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High-Low Gain Redesign for a 4 DOF Spherical Inverted Pendulum

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Abstract: We revisit a previous high-low gain control idea for a 4 DOF spherical inverted pendulum using a different approach, inspired by a nested saturation tool proposed by Marconi and Isidori, that provides explicit tuning rules to deal with certain bounded external disturbances. The update controller is a robust, decentralized and "global" controller.

1. INTRODUCTION

The pendulum is a cylindrical beam with the length $2L$ and the mass m attached to a horizontal plane via a universal joint that is driven by a planar control force $F \stackrel{\triangle}{=} (F_x, F_y)$ and sliding in the plane (see Fig. 1). The system has four degrees of freedom with the generalized coordinates $q \stackrel{\triangle}{=} (x, y, \delta, \varepsilon)$ with the translation ones: (x, y) and a pair of Euler angles (δ, ε). The whole upper space denoted by $U \stackrel{\triangle}{=} \left\{(q,\dot{q}) \in R^8 | (\delta, \varepsilon) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right\}$ is defined as the "global" region. The benchmark problem is motivated by several practical problems: *vector thrusted rockets hovering in the air, personal transporters (Segway), jugglers' balancing problems and laboratory test-benches.* Our aim is to design F such that, for any $(q(0), \dot{q}(0)) \in U$, $(q(t), \dot{q}(t)) \to 0$ as $t \to \infty$.

To achieve a "global" stability region, one could use strategies that switched between a local (or non-local) stabilizing controller and a swing-up strategy (see Albouy and Praly [2000], Shiriaev [2004]). See Liu *et al* [2007b] for a way-point tracking design with switching (see also Liu *et al* [2008a] for exact output tracking). Here, we assume that the pendulum is already swung up above the horizontal plane. Several non-local continuous stabilizing controllers (no switching) were proposed for the system Bloch *et al* [2001], Liu *et al* [2008b, 2006]. The controller of controlled Lagrangians Bloch *et al* [2001] (see [Liu , 2006b, Chapter 7] for details) yielded some non-local "bounded" stabilizing region but it suffered poor robustness using the parameters we attempted (see Liu *et al* [2007a]). A "semi-global" decentralized stabilizing controller was proposed in Liu *et al* [2006] (see also Liu *et al* [2008c]) based on Lyapunov theory of singular perturbed systems. Although the robustness was guaranteed by an associated Lyapunov function, it might be deteriorated when a larger domain of attraction was attempted. In Liu *et al* [2008b], a "global" high-low gain control idea that improved Liu *et al* [2005] was proposed for the pendulum through identifying some appropriate upper triangular form, where a high-gain controller was used to regulate angular dynamics and a low gain controller was used to regulate the rest of the dynamics by applying the nested saturation tool Teel [1996]. However, the tuning rules are implicitly dealing with the disturbance.

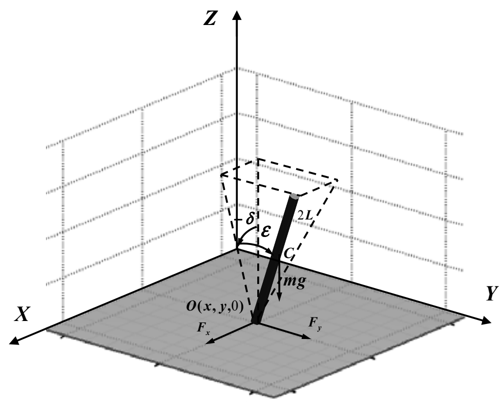

Fig. 1. The spherical inverted pendulum

In this paper, we redesign the high and low gain controller Liu *et al* [2008b] inspired by a robust nested saturation procedure in [Isidori *et al* , 2003, Appendix C] and Marconi & Isidori [2001] (see Arcak *et al* [2001], Kaliora & Astofi [2004] for different approaches) such that it provides explicit tuning rules for the design parameters at the presence of certain bounded disturbances. The controller is decentralized based on the structure of two interconnected chains of integrators Liu *et al* [2006, 2008c]) and yields a "global" domain of attraction inherit form Liu *et al* [2008b]. The effectiveness of the controller is evaluated through computer simulations.

The paper is organized as follows. In Section 2, we recall the model and the decoupled dynamics in Liu *et al* [2006, 2008c]. In Section 3, we present our main result. Some simulations are given in Section 4. Final observation is given in Section 5.

Notations: For a piecewise-continuous function $u(t)$: $[0, \infty) \to R^m$, define $\|u(\cdot)\|_a = \limsup_{t\to\infty} \{\max_{1\le i\le m} |u_i(t)|\}$ the asymptotic "norm" of $u(\cdot)$. The set of $u(t)$, endowed with the supremum norm $\|u(\cdot)\|_\infty = \sup_{t\geq 0}\|u(t)\|$, is denoted by L^m_∞. "$c(\cdot)$" and "$s(\cdot)$" represent $\cos(\cdot)$ and $\sin(\cdot)$ respectively and $(x_1, x_2) \triangleq (x_1^T, x_2^T)^T$ is used for convenience. With respect to $(q, \dot{q})$, we define a set of new coordinates for the system: $X_1 \triangleq x - \frac{4L}{3}\delta$, $X_2 \triangleq y + \frac{4L}{3}\varepsilon$, $X_3 \triangleq \dot{x} - \frac{L(1+\frac{1}{3}c^2(\varepsilon))}{c(\delta)}\dot{\delta} - \frac{Ls(\delta)s(\varepsilon)(\frac{1}{3}+c^2(\delta))}{c^2(\delta)c(\varepsilon)}\dot{\varepsilon}$, $X_4 \triangleq \dot{y} + \frac{L(\frac{1}{3}+c^2(\delta))}{c(\delta)c(\varepsilon)\dot{\varepsilon}}$, $X_5 \triangleq \tan(\delta)$, $X_6 \triangleq \tan(\varepsilon)$, $X_7 \triangleq (1+\tan^2(\delta))\dot{\delta}$, $X_8 \triangleq (1+\tan^2(\varepsilon))\dot{\varepsilon}$.

We refer to a saturation function with a shape like "$\_/^{-}$" as a mapping $\sigma : R \to R$ which enjoys the properties: (i) $\sigma'(s) \triangleq |d\sigma(s)/ds| \leq 2$ for all s; (ii) $s\sigma(s) > 0$ for all $s \neq 0$, $\sigma(0) = 0$; (iii) $\sigma(s) = sgn(s)$ for $|s| \geq 1$; (iv) $|s| < |\sigma(s)| < 1$ for $|s| < 1$.

2. PRIOR RESULTS

2.1 The Model

We review the equations of motion in Liu *et al* [2008b, 2006] for our system

$$\mathbf{D}(q)\cdot\ddot{q} + \mathbf{C}(q,\dot{q})\cdot\dot{q} + \mathbf{G}(q) = \mathbf{Q}, \tag{1}$$

where

$$\mathbf{G}(q) = \begin{pmatrix} 0 \\ 0 \\ -mgLs(\delta)c(\varepsilon) \\ -mgLc(\delta)s(\varepsilon) \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} F_x + v_{f_1} \\ F_y + v_{f_2} \\ v_{f_3} \\ v_{f_4} \end{pmatrix},$$

$$\mathbf{D}(q) = m\times \begin{pmatrix} 1 & 0 & -Lc(\delta) & 0 \\ 0 & 1 & -Ls(\delta)s(\varepsilon) & Lc(\varepsilon)c(\delta) \\ -Lc(\delta) & -Ls(\delta)s(\varepsilon) & L^2(1+1/3c(\varepsilon)^2) & 0 \\ 0 & Lc(\varepsilon)c(\delta) & 0 & L^2(1/3+c^2(\delta)) \end{pmatrix}$$

$$\mathbf{C}(q,\dot{q}) = \begin{pmatrix} 0 & 0 & mL\dot{\delta}s(\delta) \\ 0 & 0 & -mL(\dot{\delta}s(\varepsilon)c(\delta)+\dot{\varepsilon}c(\varepsilon)s(\delta)) \\ 0 & 0 & -1/3mL^2\dot{\varepsilon}c(\varepsilon)s(\varepsilon) \\ 0 & 0 & 1/3mL^2\dot{\delta}c(\varepsilon)s(\varepsilon) - mL^2\dot{\varepsilon}c(\delta)s(\delta) \end{pmatrix}$$
$$\begin{pmatrix} 0 \\ -mL(\dot{\varepsilon}s(\varepsilon)c(\delta)+\dot{\delta}c(\varepsilon)s(\delta)) \\ -1/3mL^2\dot{\delta}c(\varepsilon)s(\varepsilon) + mL^2\dot{\varepsilon}c(\delta)s(\delta) \\ -mL^2\dot{\delta}c(\delta)s(\delta) \end{pmatrix},$$

with $v_f = (v_{f_1}, \ldots, v_{f_4})$ a collection of external forces.

2.2 The Decoupled Dynamics

Because $\mathbf{D}(q)$ is invertible in U, the dynamics (1) can be written as follows

$$\ddot{q} \triangleq \begin{pmatrix} H_{11} \\ H_{21} \end{pmatrix} F + \begin{pmatrix} H_{12} \\ H_{22} \end{pmatrix} + \mathbf{D}^{-1}(q)v_f, \tag{2}$$

where H_{ij}, $i = 1, 2$ and $j = 1, 2$, with the arguments $(q, \dot{q})$ are nonlinear terms derived from (1) and $H_{21} \in R^{2\times 2}$ is invertible on U (see Liu *et al* [2008b] for the entries). The following result converts the dynamics (2) with $v_f = 0$ to two perturbed chains of integrators.

Lemma 2.1. Liu *et al* [2006, 2008c] Apply a mapping $T : (q, \dot{q}) \in U \mapsto \mathrm{X} \in R^8$ defined as $\mathrm{X} \triangleq (X_1, \ldots, X_8)$ and, *then*, take a feedback transformation

$$F = H_{21}^{-1}\left(H_{31}^{-1}(u - H_{32}) - H_{22}\right) \tag{3}$$

where $u \in R^2$ is the new control variable,

$$H_{31} \triangleq \begin{pmatrix} 1+\tan^2(\delta) & 0 \\ 0 & 1+\tan^2(\varepsilon) \end{pmatrix}, H_{32} \triangleq \begin{pmatrix} 2\dot{\delta}^2\delta(1+\tan^2(\delta)) \\ 2\dot{\varepsilon}^2\varepsilon(1+\tan^2(\varepsilon)) \end{pmatrix}$$

such that the system (2) with $v_f = 0$ converts to

$$\begin{cases} \dot{X}_1 &= X_3 + \varphi_1(X_5, X_6, X_7, X_8) \\ \dot{X}_3 &= X_5 + \varphi_3(X_5, X_6, X_7, X_8) \\ \dot{X}_5 &= X_7, \ \dot{X}_7 = u_1 \end{cases} \tag{4}$$

$$\begin{cases} \dot{X}_2 &= X_4 + \varphi_2(X_5, X_6, X_7, X_8) \\ \dot{X}_4 &= X_6 + \varphi_4(X_5, X_6, X_7, X_8) \\ \dot{X}_6 &= X_8, \ \dot{X}_8 = u_2 \end{cases} \tag{5}$$

where let $s \triangleq (X_5, \ldots, X_8)$

$$\varphi_1(s) = \frac{L}{g}\left(\left(-\frac{4}{3(1+X_5^2)} + \frac{\left(1+\frac{1}{3(1+X_6^2)}\right)}{(1+X_5^2)^{1/2}}\right)X_7 + \left(\frac{-X_5X_6(1+X_5^2)^{1/2}\left(\frac{1}{3}+\frac{1}{1+X_5^2}\right)}{1+X_6^2}\right)X_8\right),$$

$$\varphi_2(s) = \frac{L}{g}\left(\frac{1}{(1+X_6^2)}\left(\frac{4}{3} - \frac{1}{3(1+X_5^2)^{1/2}} - \frac{1}{(1+X_5^2)^{3/2}}\right)X_8\right),$$

$$\varphi_3(s) = X_5\left(\sqrt{1+X_6^2}-1\right) + \frac{L}{g}\left(\frac{X_5X_8^2\left((4+X_5^2)(1+X_6^2)-3\right)}{3(1+X_6^2)^2(1+X_5^2)^{1/2}} + \frac{2X_5X_7^2(2+X_6^2)}{3(1+(X_5)^2)^{3/2}(1+X_6^2)} - \frac{X_8X_7X_6(1+2X_5^2)(4+X_5^2)}{3(1+X_5^2)^{3/2}(1+X_6^2)}\right),$$

$$\varphi_4(s) = \frac{L}{g}\left(\left(\frac{X_8^2X_6(1+X_5^2)^{1/2}\left(\frac{1}{3}+\frac{1}{1+X_5^2}\right)}{(1+X_6^2)^{3/2}}\right) + \frac{X_7X_8X_5\left(\frac{1}{3}+\frac{1}{1+X_5^2}\right)}{(1+X_5^2)^{1/2}(1+X_6^2)^{1/2}} - \frac{X_7^2X_6}{3(1+X_5^2)^{3/2}(1+x_6^2)^{1/2}}\right),$$

and $\lim_{\|s\|\to 0}\frac{\|\varphi_i(s)\|}{\|s\|} = 0$, $i = 1, \ldots, 4$, that is, $\varphi_i(s)$ are high order terms about s. For convenience, we let $x_a \triangleq (X_1, X_3, X_5, X_7)$ and $x_b \triangleq (X_2, X_4, X_6, X_8)$ and rewrite (4) and (5) as

$$\dot{x}_a = f_a(x_a, x_b) + (0, 0, 0, u_1)^T \tag{6}$$

$$\dot{x}_b = f_b(x_a, x_b) + (0, 0, 0, u_2)^T \tag{7}$$

where $f_a(\cdot)$ and $f_b(\cdot)$ are uncontrolled dynamics, u_1 and u_2 are actual intermediate control signal,

Remark 1. In Lemma 2.1, we first apply a change of coordinates that annihilates F in the dynamics of (X_1, X_3) and (X_2, X_4) without any restriction on F. So, it holds at the presence of the matched disturbance (v_{f_1}, v_{f_2}). However, at the presence of the whole $\mathbf{D}^{-1}(q)v_f$ ($\mathbf{D}^{-1}(q)$ is non-singular), (2) should convert to (6) and (7) *plus* some *vanishing*, lossy pertubation terms in a form $p_1(X)\cdot p_2(v_f)$ with $p_2(v_f)|_{v_f=0} = 0$ and $\|p_1(X)\|_\infty < \infty$ for $\|X\|_\infty < \infty$. To capture the effect of disturbance, we let the transformed system (6) and (7)

$$\dot{x}_a = f_a(x_a,\, x_b) + (0,\, \bar{v}_{a2},\, 0,\, \bar{v}_{a4} + u_1)^T \qquad (8)$$

$$\dot{x}_b = f_b(x_a,\, x_b) + (0,\, \bar{v}_{b2},\, 0,\, \bar{v}_{b4} + u_2)^T \qquad (9)$$

where the new terms: $\bar{v}_{a2}$, $\bar{v}_{b2}$, $\bar{v}_{a4}$, and $\bar{v}_{b4}$ are unmodelled dynamics. Because the new terms vanish as $v_f = 0$, we *assume* that they are bounded with respect to a special class v_f such as $\|p_1(X) \cdot p_2(v_f)\|_\infty < d$ for $X \in R^8$ and a scalar d.

3. MAIN RESULT

With reference to (8) and (9), our control objective is reduced to assign a high-low gain control function to (u_1, u_2) such that the close loop system is ISS from the disturbances with restriction.

By referring to system (8)-(9), we choose the control laws

$$u_1 = -K_P(K_D X_7 + X_5 + \Lambda_1), \qquad (10)$$

$$u_2 = -K_P(K_D X_8 + X_6 + \Lambda_2), \qquad (11)$$

with

$$\Lambda_1 = \lambda_2 \sigma\left(\frac{K_2}{\lambda_2}\left(X_3 + \lambda_1 \sigma\left(\frac{K_1}{\lambda_1} X_1\right)\right)\right),$$

$$\Lambda_2 = \lambda_2 \sigma\left(\frac{K_2}{\lambda_2}\left(X_4 + \lambda_1 \sigma\left(\frac{K_1}{\lambda_1} X_2\right)\right)\right),$$

where K_P, K_D, K_i, λ_i are positive design parameters. It can be shown that the previous control laws can be tuned so as to achieve ISS of the closed-loop system without restrictions on the initial state, arbitrary large restrictions on the inputs $(\bar{v}_{a,4}, \bar{v}_{b,4})$ and sufficiently small restrictions on the inputs $(\bar{v}_{a,2}, \bar{v}_{b,2})$. This is precisely stated in the next proposition.

Proposition 3.1. Let Δ be a positive arbitrary number. Let the control laws be chosen as in (10)-(11) with (λ_i, K_i), $i = 1, 2$ chosen so that

$$\lambda_1 = \epsilon \lambda_1^\star, \quad \lambda_2 = \epsilon^2 \lambda_2^\star, \quad K_1 = \epsilon K_1^\star, \quad K_2 = \epsilon K_2^\star \quad (12)$$

where ϵ is a positive design parameters and $(\lambda_i^\star, K_i^\star)$ are such that

$$\begin{array}{ll} \dfrac{\lambda_2^\star}{K_2^\star} < \dfrac{\lambda_1^\star}{4}, & 24\dfrac{K_1^\star}{K_2^\star} < \dfrac{1}{6}, \\ v_{1,M}^\star \le \dfrac{\lambda_1^\star}{4}, & v_{2,M}^\star + 4K_1^\star \lambda_1^\star < \dfrac{\lambda_2^\star}{4} \end{array} \qquad (13)$$

for some positive $(v_{1,M}^\star, v_{2,M}^\star)$. Then there exist $K_D^\star > 0$, $\epsilon^\star > 0$, $K_P(K_D, \epsilon, \Delta)$ such that for all positive $K_D \le K_D^\star$, $\epsilon \le \epsilon^\star$ and $K_P \ge K_P^\star$, system (8)-(9) in closed-loop with (10)-(11) is ISS with restrictions $(\epsilon^2 v_{1,M}^\star/4, \Delta, \epsilon^2 v_{1,M}^\star/4, \Delta)$ on the inputs $(\bar{v}_{a,2}, \bar{v}_{a,4}, \bar{v}_{b,2}, \bar{v}_{b,4})$ and no restriction on the initial state.

Remark 2. As shown in Marconi & Isidori [2001], it is always possible to choose the variables $(\lambda_i^\star, K_i^\star)$ so that (13) hold for some positive $(v_{1,M}^\star, v_{2,M}^\star)$. As a matter of fact, take

$$\lambda_1^\star = \kappa c_1\,, \quad K_1^\star = \kappa \ell\,, \quad \lambda_2^\star = \kappa^2 c_2\,, \quad K_2^\star = \kappa \ell^2 \quad (14)$$

with

$$c_2 = \frac{\nu\, \ell^2}{4}\, c_1\,, \qquad (15)$$

where κ, c_1 and ν are arbitrary positive coefficients with $0 < \nu < 1$ and ℓ is a positive design parameter. This particular choice renders the first two inequalities in (13) fulfilled for any $\ell > 0$ by inspection. Furthermore, simple computations show that also the last two inequalities in (13) are satisfied for sufficiently small $v_{i,\mathrm{M}}$ if ℓ is chosen sufficiently large. $\lhd$

In the next part of the section we prove the previous proposition. The idea is to study the feedback interconnection of systems (8) and (9) controlled via u_1 and u_2 by means of small gain arguments. Instrumental in the stability analysis is the study of the ISS properties of a system of the form

$$\begin{aligned} \dot{\eta}_1 &= \eta_2 + h_1(\eta_3, \eta_4) + v_1 \\ \dot{\eta}_2 &= \eta_3 + h_2(\eta_3, \eta_4) + v_2 \\ \dot{\eta}_3 &= \eta_4 \\ \dot{\eta}_4 &= -K_P(K_D \eta_4 + \eta_3 + \Lambda) + v_3 \\ \Lambda &= \lambda_2 \sigma\left(\frac{K_2}{\lambda_2}\left(\eta_2 + \lambda_1 \sigma\left(\frac{K_1}{\lambda_1}\eta_1\right)\right)\right) \end{aligned} \qquad (16)$$

with inputs (v_1, v_2, v_3) in which $h_1(\cdot, \cdot)$ and $h_2(\cdot, \cdot)$ are higher order functions of their arguments vanishing at the origin.

The main asymptotic properties of system (16) are presented in the next two claims.

Claim 1. *Consider system*

$$\begin{aligned} \dot{\eta}_3 &= \eta_4 \\ \dot{\eta}_4 &= -K_P(K_D \eta_4 + \eta_3 + \Lambda) + v_3 \end{aligned} \qquad (17)$$

with inputs (Λ, v_3). For any $K_D > 0$ there exist a $\gamma > 0$ and a $K_P^\star$ such that for any $K_P \ge K_P^\star$, system (17) is ISS with respect to the inputs (Λ, v_3) without restrictions and with asymptotic gain $(\gamma, \gamma/K_P)$, namely

$$\|(\eta_3, \eta_4)\|_a \le \frac{\gamma}{K_P} \max\{K_P \|\Lambda\|_\infty,\, \|v_3\|_\infty)\}\,.$$

Proof Consider the change of variables $\eta_4 \mapsto \tilde{\eta}_4 := \eta_4 + \eta_3/K_D$ which transforms system (17) into

$$\begin{aligned} \dot{\eta}_3 &= -\frac{1}{K_D}\eta_3 + \tilde{\eta}_4 \\ \dot{\tilde{\eta}}_4 &= -K_P(K_D \tilde{\eta}_4 + \Lambda) + v_3 - \frac{1}{K_D^2}\eta_3 + \frac{1}{K_D}\tilde{\eta}_4 \end{aligned}$$

From this the result follows by standard Lyapunov arguments which, for sake of compactness, are not repeated. $\square$

Note that, as a consequence of the previous claim which states ultimate boundedness of the state (η_3, η_4), it is possible to argue the existence of a positive number L_H and time T such that $\|h_i(\eta_3(t), \eta_4(t))\| \le L_H \|(\eta_3(t), \eta_4(t))\|$ for all $t \ge T$, with L_H an upper bound of the Lipschitz constants of the functions $h_i(\cdot)$. Indeed, from now on, we shall take advantage of the previous bound in our analysis (by assuming, without loss of generality, that $T = 0$).

Claim 2. *Consider system (16) and let (λ_i, K_i), $i = 1, 2$ be chosen so that (12)-(13) are fulfilled for some positive $(v_{1,M}^\star, v_{2,M}^\star)$. Then there exist $K_D^\star > 0$ and $\epsilon^\star > 0$ and, for any positive $K_D \le K_D^\star$, $\epsilon \le \epsilon^\star$ and Δ, there exists a $K_P^\star(K_D, \Delta, \epsilon) > 0$ such that for any $K_P \ge K_P^\star$ system (16) is ISS with restrictions $(\epsilon \frac{v_{1,M}^\star}{2}, \epsilon^2 \frac{v_{2,M}^\star}{2}, \Delta)$ on the inputs (v_1, v_2, v_3), no restrictions on the initial state, and with the following asymptotic bounds on the state*

$$\begin{aligned} \|\eta_1\|_a &\le \rho \max\{\frac{1}{\epsilon}\|v_1\|_\infty \ \frac{1}{\epsilon^2}\|v_2\|_\infty \ \frac{1}{\epsilon^2 K_P}\|v_3\|_\infty\} \\ \|\eta_2\|_a &\le \rho \max\{\|v_1\|_\infty \ \frac{1}{\epsilon}\|v_2\|_\infty \ \frac{1}{\epsilon K_P}\|v_3\|_\infty\} \\ \|(\eta_3, \eta_4)\|_a &\le \rho \max\{\epsilon\|v_1\|_\infty \ \|v_2\|_\infty \ \frac{1}{K_P}\|v_3\|_\infty\} \end{aligned} \qquad (18)$$

where ρ is a fixed positive number.

Proof Consider the change of variables

$$\eta_1 \mapsto z_1 = \eta_1 \qquad \eta_2 \mapsto z_2 = \eta_2 + \lambda_1 \sigma(\frac{K_1}{\lambda_1}\eta_1)$$

$$\eta_3 \mapsto z_3 = \eta_3 + \lambda_2 \sigma(\frac{K_2}{\lambda_2} z_2) \qquad \eta_4 \mapsto z_4 = \eta_4 + \frac{1}{K_D} z_3 \,.$$

System (16) in the new coordinates can be seen as the interconnection of two subsystems. The first is a system of the form

$$\begin{aligned} \dot z_1 &= -\lambda_1 \sigma(\frac{K_1}{\lambda_1} z_1) + z_2 + h_1(-\lambda_2 \sigma(\frac{K_2}{\lambda_2} z_2), 0) \\ &\qquad + \Delta h_1(z_3, z_4, t, K_D) + v_1 \\ \dot z_2 &= -\lambda_2 \sigma(\frac{K_2}{\lambda_2} z_2) + K_1 \sigma'(\cdot) \dot z_1 + h_2(-\lambda_2 \sigma(\frac{K_2}{\lambda_2} z_2), 0) \\ &\qquad + z_3 + \Delta h_2(z_3, z_4, t, K_D) + v_2 \end{aligned} \tag{19}$$

with inputs $(\Delta h_1(z_3, z_4, t, K_D), z_3 + \Delta h_2(z_3, z_4, t, K_D))$ and (v_1, v_2) and output

$$\begin{aligned} y_z &:= K_2 \sigma'(\frac{K_2}{\lambda_2} z_2)[\dot z_2 - z_3 - \Delta h_2(z_3, z_4, t, K_D) - \\ &\qquad K_1 \sigma'(\cdot) \Delta h_1(z_3, z_4, t, K_D)] \end{aligned}$$

where Δh_i, $i = 1, 2$, are higher order functions vanishing at $(z_3, z_4) = (0, 0)$ for all t and K_D defined as [1]

$$\begin{aligned} \Delta h_i(z_3, z_4, t, K_D) &:= h_i(z_3 - \lambda_2 \sigma(\frac{K_2}{\lambda_2} z_2(t)), z_4 - \frac{1}{K_D} z_3) \\ &\qquad - h_i(-\lambda_2 \sigma(\frac{K_2}{\lambda_2} z_2(t)), 0)\,. \end{aligned}$$

The second subsystem is a system of the form

$$\begin{aligned} \dot z_3 &= -\frac{1}{K_D} z_3 + z_4 - K_2 \sigma'(\cdot) L(z_3, z_4, t, K_D) + y_z \\ \dot z_4 &= -K_P K_D z_4 + \frac{1}{K_D}\Big[-\frac{1}{K_D} z_3 + z_4 - \\ &\qquad K_2 \sigma'(\cdot) L(z_3, z_4, t, K_D) + y_z\Big] + v_3 \end{aligned} \tag{20}$$

with inputs (y_z, v_3) and output (z_3, z_4) where $L(\cdot)$ is a Locally Lipschitz function vanishing at $(z_3, z_4) = (0, 0)$ defined as

$$\begin{aligned} L(z_3, z_4, t, K_D) &:= z_3 + \Delta h_2(z_3, z_4, t, K_D) + \\ &\qquad K_1 \sigma'(\cdot) \Delta h_1(z_3, z_4, t, K_D)\,. \end{aligned}$$

Note that, by definition, it is possible to argue the existence of a fixed positive constant $\bar L$ such that $\|L(\cdot)\| \le \bar L \|(\frac{z_3}{K_D}, z_4)\|$. As far as the first subsystem is concerned, the arguments in Isidori *et al* [2003] can be used to prove that there exists an $\epsilon_1^\star > 0$ such that for all positive $\epsilon \le \epsilon_1^\star$ such a system is ISS with restrictions $(\epsilon \frac{v_{1,M}^\star}{2}, \epsilon^2 \frac{v_{2,M}^\star}{2})$ and $(\epsilon \frac{v_{1,M}^\star}{2}, \epsilon^2 \frac{v_{2,M}^\star}{2})$ on the inputs $(\Delta h_1(\cdot), z_3 + \Delta h_2(\cdot))$ and (v_1, v_2), no restrictions on the initial state and the following asymptotic bounds on the states

$$\begin{aligned} \|z_1\|_a &\le c_1 \max\{\frac{1}{K_1}\|\Delta h_1\|_a,\ \frac{1}{K_1 K_2}\|z_3 + \Delta h_2\|_a, \\ &\qquad \frac{1}{K_1}\|v_1\|_a\,,\ \frac{1}{K_1 K_2}\|v_2\|_a\} \\ \|z_2\|_a &\le c_2 \max\{\frac{K_1}{K_2}\|\Delta h_1\|_a,\ \frac{1}{K_2}\|z_3 + \Delta h_2\|_a, \\ &\qquad \frac{K_1}{K_2}\|v_1\|_a\,,\ \frac{1}{K_2}\|v_2\|_a\} \end{aligned}$$

with c_1, c_2 fixed positive numbers. In particular, by definition of y_z and by bearing in mind the definition of saturation function and (12), it turns out that the following asymptotic bound on y_z can be computed

$$\begin{aligned} \|y_z\|_a &\le c_3 \max\{\epsilon\|\Delta h_1\|_a\,,\ \|z_3 + \Delta h_2\|_a\,,\ \epsilon\|v_1\|_a\,,\ \|v_2\|_a\} \\ &\le c_3 \max\{\epsilon\|\Delta h_1\|_a\,,\ 2\|z_3\|_a,\ 2\|\Delta h_2\|_a\,, \\ &\qquad \epsilon\|v_1\|_a\,,\ \|v_2\|_a\} \end{aligned} \tag{21}$$

where c_3 is a fixed positive number. Furthermore, the following bound on y_z can be computed (in the computation it is argued that $|z_2| \le \lambda_2/K_2$ otherwise $\sigma'(K_2 z_2/\lambda_2) \equiv 0$ by definition of saturation function)

$$\begin{aligned} \|y_z\|_\infty &\le K_2\,[\lambda_2 + L_h\lambda_2 + \|v_2\|_\infty + \\ &\qquad K_1(\lambda_1 + \frac{\lambda_2}{K_2} + L_h\lambda_2 + \|v_1\|_\infty)\Big] \\ &\le \Gamma_1\epsilon^2 + \Gamma_2\epsilon^3 + \Gamma_4\epsilon^4 + \Gamma_5\epsilon^2\|v_1\|_\infty + \Gamma_6\epsilon\|v_2\|_\infty \end{aligned} \tag{22}$$

where L_h is un upper bound of the Lipschitz constants of $h_1(\cdot, 0)$ and $h_2(\cdot, 0)$ and $\Gamma_i > 0$ are fixed positive numbers. This, in turn, implies that

$$(\|v_1\|_\infty\,,\ \|v_2\|_\infty) \le (\epsilon \frac{v_{1,M}^\star}{2}\,,\ \epsilon^2 \frac{v_{2,M}^\star}{2}) \ \Rightarrow\ \|y_z\|_\infty \le R\epsilon^2 \tag{23}$$

for some fixed $R > 0$.

As far as the second subsystem (20) is concerned, standard ISS Lyapunov arguments can be used to prove that there exists an $\epsilon_2^\star > 0$ and, for any $\gamma > 0$, there exists a $K_D^\star(\gamma) > 0$ and a $K_P^\star(K_D, \gamma) > 0$ such that for any positive $K_D \le K_D^\star(\gamma)$, $K_P \ge K_P^\star(K_D, \gamma)$ and [2] $\epsilon \le \epsilon_2^\star$ the system in question is ISS without any kind of restriction and with the following asymptotic bound on the state

$$\|(z_3, z_4)\|_a \le \gamma \max\{\|y_z\|_a\,,\ \frac{1}{K_P}\|v_3\|_a\}\,. \tag{24}$$

By the previous results it is possible to carry out the stability analysis of the overall system (19), (20) by means of small gain arguments as addressed in the following.

First of all we observe that system (16) has not finite escape time as it behaves as a linear system driven by bounded inputs (to this respect the terms $h_1(\cdot)$ and $h_2(\cdot)$ can be regarded as bounded inputs due to claim 1 as $\|\Lambda\| \le \lambda_2$ by definition of saturation function).

We prove now that the restrictions $(\epsilon v_{1,M}^\star/2,\ \epsilon^2 v_{2,M}^\star/2)$ on the inputs $(\Delta h_1, z_3 + \Delta h_2)$ of system (19) are fulfilled in finite time and that the small gain conditions associated to the interconnection (19), (20) are satisfied for a proper tuning of the design parameters. To this purpose note that, in order to have the restrictions fulfilled in finite time, it is sufficient to prove that (assuming without loss of generality that $\epsilon < 1$)

$$\|z_3\|_a \le \epsilon^2 \frac{v_{2,M}^\star}{8} \qquad \|\Delta h_i\|_a \le \epsilon^2 \frac{\min\{v_{1,M}^\star,\ v_{2,M}^\star\}}{8} := r\epsilon^2\,. \tag{25}$$

We analyze the overall system by focusing first on the interconnection taking place through the input z_3 of (19) and y_z of (20). Fix $\gamma > 0$ so that $\gamma \le \frac{v_{2,M}^\star}{8R}$ and $\gamma \le \frac{1}{2c_3}$ and $K_D^\star(\gamma)$, $K_{P1}^\star(K_D, \gamma)$ and $\epsilon_2^\star$ according to the previous considerations so that for any positive $K_D \le K_D^\star(\gamma)$, $K_P \ge K_{P1}^\star(K_D, \gamma)$ and $\epsilon \le \epsilon_2^\star$ the bound (24) holds true. This fact, along with (21) and (23) (and by using $\|z_3\| \le \|(z_3, z_4)\|$) yield that the small gain condition linked to the input z_3 of (19) is satisfied and the first inequality in (25) is fulfilled provided that $\epsilon \le \min\{\epsilon_1^\star,\ \epsilon_2^\star\}$, $\|v_3\|_\infty \le \Delta$ and

[1] Note that, by taking advantage from the definition of saturation function, the z_2 entry in Δ_{h_i} is considered as a time-varying bounded signal.

[2] The small value of ϵ is required to get rid of the presence of K_D in the definition of L.

$K_P \geq \max\{K^\star_{P1}\ K^\star_{P2}\}$ with $K^\star_{P2} \geq \frac{8\gamma\Delta}{\epsilon^2 v^\star_{2,M}}$ where Δ is an arbitrary positive number.

From now on, we consider the design parameter K_D fixed *once for all* so that $K_D \geq K^\star_D(\gamma)$ and we pass to analyze the interconnection thorough the inputs $(\Delta h_1, \Delta h_2)$ of (19) and y_z of (20). To this respect note that the fact that the functions $\Delta h_i(\cdot)$ are higher order, imply that for any $\nu > 0$ there exists a $\delta_\nu > 0$ such that $\|(z_3, z_4)\| \leq \delta_\nu \Rightarrow \|\Delta_i(z_3, z_4)\| \leq \nu\|(z_3, z_4)\|$, $i = 1, 2$. Now fix $\nu > 0$ so that $\nu \leq \min\{\frac{r}{\gamma R}, \frac{1}{2c_3\gamma}\}$ and δ_ν accordingly. Furthermore, with Δ an arbitrary positive number, let $\epsilon^\star_3 \leq \frac{\delta_\nu}{2R\gamma}$ and $K^\star_{P3}(\epsilon) \geq \max\{\frac{2\gamma\Delta}{\delta_\nu},\ \frac{\nu\gamma\Delta}{\epsilon^2 r}\}$. By bearing in mind (21), (23) and (24) and assuming without loss of generality $\epsilon \leq 1$, it turns out that for all $\epsilon \leq \min\{\epsilon^\star_1,\ \epsilon^\star_2,\ \epsilon^\star_3\}$ and $K_P \geq \max\{K^\star_{P1},\ K^\star_{P2}(\epsilon)\}$ the second relation (25) is satisfied (namely the restriction on the inputs $(\Delta h_1, \Delta h_2)$ of (19) are fulfilled in finite time) and the small gain conditions linked to the inputs $(\Delta h_1, \Delta h_2)$ of (19) are fulfilled.

According to the results in Isidori *et al* [2003], the previous considerations guarantee that the overall system is ISS with restrictions $(\epsilon\frac{v^\star_{1,M}}{2}, \epsilon^2\frac{v^\star_{2,M}}{2}, \Delta)$ on the inputs (v_1, v_2, v_3) and asymptotic bound which, by gain composition, by bearing in mind the definition of the z_i and of saturation function, can be estimated as in (18) (end proof Claim 2). □

The results in Claim 1 and Claim 2 contain all what is needed to prove Proposition 3.1. To this purpose note, first of all, that claim 1, applied respectively to the last two equations of (8) and (9) with u_1 and u_2 chosen as in (10)-(11), yields that the state variable (X_5, X_7) and (X_6, X_8) are ultimately bounded and

$$\begin{aligned} \|(X_5, X_7)\|_a &\leq \bar\gamma \max\{\epsilon^2,\ \frac{\Delta}{K_P}\} \\ \|(X_6, X_8)\|_a &\leq \bar\gamma \max\{\epsilon^2,\ \frac{\Delta}{K_P}\} \end{aligned} \tag{26}$$

for some positive $\bar\gamma$. This guarantees that system (8)-(9) does not have finite escape time as it behaves as a linear system driven by bounded inputs.

By bearing in mind these facts, note that system (8)-(9) can be interpreted as the feedback interconnection of a first subsystem

$$\begin{aligned} \dot X_1 &= X_3 + \varphi_1(X_5, 0, X_7, 0) + \Delta\varphi_1(X_6, X_8, t) \\ \dot X_3 &= X_5 + \varphi_3(X_5, 0, X_7, 0) + \Delta\varphi_3(X_6, X_8, t) + \bar v_{a2} \\ \dot X_5 &= X_7 \\ \dot X_7 &= u_1 + \bar v_{a4} \end{aligned} \tag{27}$$

with[3] $\Delta\varphi_i(X_6, X_8, t) = \varphi_i(X_5(t), X_6, X_7(t), X_8) - \varphi_i(X_5(t), 0, X_7(t), 0)$, $i = 1, 3$, which is regarded as a system with inputs $(\Delta\varphi_1, \Delta\varphi_3)$ and $(\bar v_{a2}, \bar v_{a4})$ and output (X_5, X_7), and a second subsystem

$$\begin{aligned} \dot X_2 &= X_4 + \varphi_2(0, X_6, 0, X_8) + \Delta\varphi_2(X_5, X_7, t) \\ \dot X_4 &= X_6 + \varphi_4(0, X_6, 0, X_8) + \Delta\varphi_4(X_5, X_7, t) + \bar v_{b2} \\ \dot X_6 &= X_8 \\ \dot X_8 &= u_2 + \bar v_{b4} \end{aligned} \tag{28}$$

with $\Delta\varphi_i(X_5, X_7, t) = \varphi_i(X_5, X_6(t), X_7, X_8(t)) - \varphi_i(0, X_6(t), 0, X_8(t))$, $i = 2, 4$, which is regarded as a system with inputs $(\Delta\varphi_2, \Delta\varphi_4)$ and $(\bar v_{b2}, \bar v_{b4})$ and output (X_6, X_8). Note that $\Delta\varphi_i(\cdot, \cdot, t)$ are higher order functions in their arguments for all $t \geq 0$ and vanishing at the origin. We shall study such a interconnection by small gain arguments. To this respect note that either system (27) and (28) are described in the form (16) with $\eta_i = X_{2i-1}$, $h_i(\cdot) = \varphi_{2i-1}$, $v_1 = \Delta\varphi_1 + v_{a1}$, $v_2 = \Delta\varphi_3$ and $v_3 = v_{a4}$ in the case of system (27) and $\eta_i = X_{2i}$, $h_i(\cdot) = \varphi_{2i}$, $v_1 = \Delta\varphi_2 + v_{b1}$, $v_2 = \Delta\varphi_4$ and $v_3 = v_{b4}$ in the case of system (28). Thus, by the previous claim 2, it follows that there exist $K^\star_D > 0$ and $\epsilon^\star_1 > 0$ and, for all positive $K_D \leq K^\star_D$, $\epsilon \leq \epsilon^\star_1$ and Δ, a $K^\star_{P1}(K_D, \Delta, \epsilon)$ such that for any $K_P \geq K^\star_{P1}$ the two subsystems are ISS with the same restrictions $(\epsilon\frac{v^\star_{1,M}}{2}, \epsilon^2\frac{v^\star_{2,M}}{2}, \Delta)$ on the inputs $(\Delta\varphi_1, \Delta\varphi_3 + v_{a2}, v_{a4})$ and $(\Delta\varphi_2, \Delta\varphi_4 + v_{b2}, v_{b4})$, no restrictions on the initial state and the following asymptotic bounds hold true

$$\begin{aligned} \|(X_5, X_7)\|_a &\leq \rho \max\{\epsilon\|\Delta\varphi_1\|_a, \|\Delta\varphi_3 + v_{a2}\|_a, \frac{1}{K_P}\|v_{a4}\|_a\} \\ &\leq \bar\rho \max\{\epsilon\|\Delta\varphi_1\|_a\,, \|\Delta\varphi_3\|_a\,, \|v_{a2}\|_a\,, \\ &\qquad \frac{1}{K_P}\|v_{a4}\|_a\} \\ \|(X_6, X_8)\|_a &\leq \bar\rho \max\{\epsilon\|\Delta\varphi_2\|_a\,, \|\Delta\varphi_4\|\,, \|v_{b2}\|_a\,, \\ &\qquad \frac{1}{K_P}\|v_{b4}\|_a\}\,. \end{aligned} \tag{29}$$

where $\bar\rho$ is a fixed positive number. In the final part of the section we prove that, by a proper choice of the design parameters, the restrictions of two subsystems are fulfilled in finite time and that the small gain conditions are satisfied. To this purpose note that the restrictions are fulfilled in finite time if $\|\bar v_{j2}\|_\infty \leq \epsilon^2 v^\star_{1,M}/4$, $\|\bar v_{j4}\|_\infty \leq \Delta$, $j = a, b$, and (assuming without loss of generality $\epsilon < 1$)

$$\|\Delta\varphi_i\|_a \leq \frac{\epsilon^2}{8}\max\{v^\star_{1,M},\ v^\star_{2,M}\} := r\epsilon^2 \quad i = 1, \ldots, 4\,. \tag{30}$$

Since $\Delta\varphi_i$ are higher order, it follows that for any $\nu > 0$ there exists a $\delta_\nu > 0$ such that $\|(X_{5+i}, X_{7+i})\| \leq \delta_\nu \Rightarrow \|\Delta\varphi_{2-i}(\cdot)\| \leq \nu\|(X_{5+i}, X_{7+i})\|$ and $\|\Delta\varphi_{4-i}(\cdot)\| \leq \nu\|(X_{5+i}, X_{7+i})\|$, $i = 0, 1$. Now fix ν so that $\nu \leq \min\{\frac{r}{\bar\gamma}, \bar\rho\}$ and δ_v accordingly. Furthermore let $\epsilon^\star_2 = \frac{\delta_\nu}{2\bar\gamma}$, $K^\star_{P2}(\epsilon) = \max\{\frac{\bar\gamma\Delta\nu}{r\epsilon^2},\ \frac{2\bar\gamma\Delta}{\delta_\nu}\}$. By bearing in mind (26) and (29), it turns out that for any positive $\epsilon \leq \min\{\epsilon^\star_1,\ \epsilon^\star_2\}$ and $K_P \geq \max\{K^\star_{P1},\ K^\star_{P2}\}$ relations (30) are satisfied (namely the restrictions are fulfilled in finite time) and the small gain conditions are satisfied. From this, the claim of Proposition 3.1 follows by means of the appropriate small gain theorem (see, for instance, Isidori *et al* [2003]).

4. SIMULATION RESULTS

By going through the proof of the main result in the previous section, it is possible to identify a practical procedure for tuning the design parameters which articulates in the following five steps:

(1) Fix $(\lambda^\star_i, K^\star_i)$ in such a way that (13) is satisfied by using the design procedure described in the remark just after the proposition.
(2) Choose (λ_i, K_i) as in (12) in terms of the design parameter ϵ yet to be chosen.
(3) Compute the value of $K^\star_D$ and preliminary values for $K^\star_P$ and $\epsilon^\star$ by working on system (20) (see the reasonings in the proof of Claim 2). In particular $K^\star_D$ must be sufficiently small and, accordingly, $K^\star_P$ sufficiently large so that system (20) has a sufficiently

[3] Note that, by ultimate boundedness of last two state variables of (8), the (X_5, X_7) entries in $\Delta\varphi_i$ can be regarded as time-varying bounded "exogenous" signals.

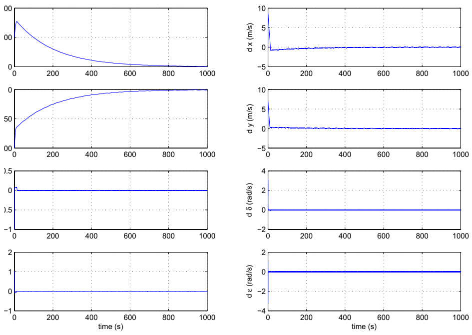

Fig. 2. Simulation results of the closed loop system (1) under certain disturbances

small asymptotic gain γ. From a practical viewpoint one should simulate system (20) by calibrating the design parameters until the asymptotic effect of the inputs (y_z, v_3) (which could be taken constant in this calibration phase) on the state is sufficiently small. The function h_i in system (20) should be taken according to the consideration after (28).

(4) Once obtained ISS of system (20) with "small" asymptotic gains, move to the interconnection (19)-(20). Alternatively it is possible to simulate system (27) by considering the term $\Delta\varphi_1$ and $\Delta\varphi_3$ as exogenous small (i.e. whose amplitude decreases with ϵ^2) inputs. By possibly decreasing further K_D and ϵ and, accordingly, increasing K_P it is possible to obtain ISS of the overall interconnection as proved in Claim 2. From now the value of K_D is *fixed*.

(5) Complete the design by decreasing ϵ and, accordingly, increasing K_P by making the overall system (27)-(28) ISS.

A detailed tuning example is not given here due to the space limitation.

Fig. 2 shows a simulation result under some small external disturbances and a set of initial conditions: $x = 100(m)$, $\dot{x} = 1(m/s)$, $y = -100(m)$, $\dot{y} = 1(m/s)$, $\delta = -1(rad)$, $\dot{\delta} = 1(rad/s)$, $\varepsilon = 1(rad)$, $\dot{\varepsilon} = 1(rad/s)$ (starting from a non-local domain).

Remark 3. The simulation result is similar to Liu *et al* [2008b]. Practically, it is very hard to quantify how much is improved from the original design since they are based on a similar design idea. In fact, our main contribution here is to make the tuning rules more explicit, in particular, in dealing with the disturbances. Besides the similarity, the specific design procedure of the low gain part here is in a flavor of combining the backstepping tool with saturations (see the proof of **Claim 2**) while the corresponding part in Liu *et al* [2008b] is derived by directly applying classical forwarding tool Teel [1996].

5. CONCLUSION

The modified high and low gain controller proposed here improves the original design in Liu *et al* [2008b] as follows: less layers in nested saturations are used; explicit tuning rules are provided to deal with the disturbances.

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