terms of a finite number of LMIs. A robust state-feedback existence lemma is derived from stability condition. As the resulting conditions are non-convex we provide tractable conditions using the cone complementary algorithm which is known to be efficient in practice but does not guarantee global convergence. From the solutions of the stabilizability conditions, we provide how construct the controller matrix either using an explicit formulation or an implicit one through SDP. Due to linear fractional representation of the uncertain system it is possible to tackle a wider class of uncertain system (such as polynomial or even rational dependence onto uncertain parameters) and we show that the proposed approach leads to better results.

Appendix A. PROOF OF THEOREM 3.1

Consider the following Lyapunov-Krasovskii functional

$$V = x^T(t)Px(t) + \int_{t-h(t)}^{t} x^T(\theta)Qx(\theta)d\theta \ldots + \int_{-h_M}^{0}\int_{t+\theta}^{t} \dot{x}(\eta)^T R\dot{x}(\eta)d\eta d\theta$$

and the uncertain system

$$\begin{aligned}\dot{x}(t) &= \mathscr{A}(\Delta)x(t) + \mathscr{A}_h(\Delta)x(t-h(t)) + \mathscr{B}_1(\Delta)w(t)\\ z(t) &= \mathscr{C}(\Delta)x(t) + \mathscr{C}_h(\Delta)x(t-h(t)) + \mathscr{D}_{11}(\Delta)w(t)\end{aligned}$$

Denoting $x_h(t) := x(t-h(t))$ and computing the derivative V along the trajectories solutions of the system (we drop the dependence on time and Δ for ease of simplicity) we obtain

$$\begin{aligned}\dot{V} &\le (\mathscr{A}x + \mathscr{A}_h x_h + \mathscr{B}_1 w)Px + (\star)^T + x^T Qx\\ &- (1-\mu)x_h^T Q x_h + h_M \dot{x}^T R\dot{x} \underbrace{- \int_{t-h(t)}^{t} \dot{x}(\theta)^T R\dot{x}(\theta)d\theta}_{\mathcal{I}}\end{aligned}$$

Using the Jensen's inequality on the integral term leads to

$$\mathcal{I} \le -h_M^{-1}\left(\int_{t-h(t)}^{t} \dot{x}(\theta)\right)^T R\left(\int_{t-h(t)}^{t} \dot{x}(\theta)\right)$$

Note that the first equation is not defined for $h(t) = 0$ but it can easily be shown that the bound on $\mathcal{I}$ is well defined in t_i with $h(t_i) = 0$. The same inequality for constant time-delays is also presented in [Gouaisbaut and Peaucelle, 2006b].

Replacing the term $\dot{x}$ by its explicit expression leads to the quadratic form (where we drop the dependency on Δ for ease of simplicity

$$X^T \Xi X < 0 \tag{A.1}$$

where $X = \begin{bmatrix}x^T & x_h^T & w^T\end{bmatrix}^T$ and

$$\begin{aligned}\Xi_{11} &= \mathscr{A}^T P + P\mathscr{A} + Q + h_M\mathscr{A}^T R\mathscr{A} - h_M^{-1}R\\ \Xi_{21} &= \mathscr{A}_h^T P + h_M\mathscr{A}_h^T R\mathscr{A} + h_M^{-1}R\\ \Xi_{22} &= -(1-\mu)Q + h_M\mathscr{A}_h^T R\mathscr{A}_h - h_M^{-1}R\\ \Xi_{31} &= \mathscr{B}_1^T P + \mathscr{B}_1^T R\mathscr{A}\\ \Xi_{32} &= h_M\mathscr{B}_1^T R\mathscr{A}_h\\ \Xi_{33} &= h_M\mathscr{B}_1^T R\mathscr{B}_1\end{aligned}$$

This LMI is infinite dimensional due to the dependence on the uncertainty function Δ. This quadratic form can be expressed as

$$(\star)^T\Theta\left[\begin{array}{ccc} I & 0 & 0\\ \mathscr{A}(\Delta) & \mathscr{A}_h(\Delta) & \mathscr{B}_1(\Delta)\\ \hline 0 & I & 0\\ I & 0 & 0\\ \hline -I & I & 0\\ \mathscr{A}(\Delta) & \mathscr{A}_h(\Delta) & \mathscr{B}_1(\Delta)\end{array}\right] < 0$$

where $\Theta = \begin{bmatrix}0 & P\\ P & 0\end{bmatrix} \oplus \begin{bmatrix}-(1-\mu)Q & 0\\ 0 & Q\end{bmatrix} \oplus \begin{bmatrix}-h_M^{-1}R & 0\\ 0 & h_M R\end{bmatrix}$.

To specify the input/output $\mathcal{H}_\infty$ constraint, we add to the Lyapunov-Krasovskii functional derivative the input/output constraint $s(w, z) = -\gamma w^T w + \gamma^{-1} z^T z$ where $\gamma > 0$ is a positive scalar. Then we obtain

$$(\star)^T\Theta_\gamma\left[\begin{array}{ccc} I & 0 & 0\\ \mathscr{A}(\Delta) & \mathscr{A}_h(\Delta) & \mathscr{B}_1(\Delta)\\ \hline 0 & I & 0\\ I & 0 & 0\\ \hline -I & I & 0\\ \mathscr{A}(\Delta) & \mathscr{A}_h(\Delta) & \mathscr{B}_1(\Delta)\\ \hline 0 & 0 & I\\ \mathscr{C}_1(\Delta) & \mathscr{C}_{1h}(\Delta) & \mathscr{D}_{11}(\Delta)\end{array}\right] < 0$$

where $\Theta_\gamma = \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} \oplus \begin{bmatrix} -(1-\mu)Q & 0 \\ 0 & Q \end{bmatrix} \oplus \begin{bmatrix} -h_M^{-1}R & 0 \\ 0 & h_M R \end{bmatrix} \oplus \begin{bmatrix} -\gamma I_w & 0 \\ 0 & \gamma^{-1} I_z \end{bmatrix}$.

Then we apply the full-block $\mathcal{S}$-procedure, expand the expression of the obtained LMI and finally perform Schur complement onto quadratic term

$$-\begin{bmatrix} C_1^T & h_M A^T R \\ 0 & h_M A_h^T R \\ D_{10}^T & h_M B_0^T R \\ D_{11}^T & h_M B_1^T R \end{bmatrix} \begin{bmatrix} -\gamma^{-1} I & 0 \\ 0 & -h_M^{-1} R^{-1} \end{bmatrix} (\star)^T$$

and we obtain LMI (4). This proof is then complete.

Appendix B. PROOF OF LEMMA 4.1

First inject matrices of augmented adjoint system (6) into LMI of statement 3 of theorem 3.1. Then note that the inequality (4) can be rewritten as

$$\mathcal{M}_2 + \Theta_2 F \Theta_2^T + U^T K V + V^T K^T U < 0$$

where $\mathcal{M}_2$ is defined in (12), $\Theta_2 = \begin{bmatrix} 0 & 0 & B_0 & 0 \\ I & 0 & D_{00} & 0 \\ 0 & I & 0 & D_{00} \\ 0 & 0 & D_{10} & 0 \\ 0 & 0 & 0 & B_0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $U = \begin{bmatrix} B_u^T & D_{0u}^T & D_{1u}^T & 0 & 0 & 0 & 0 \end{bmatrix}$, $V = [P \; 0 \; 0 \; 0 \; 0 \; 0 \; h_M R]$.

Then using projection lemma [Apkarian and Adams, 1998] this nonlinear inequality is equivalent to two underlying inequalities:

$$\mathcal{N}_1^T(\mathcal{M}_2 + \Theta_2 F \Theta_2^T)\mathcal{N}_1 < 0 \quad \text{(B.1)}$$

$$\mathcal{N}_2^T(\mathcal{M}_2 + \Theta_2 F \Theta_2^T)\mathcal{N}_2 < 0 \quad \text{(B.2)}$$

LMI (B.2) is obviously (10) with $\mathcal{N}_2 = Ker(U)$. Now consider $PP_1 + h_M R P_2 = 0$ then we have

$$\mathcal{N}_1 := Ker(V) = \left[\begin{array}{c|c} P_1 & 0 \\ \hline 0 & I \\ \hline P_2 & 0 \end{array}\right]$$

As $P, R > 0$ (hence nonsingular), there exists an infinite number of couples (P_1, P_2) such that $PP_1 + h_M R P_2 = 0$. Let $P_1 = I$ and we obtain $P_2 = -h_M^{-1} R^{-1} P$. Use this basis to project in inequality (B.1). The result is matrix inequality (9) modulo some rows/columns permutations. This concludes the proof.

REFERENCES

P. Apkarian and R.J. Adams. Advanced gain-scheduling techniques for uncertain systems. *IEEE Transactions on Automatic Control*, 6:21–32, 1998.

A. Bensoussan, G. Da Prato, M.C. Delfour, and S.K. Mitter. *Representation and Control of Infinite Dimensional Systems - 2^{nd} Edition*. Springer, 2006.

W. H. Chen and W. X. Zheng. On improved robust stabilization of uncertain systems with unknown input delays. *Automatica*, 42:1067–1072, 2006.

E. Fridman. Stability of systems with uncertain delays: a new 'complete' lyapunov-krasovskii functional. *IEEE Transactions on Automatic Control*, 51:885–890, 2006a.

E. Fridman. Descriptor discretized lyapunov functional method: Analysis and design. *IEEE Transactions on Automatic Control*, 51:890–897, 2006b.

L. El Ghaoui, F. Oustry, and M. Ait Rami. A cone complementary linearization algorithm for static output-feedback and related problems. *IEEE Transactions on Automatic Control*, 42:1171–1176, 1997.

F. Gouaisbaut and D. Peaucelle. Stability of time-delay systems with non-small delay. In *Conference on Decision and Control, San Diego, California*, 2006a.

F. Gouaisbaut and D. Peaucelle. Delay depependent robust stability of time delay-systems. In 5^{th} *IFAC Symposium on Robust Control Design*, Toulouse, France, 2006b.

K. Gu, V.L. Kharitonov, and J. Chen. *Stability of Time-Delay Systems*. Birkhäuser, 2003.

Y. He, M. Wu, and J-H. She adn G-P. Liu. Parameter-dependent lyapunov functional for stability of time-delay systems with polytopic type uncertainties. *IEEE Transactions on Automatic Control*, 49:828–832, 2004.

T. Iwasaki and S. Hara. Well-posedness of feedback systems: insight into exact robustness analysis and approximate computations. *IEEE Transactions on Automatic Control*, 43:619–630, 1998.

T. Iwasaki and R. E. Skelton. All controllers for the general $\mathcal{H}_\infty$ control probblems: Lmi existence conditions and state-space formulas. *Automatica*, 30(8):1307–1317, 1994.

T. Iwasaki and R.E. Skelton. A unified approach to fixed order controller design via linear matrix inequalities. *Mathematical Problems in Engineering*, 1:59–75, 1995.

S.-I. Niculescu. *Delay effects on stability. A robust control approach*, volume 269. Springer-Verlag: Heidelbeg, 2001.

P. Park. A delay-dependent stability criterion for systems with uncertain time-invariant delays. *IEEE Transactions on Automatic Control*, 44(4):876–877, 1999.

C. W. Scherer. LPV control and full block multipliers. *Automatica*, 37:361–375, 2001.

C. W. Scherer and C. W. J. Hol. Matrix sum-of-squares relaxations for robust semi-definite programs. *Mathematical Programming Series B*, 107:189–211, 2006.

V. Suplin, E. Fridman, and U. Shaked. $\mathcal{H}_\infty$ control of linear uncertain time-delay systems - a projection approach. *IEEE Transactions on Automatic Control*, 51:680–685, 2006.

F. Wu. An unified framework for LPV system analysis and control synthesis. In *Conference on Decision and Control*, 2000.

F. Wu. Robust quadratic performance for time-delayed uncertain linear systems. *International Journal of Robust and Nonlinear Systems*, 13:153–172, 2003.

F. Wu. A generalized LPV system analysis and control synthesis framework. *International Journal of Control*, 74:745–759, 2001.

S. Xu and J. Lam. On equivalence and efficiency of certain stability criteria for time-delay systems. *IEEE Transactions on Automatic Control*, 52(1):95–101, 2007.

S. Xu, J. Lam, and Y. Zhou. New results on delay-dependent robust h control for systems with time-varying delays. *Automatica*, 42(2):343–348, 2006.