

Synchronization of a Complex Network with Switched Coupling

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Abstract: Synchronization of a complex network with switched coupling is considered. In particular, we establish a simple method to obtain a proper coupling strength for the network. Via switched system theory, with this coupling strength, we achieve synchronization of the network without requiring synchronizability of all possible configurations. First, we convert the network into a switched system with the reduced number of subsystems which are static complex networks whose configurations are non isomorphic via partitioning all possible configurations generated from the switched coupling. Then, we obtain a finite number of coupling strength candidates from the subsystems. By applying the *average dwell time approach*, synchronization of the network is indeed achieved with the coupling strength chosen among the candidates.

1. INTRODUCTION

A complex network is a large scale system which consists of nodes and links represented by a graph with a complex structure (connection topology). The nodes are the elements of the system and the links represent the interactions among them. Examples include the world wide web (WWW), food and metabolic pathways, power grid and the Internet (Wang and Chen, 2002). In spite of its complex structure and large number of nodes, one emerging property from complex networks is *synchronization*, i.e. *coherence of the nodes* in the networks. Synchronization can be easily observed from fireflies flashing in unison, heart cells beating in rhythm to generators' rotating with the same frequency in a power network.

One of the major interests in the study on synchronization of a complex network is obtaining a proper coupling strength which characterizes how strongly the nodes are connected and interact in the network. So, it has been theoretically and numerically explored by (Pecora and Carroll, 1998; Wang and Chen, 2002; Newman, 2003; Fan and Wang, 2005). These results provide various synchronization criteria including a lower bound for the coupling strength in a complex network at which synchronization will occur.

Recently, complex networks with switched coupling which involves link switchings, have demanded attention due to applications of complex networks to telecommunication networks and power systems, etc. The above networks switch links over time in order to control communication and generated electricity flows subject to circumstance changes. However, there were a few works done and the investigation on synchronization of such networks was mostly focused on fast switching topology (Belykh et al., 2004; Stilwell et al., 2006) for a given coupling strength. As a control point of view, the fast switching property may not be always desirable for control design. In addition, how

to choose a proper coupling strength for networks with switching topology will be a challenging task comparing to static networks.

In this paper, we tackle the synchronization problem of a complex network with switched coupling and provide a criterion on how to determine its coupling strength via switched system theory. To do so, we firstly convert the network into a switched system whose subsystems are static networks with non isomorphic connection configurations obtained by partitioning of all possible connection configurations from the switching topology of the network. Secondly, we obtain the finite number of coupling strength candidates from each subsystem and choose one among them. Finally, with this coupling strength, we achieve stability of the switched system via the *average dwell time approach* (Zhai et al., 2001) which implies synchronization of the original network. By doing this, we achieve synchronization of the network without requiring a fast switching strategy and synchronizability of all possible network configurations.

The organization of this paper is as follows: In Section 2, we formulate our model and address necessary assumptions, and in Section 3, we analyze the synchronization problem of a complex network with switched coupling in terms of the stability problem of a switched system. In Section 4, we show an example of a complex network consisting of four Chua's chaotic oscillators. Finally, in Section 5, we discuss the results of the paper and provide possible future research areas.

2. PROBLEM FORMULATION

Consider a dynamical network consisting of N identical linearly and diffusively coupled nodes, with each node being an n -dimensional dynamical system. The state equations of the network are given by

$$\dot{x}_i = f(x_i) + c \sum_{j=1}^N a_{ij}(t) \Gamma x_j, \quad i = 1, \dots, N, \quad (1)$$

where $x_i = (x_{i1}, \dots, x_{in}) \in \mathcal{R}^n$ is the state of the node i , $f \in C^1[\mathcal{R}^n, \mathcal{R}^n]$, $c > 0$ is the coupling strength, and $\Gamma = \text{diag}(r_1, \dots, r_n)$ is the inner coupling matrix with $r_i = 1$ for a particular i and $r_j = 0$ for $i \neq j$, which means that two coupled nodes are linked through their i^{th} state variables. Here, $a_{ij}(t)$ is given by

$$a_{ij}(t) = \begin{cases} 1 & \text{if the nodes } i \text{ and } j \text{ are connected at } t, \\ 0 & \text{otherwise,} \end{cases} \quad (2)$$

and

$$a_{ii}(t) = - \sum_{j=1, i \neq j}^N a_{ij}(t) \text{ for all } t. \quad (3)$$

For the network (1), we are particularly interested in *synchronization* which is stated in the following definition:

Definition 1. The network (1) is said to be synchronized if

$$x_1(t) = x_2(t) = \dots = x_N(t) = s(t) \text{ as } t \rightarrow \infty \quad (4)$$

where $s(t) \in \mathcal{R}^n$ is the solution of an isolated node, namely, $\dot{s}(t) = f(s(t))$.

Let $A(t) = [a_{ij}(t)]_{N \times N}$ be the network *configuration matrix*, or *outer link matrix* of (1). Note that it is symmetric and has a zero eigenvalue by (3). This guarantees the existence of the synchronization manifold for (1). If the coupling strength is properly chosen, the network will synchronize (Wang and Chen, 2002). Thus, finding a proper coupling strength is important.

Note. (a) For convenience, we will address a network given with a node dynamics by a smooth function f , a configuration matrix G , and a coupling strength c with $\mathcal{N}(f, G, c)$. Throughout the paper, f in (1) will represent the node dynamics in all networks in later sections. (b) The network (1) is a time-varying system in a sense that its network configuration is varying according to the switchings of $a_{ij}(t)$ between 0 and 1 over time. Hence, whenever we address (1) as a time-varying network, the meaning is in the above sense.

Assumption. $a_{ij}(t)$ given in (2) switches on with the value 1 and off with the value 0 such that $A(t)$ represents a connected network at all time.

The above assumption guarantees that $A(t)$ will have zero and $n - 1$ non-zero real eigenvalues for all t . This is an important assumption for the existence of the synchronization manifold for the network $\mathcal{N}(f, A(t), c)$ (Newman, 2003).

Let $e_i(t) = x_i(t) - s(t)$ be the error between the state x_i and the synchronized state $s(t)$, $i = 1, \dots, N$. Let $e(t) = [e_1(t), \dots, e_N(t)] \in \mathcal{R}^{n \times N}$. By linearizing (1) about $s(t)$ and using $A(t)$, we have the error dynamics for synchronization

$$\dot{e}(t) = J(t)e(t) + c\Gamma e(t)A^T(t), \quad (5)$$

where $J(t) = f'(s(t)) \in \mathcal{R}^{n \times n}$ is the Jacobian of $f(x_i(t))$ at $s(t)$. If $\|e(t)\| \rightarrow 0$ as $t \rightarrow \infty$, then we achieve (4), i.e. the time-varying network (1) reaches synchronization.

Note that in a connected network with N nodes there will be at most $N(N - 1)/2$ links. Hence, the time-varying

configuration $A(t)$ is generated by permutations of $N(N - 1)/2$ numbers of links over time. Thus, we can consider all possible configurations B_k of $A(t)$ according to the values of $a_{ij}(t)$ for all i, j . Let L be the number of permutations $\pi_{N(N-1)/2}$ of $N(N - 1)/2$ numbers of links. Then, the original time-varying network is indeed a switched system of L number of subsystems which are static networks with configuration matrices B_k and whose error dynamics is given by

$$\dot{e}(t) = J(t)e(t) + c\Gamma e(t)B_k^T, \quad k = 1, \dots, L, \quad (6)$$

with a switching signal $\sigma(t) : \mathcal{R}^+ \mapsto \{1, \dots, L\}$ (Liberzon, 2003) such that $B_{\sigma(t)} = B_k$, where $L = \pi_{N(N-1)/2}$, $B_k \in \mathcal{F}(A(t))$, and $\mathcal{F}(A(t))$ is the family of all possible configurations of $A(t)$. Hence, the *synchronization* problem of (1) is indeed a *stability* problem of (6). Thus, we would like to obtain a proper coupling strength as mentioned before and stability conditions for the switched system.

3. STABILITY ANALYSIS

In this section, we will firstly reduce the number of subsystems of interest by partitioning $\mathcal{F}(A(t))$ into equivalence classes to obtain coupling strength candidates for c in (6). Then we analyze the stability of the switched system of the reduced number of subsystems. This reduction is due to the identical node dynamics.

3.1 Equivalence Classes of $\mathcal{F}(A(t))$ and Coupling Strength Candidates

For each x_i , in (1) the node dynamics given by f is identical. Hence, although $\mathcal{F}(A(t))$ has L number of configurations of $A(t)$, there must be isomorphic configurations. The topologically isomorphic configurations will have the same eigenvalues; they will generate the same stability results since the node dynamics is identical for each node. Thus, we only need to focus on non isomorphic configurations for stability analysis. In this section, we will partition $\mathcal{F}(A(t))$ into equivalence classes by an equivalence relation which is closely related to the coupling strength and the topology of each B_k in (6).

For isomorphic equivalence classes we have the following proposition:

Proposition 2. The family of all possible configurations $\mathcal{F}(A(t))$ is partitioned into equivalence classes by topological isomorphism as follows:

$$[B_k] = \{B'_k | B_k \text{ and } B'_k \text{ are topologically same, } B'_k \in \mathcal{F}(A(t))\}, \quad (7)$$

for $k = 1, \dots, M$. Then, $\mathcal{F}(A(t)) = \bigcup_{k=1}^M [B_k]$.

Hence, we have

$$\dot{e}(t) = J(t)e(t) + c\Gamma e(t)B_k^T, \quad (8)$$

for only $k = 1, \dots, M$ with a switching signal $\sigma(t) : \mathcal{R}^+ \mapsto \{1, \dots, M\}$ as in (6), where typically $M \ll L$, and B_k is the representative of $[B_k]$. Let

$$\mathcal{B} = \{B_1, \dots, B_M\} \quad (9)$$

be a *core configuration set* in $\mathcal{F}(A(t))$. Thus, \mathcal{B} is a set of the reduced number of subsystems of (8) to consider. For synchronization of the network $\mathcal{N}(f, B_k, c)$ for B_k

from (8), we need to determine the coupling strength c as mentioned in Sec. 1. In fact, this is closely related to the second largest eigenvalue of B_k .

Note that for each k , B_k is a real symmetric matrix and has zero eigenvalue. Thus, there is a unitary matrix $U_k \in \mathcal{R}^{N \times N}$ such that $B_k^T U_k = U_k \Lambda_k$ where $\Lambda_k = \text{diag}(\lambda_{1k}, \dots, \lambda_{Nk})$, $0 = \lambda_{1k} > \lambda_{2k} \geq \dots \geq \lambda_{Nk}$, and $\lambda_{1k}, \dots, \lambda_{Nk}$ are eigenvalues of B_k , $k = 1, \dots, M$. Using nonsingular transformation $e(t)U_k = h_k(t)$, the system (8) becomes

$$\dot{h}_k(t) = J(t)h_k(t) + c\Gamma h_k(t)\Lambda_k, \text{ for } k = 1, \dots, M, \quad (10)$$

i.e., we have for each $k = 1, \dots, M$,

$$\dot{h}_{ik}(t) = (J(t) + c\lambda_{ik}\Gamma)h_{ik}(t), \text{ for } i = 1, \dots, N. \quad (11)$$

Note that for $i = 1$ we have the variational equation for the synchronization manifold ($\lambda_{1k} = 0$), and for all other i h_{ik} is transverse to the synchronization manifold. We want these variations damp out. The choice of a proper coupling strength for (11) plays a key role as stated in the following lemma:

Lemma 3. (Wang and Chen, 2002) Suppose there exist a symmetric positive definite matrix $R_{ik} > 0$ and a constant $\bar{d} < 0$ such that

$$[J(t) + d_{ik}\Gamma]^T R_{ik} + R_{ik}[J(t) + d_{ik}\Gamma] \leq -Q_{ik}, \quad (12)$$

for a symmetric positive definite matrix $Q_{ik} > 0$ and for all $d_{ik} \leq \bar{d}$. If

$$c\lambda_{2k} \leq \bar{d}, \quad (13)$$

then, (11) is exponentially stable for $i = 2, \dots, N$, and for each $k = 1, \dots, M$. Consequently, (6) is exponentially stable, i.e. the network $\mathcal{N}(f, B_k, c)$ synchronizes in the sense of (4) for each $k = 1, \dots, M$.

Note that the condition in (13) is equivalent to

$$c \geq \left\lceil \frac{\bar{d}}{\lambda_{2k}} \right\rceil. \quad (14)$$

Remark. If the coupling strength c is at least $\lceil \bar{d}/\lambda_{2k} \rceil$, $k = 1, \dots, M$, then the $N-1$ decoupled systems in (11) are exponentially stable. Hence (6) achieves stability for each k , i.e. (4) is accomplished for each k . For each $B_k \in \mathcal{B}$, we can obtain the lower bounds for c via Lemma 3.

Definition 4. Let ρ be an equivalence relation given by

$$\rho : A \equiv B \text{ if } \lambda_2(A) = \lambda_2(B), \quad (15)$$

where $A, B \in \mathcal{R}^{N \times N}$, $\lambda_2(A)$ and $\lambda_2(B)$ are the second largest eigenvalues of A and B , respectively. If $A \equiv B$ by ρ , then, $\mathcal{N}(f, A, c_A)$ and $\mathcal{N}(f, B, c_B)$ are said to have the same synchronizability, where c_A and c_B are the coupling strengths.

Remark.

(i) From Definition 4, for $k \neq l \in \{1, \dots, M\}$, B_k and B_l are topologically different in \mathcal{B} , but they can have the same lower bounds, i.e., the same synchronizability if $\lambda_2(B_k) = \lambda_2(B_l)$. Hence, we have at most M candidates of lower bounds for c , say,

$$c_1 \geq \dots \geq c_M, \quad (16)$$

where $c_k = \lceil \bar{d}/\lambda_{2k} \rceil$, $c_1 = \max_{1 \leq k \leq M} \lceil \bar{d}/\lambda_{2k} \rceil$, and $c_M = \min_{1 \leq k \leq M} \lceil \bar{d}/\lambda_{2k} \rceil$.

(ii) For example, in the K_4 configuration matrix, which is a complete graph with four nodes and three links per node,

there are totally 38 configurations according to the link switching in (2). By the topological isomorphic relation in Proposition 2 there are six equivalence classes and hence, we have $\mathcal{B} = \{B_1, \dots, B_6\}$. Moreover, by the equivalence relation ρ in (4), we have four equivalence classes in \mathcal{B} . Thus, we need to consider six different subsystems and four lower bounds candidates for c .

As a convention, we assume that each network $\mathcal{N}(f, B_k, c)$ chooses the lower bound c_k in (16) for c . Now, we have a switched system in the form (8) with M number of subsystems with at most M coupling strength candidates in (16). In the following section, we will establish a stability condition for this system which will guarantee synchronization of $\mathcal{N}(f, A(t), c)$.

3.2 Synchronization via Switching Principle

In the switched system (8), the coupling strength is not yet determined. If we choose $c = c_1$, the largest lower bound, then each subsystem will be asymptotically stable, i.e. $\mathcal{N}(f, B_k, c_1)$ synchronizes for all $k = 1, \dots, M$. However, it does not imply that the switched system (8) will be stable depending on the switching nature (Liberzon, 2003; Kim et al., 2006). Hence, we need a switching scheme to stabilize the switched system with a proper coupling strength. Moreover, the choice $c = c_1$ may not be necessary for some $\mathcal{N}(f, B_k, c)$ whose coupling strength can be lower than c_1 to achieve synchronization.

To compromise between the choice of a coupling strength and the stability of the switched system, we propose the following procedure, which is our main focus in this paper:

- (P1) Choose a lower bound, say c_o for c among $\{c_1, \dots, c_M\}$ and let $c = c_o$.
- (P2) Partition the core configuration set $\mathcal{B} = \{B_1, \dots, B_M\}$ into two classes, namely, $\mathcal{U} = \{B_1, \dots, B_r\}$ whose networks' coupling strength candidates are greater than c_o , and $\mathcal{S} = \{B_{r+1}, \dots, B_M\}$ whose networks' coupling strength candidates are less than or equal to c_o . Note that $c_o = c_{r+1}$.
- (P3) Let the networks $\mathcal{N}(f, B_k, c_o)_{k=r+1}^M$ be active relatively longer in a sense of *dwell time* than the networks $\mathcal{N}(f, B_k, c_o)_{k=1}^r$.

Note that for every $B_k \in \mathcal{S}$, $\mathcal{N}(f, B_k, c_o)$ is synchronizable for sure by Lemma 3. However, for $B_k \in \mathcal{U}$, $\mathcal{N}(f, B_k, c_o)$ may or may not be synchronizable since Lemma 3 only provides a sufficient condition for synchronizability.

Then, by Lemma 3, and via (P1) and (P2), the synchronization problem in $\mathcal{N}(f, B_k, c_o)$ for $B_k \in \mathcal{S}$ or \mathcal{U} turns to the stability problem of (8) given by:

$$\begin{aligned} \dot{e}(t) &= J(t)e(t) + c_o\Gamma e(t)B_k^T \\ &\begin{cases} B_k \in \mathcal{U}, k = 1, \dots, r, \\ B_k \in \mathcal{S}, \text{ Hurwitz stable}, k = r + 1, \dots, M. \end{cases} \end{aligned} \quad (17)$$

Applying (P3) to (17) means that the Hurwitz stable subsystems are active relatively longer than the rest of subsystems whose B_k belong to \mathcal{U} . We will show stability of (17) via (P3). (P3) is chosen according to the *average dwell time approach* in (Zhai et al., 2001). For stability analysis we need the following definition:

Definition 5. (Liberzon, 2003) For any $T_2 > T_1 \geq 0$, let $N(T_1, T_2)$ denote the number of switchings over (T_1, T_2) . If $N(T_1, T_2) \leq N_o + (T_2 - T_1)/T_a$ holds for $T_a, N_o \geq 0$, then T_a is called average dwell time.

To analyze the stability of (17) more easily we consider $e(t) = [e_1^T(t), \dots, e_N^T(t)]^T \in \mathcal{R}^{nN \times 1}$ where $e_i(t) = [e_{i1}(t), \dots, e_{in}(t)]^T \in \mathcal{R}^{n \times 1}$, $i = 1, \dots, N$. Then, (17) can be written as

$$\begin{aligned} \dot{e}(t) &= \tilde{J}(t)e(t) + c_o \tilde{B}_k \tilde{\Gamma} e(t) \\ &= (\tilde{J}(t) + c_o \tilde{B}_k \tilde{\Gamma})e(t), \quad k = 1, \dots, M, \end{aligned} \quad (18)$$

where

$$\tilde{J}(t) = \text{diag}(J(t), \dots, J(t)), \quad \tilde{\Gamma} = \text{diag}(\Gamma, \dots, \Gamma) \in \mathcal{R}^{nN \times nN}, \quad (19)$$

and

$$\tilde{B}_k = [\text{diag}(a_{ij})_{n \times n}] \in \mathcal{R}^{nN \times nN} \quad (20)$$

Note that \tilde{B}_k is symmetric and has zero row sum. Let $\tilde{J}(t) + c_o \tilde{B}_k \tilde{\Gamma}$ be $H_k(c_o) \in \mathcal{R}^{nN \times nN}$. Let $\tilde{\mathcal{S}}$ be a set of $H_k(c_o)$ for $k = r + 1, \dots, M$, and $\tilde{\mathcal{U}}$ be a set of $H_k(c_o)$ for $k = 1, \dots, r$. Then, (18) is written as:

$$\begin{aligned} \dot{e}(t) &= H_k(c_o)e(t) \\ &\begin{cases} H_k(c_o) \in \tilde{\mathcal{U}}, & k = 1, \dots, r, \\ H_k(c_o) \in \tilde{\mathcal{S}}, \text{ Hurwitz stable}, & k = r + 1, \dots, M. \end{cases} \end{aligned} \quad (21)$$

Then, there exist $\eta_k > 0$ such that $H_k(c_o) - \eta_k I$ ($k \leq r$) and $H_k(c_o) + \eta_k I$ ($k > r$) are Hurwitz stable. Hence, there exist symmetric positive definite matrices P_k , $k = 1, \dots, M$, such that

$$\begin{cases} (H_k(c_o) - \eta_k I)^T P_k + P_k (H_k(c_o) - \eta_k I) < 0 & k \leq r, \\ (H_k(c_o) + \eta_k I)^T P_k + P_k (H_k(c_o) + \eta_k I) < 0 & k > r. \end{cases} \quad (22)$$

Let the total activation times for Hurwitz stable subsystems and the rest of subsystems whose $H_k(c_o)$ are in $\tilde{\mathcal{U}}$ be $T^+(t)$ and $T^-(t)$ on $[t_o, t)$, respectively. Let

$$\begin{cases} \eta^+ = \max_{1 \leq k \leq r} \eta_k, & H_k(c_o) - \eta_k I \text{ is Hurwitz stable}, \\ \eta^- = \min_{r+1 \leq k \leq M} \eta_k, & H_k(c_o) + \eta_k I \text{ is Hurwitz stable}. \end{cases} \quad (23)$$

Choose $\eta^* \in (\eta, \eta^-)$ for any $\eta \in (0, \eta^-)$. Then, (P3) is formally addressed as follows:

(P3) : Determine a subsystem k so that

$$\frac{T^-(t)}{T^+(t)} \geq \frac{\eta^+ + \eta^*}{\eta^- - \eta^*} \quad (24)$$

holds for any $t > t_o$. Then, we have the following theorem on stability of (21):

Theorem 6. Under (P3) there exists $T_a^* > 0$ such that the switched system (21) is exponentially stable for any average dwell time $T_a \geq T_a^*$, where T_a satisfies the condition in Definition 5. Consequently, the time-varying network $\mathcal{N}(f, A(t), c_o)$ in (1) synchronizes in the sense of (4), where c_o is the coupling strength determined by (P1).

Proof. The proof is essentially similar to that in (Zhai et al., 2001). Hence, many of the details are omitted.

Let $V_k(t) = e^T(t)P_k e(t)$ with P_k from (22). Then, we have

$$\dot{V}_k \leq \begin{cases} 2\eta_k V_k, & k \leq r, \\ -2\eta_k V_k, & k > r. \end{cases} \quad (25)$$

Let $V(t) = e^T(t)P_{\sigma(t)}e(t)$, where $\sigma : \mathcal{R} \mapsto \{1, \dots, M\}$ is a switching signal. Then, for $t \in [t_l, t_{l+1})$ we have

$$V(t) \leq \begin{cases} e^{2\eta^+(t-t_l)} V(t_l) & \text{for some } k \leq r, \\ e^{-2\eta^-(t-t_l)} V(t_l) & \text{for some } k > r. \end{cases} \quad (26)$$

Note that there exists $\mu > 0$ such that $V_k(e(t)) \leq \mu V_l(e(t))$ for all $e(t)$ and all $(k \neq l)$. Noting that $V(t_l) \leq \mu V(t_l^-)$ holds on switching incident t_l , from (26), by using induction we obtain

$$V(t) \leq \mu^{N(t_o, t)} e^{2\eta^+ T^+(t) - 2\eta^- T^-(t)} V(t_o) \quad (27)$$

Then, we have

$$\begin{aligned} \|e(t)\| &\leq \sqrt{\frac{\alpha_2}{\alpha_1}} e^{\eta^+ T^+(t) - \eta^- T^-(t) + \frac{\ln \mu}{2} N(t_o, t)} \|e(t_o)\| \\ &\leq \sqrt{\frac{\alpha_2}{\alpha_1}} e^{\frac{\ln \mu}{2} N(t_o, t) - \eta^*(t-t_o)} \|e(t_o)\|, \end{aligned} \quad (28)$$

where $\alpha_1 = \min_{1 \leq k \leq M} \lambda_{\min}(P_k)$, $\alpha_2 = \max_{1 \leq k \leq M} \lambda_{\max}(P_k)$, and $\mu = \sup_{k, l \in [1, M]} \lambda_{\max}(P_k) / \lambda_{\min}(P_l)$. Here, $\lambda_{\max}(P_k)$ ($\lambda_{\min}(P_k)$) means the maximum (minimum) eigenvalue of P_k . When $\mu = 1$, the exponential stability is trivial. When $\mu > 1$, we apply the average dwell time condition in Definition 5. Hence, we have

$$\|e(t)\| \leq \sqrt{\frac{\alpha_2}{\alpha_1}} e^{\alpha - \eta(t-t_o)} \quad (29)$$

since we can choose $N_o = \frac{2\alpha}{\ln \mu}$ and $T_a^* = \frac{\ln \mu}{2(\eta^* - \eta)}$ for any $\alpha > 0$. \square

Remark.

(i) Although Theorem 6 provides the stability of (21) over the core configuration set \mathcal{B} in $\mathcal{F}(A(t))$, this implies the stability of (21) over $\mathcal{F}(A(t))$ due to the isomorphic configurations in $\mathcal{F}(A(t))$ and the identical node dynamics in the network $\mathcal{N}(f, A(t), c)$. Thus, synchronization of $\mathcal{N}(f, A(t), c)$ can be checkable just via the stability of (21) over \mathcal{B} using the average dwell time condition.

(ii) Theorem 6 provides a criterion on synchronization for the time-varying network $\mathcal{N}(f, A(t), c)$ in (1) when it has a moderate coupling strength such as c_o . In other words, if $\mathcal{N}(f, A(t), c)$ changes its configuration according to the average dwell time condition in Theorem 6 induced from (P3) in (24), then the necessary coupling strength for synchronization does not need to be large enough to stabilize (synchronize) all configurations B_k , $k = 1, \dots, M$.

Proposition 7. The parameters for (P3) in (24) can be obtained from the reduced matrices $F_{k, N-1}(c_o) = \text{diag}(J(t) + c_o \lambda_{2k} \Gamma, \dots, J(t) + c_o \lambda_{Nk} \Gamma) \in \mathcal{R}^{n(N-1) \times n(N-1)}$ for $k = 1, \dots, M$, where $\lambda_{2k}, \dots, \lambda_{Nk}$ are the eigenvalues of B_k in (8).

Proof. Consider (10). Let $h_k(t) = [h_{1k}^T(t), \dots, h_{Nk}^T(t)]^T \in \mathcal{R}^{nN \times 1}$ where $h_{ik}(t) \in \mathcal{R}^{n \times 1}$, $i = 1, \dots, N$, $k = 1, \dots, M$. Then, we have

$$\dot{h}_k = F_k(c_o)h_k(t), \quad (30)$$

where $F_k(c_o) = \text{diag}(J(t), J(t) + c_o \lambda_{2k} \Gamma, \dots, J(t) + c_o \lambda_{Nk} \Gamma) \in \mathcal{R}^{nN \times nN}$. Note that $\dot{h}_{1k} = J(t)h_{1k}$ is the variational equation for the synchronization manifold. Thus,

$$\{h_k(t) : \dot{h}_k = F_k(c_o)h_k(t)\} = \{(0, \cdot, \cdot, \dots, \cdot) \in \mathcal{R}^{nN \times 1}\}. \quad (31)$$

Since each $h_{ik} \rightarrow 0$ as $t \rightarrow \infty$, $i = 2, \dots, N$ by the choice of $c = c_o$, $\{(0, \dots, 0)\} \in \mathcal{R}^{nN \times 1}$ is the invariant set. Thus, we can conclude (30) is exponentially stable. By (31), the fact that $\{(0, \dots, 0)\} \in \mathcal{R}^{nN \times 1}$ is invariant, and $h_{1k} = 0$ for all t , the decay rate of the system (30) is indeed determined by the reduced matrix $F_{k,N-1}(c_o) = \text{diag}(J(t) + c_o\lambda_{2k}\Gamma, \dots, J(t) + c_o\lambda_{Nk}\Gamma) \in \mathcal{R}^{n(N-1) \times n(N-1)}$. Therefore, we can find desired parameters for (P3) by finding η_k , $k = 1, \dots, M$ from

$$\begin{cases} (F_{k,N-1}(c_o) - \eta_k I)^T P_k + P_k (F_{k,N-1}(c_o) - \eta_k I) < 0, & k \leq r, \\ (F_{k,N-1}(c_o) + \eta_k I)^T P_k + P_k (F_{k,N-1}(c_o) + \eta_k I) < 0, & k > r, \end{cases} \quad (32)$$

where $P_k \in \mathcal{R}^{n(N-1) \times n(N-1)}$, positive definite. Since (30) and (18) are equivalent for each k , the choice of η_k is valid for (18). \square

Remark. By Proposition 7 we can find the parameters for (P3) more easily than using (22).

4. ILLUSTRATIVE EXAMPLE

Consider a complex network with $N = 4$ in the form of (1) with a Chua's oscillator (Chua et al., 1993) as a dynamic node, K_4 as a base network configuration, and $\Gamma = \text{diag}(1, 0, 0)$ as an inner coupling matrix given by

$$\dot{x}_i = g(x_i) + c \sum_{j=1}^4 a_{ij}(t)\Gamma x_j, \quad (33)$$

where $x_i = (x_{i1}, x_{i2}, x_{i3}) \in \mathcal{R}^3$,

$$g(x_i) = \begin{pmatrix} \alpha(-x_{i1} + x_{i2} + f(x_{i1})) \\ x_{i1} + x_{i2} + x_{i3} \\ -\beta x_{i2} - \gamma x_{i3} \end{pmatrix}, \quad (34)$$

and

$$f(x_{i1}) = \begin{cases} -bx_{i1} - a + b & x_{i1} > 1, \\ -ax_{i1} & |x_{i1}| \leq 1, \\ -bx_{i1} + a - b & x_{i1} < -1, \end{cases} \quad i = 1, \dots, 4 \quad (35)$$

in which $\alpha = 10$, $\beta = 15$, $\gamma = 0.0385$, $a = -1.27$, and $b = -0.68$. With these parameters each node is a *chaotic attractor*. As mentioned before, K_4 has 38 configurations according to the switching links in (2). In other words, the time-varying network (33) will change its configuration matrix among 38 possible configurations. However, by Proposition 2 we have only six non isomorphic configurations to consider given by

$$\begin{aligned} & (B_1, B_2, B_3) \\ & = \left(\begin{bmatrix} -2 & 0 & 1 & 1 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -2 & 1 \\ 1 & 1 & 1 & -3 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 1 & -3 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \begin{bmatrix} -2 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \right), \\ & (B_4, B_5, B_6) \\ & = \left(\begin{bmatrix} -3 & 1 & 1 & 1 \\ 1 & -2 & 0 & 1 \\ 1 & 0 & -2 & 1 \\ 1 & 1 & 1 & -3 \end{bmatrix}, \begin{bmatrix} -2 & 1 & 1 & 0 \\ 1 & -2 & 0 & 1 \\ 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & -2 \end{bmatrix}, \begin{bmatrix} -3 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -3 \end{bmatrix} \right). \end{aligned}$$

Note that $\lambda_2(B_6) = -4$, $\lambda_2(B_5) = \lambda_2(B_4) = -2$, $\lambda_2(B_3) = \lambda_2(B_2) = -1$, and $\lambda_2(B_1) = -0.5858$. Hence, we have four coupling strength candidates as $\{2.66, 5.32, 10.61, 18.14\}$ by Lemma 3. Then, we choose $c_o = 5.32$ for c . Note

that in this example, the Jacobian $J(t)$ is a constant matrix. Hence, we can say that the networks $\mathcal{N}(g, B_k, c_o)$, $k = 1, \dots, 3$ are not synchronized but the other three are by checking eigenvalues of $J(t) + c_o\lambda_{2k}\Gamma$, $k = 1, \dots, 6$. By Proposition 7, we obtain η_k and then $\eta^+ = 0.3$, $\eta^- = 0.05$, $\eta = 0.01$, and $\eta^* = 0.035$. Also, $\mu = 141.7473$ from P_k in (32) in Proposition 7. Thus, the average dwell time bound is $T_a^* = 99.0898$. Taking $T_a = 100 > T_a^*$, by Theorem 6, we obtain the exponential stability of the switched system consisting of $\mathcal{N}(g, B_k, c_o)$, $k = 1, \dots, 6$. Therefore, the original time-varying network (33) achieves synchronization in the sense of (4). This is confirmed in Figure 1. In the following figures, we plot the aggregated error defined by

$$z(t) = \sum_{j=2}^4 (x_{11} - x_{j1}) + \sum_{j=2}^4 (x_{12} - x_{j2}) + \sum_{j=2}^4 (x_{13} - x_{j3}).$$

If (33) is synchronized via the average dwell time condition in Theorem 6, then $z(t) \rightarrow 0$. As we can see in Figure 1, the aggregated error converges to zero which confirms synchronization of the network in (33) with $c = c_o = 5.3$.

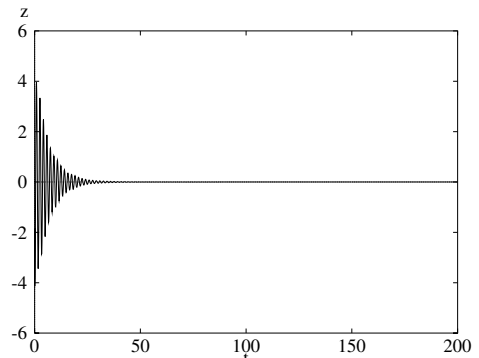


Fig. 1. Numerical simulation of the aggregated error z vs. time t from the switched system consisting of $\mathcal{N}(g, B_k, c_o)$, $k = 1, \dots, 6$ with the coupling strength $c_o = 5.3$.

As shown in Figure 2, when $c_o = 3$ is close to 2.66, the network still synchronizes since $\mathcal{N}(g, B_6, 3)$ is synchronizable whereas the rest is not. Thus, the average dwell time approach can still be applicable and hence the network (33) achieves synchronization. However, when

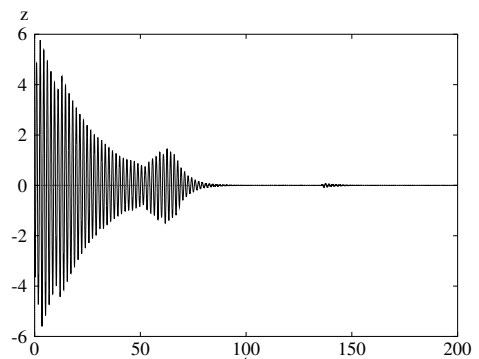


Fig. 2. Numerical simulation of the aggregated error z vs. time t from the switched system consisting of $\mathcal{N}(g, B_k, c_o)$, $k = 1, \dots, 6$ with the coupling strength $c_o = 3$.

$c_o < 2.66$, the network does not synchronizes since all networks $\mathcal{N}(g, B_k, c_o)$ are not synchronizable. As shown in Figure 3, the aggregated error blows out. Moreover, even

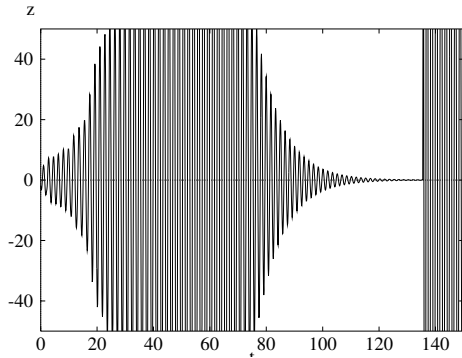


Fig. 3. Numerical simulation of the aggregated error z vs. time t from the switched system consisting of $\mathcal{N}(g, B_k, c_o)$, $k = 1, \dots, 6$ with the coupling strength $c_o = 2$. The error blows out at $t = 158$.

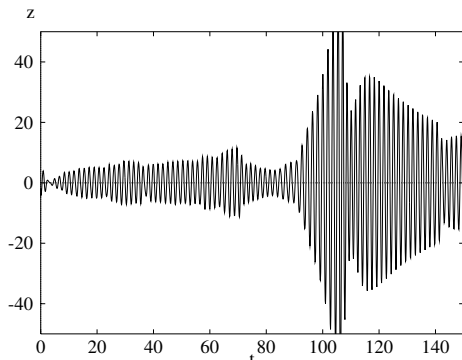


Fig. 4. Numerical simulation of the aggregated error z vs. time t from the switched system consisting of $\mathcal{N}(g, B_k, c_o)$, $k = 1, \dots, 6$ with the coupling strength $c_o = 5.3$, $T^- = 1.5$ and $T^+ = 33.75$. (P3) in (24) is violated and so is the average dwell time condition. The error becomes unbounded from $t = 240$.

when $c = c_o = 5.3$, if the average dwell time condition is violated, then the network does not synchronize as shown in Figure 4. Thus, we conclude as follows: (a) the time-varying network (33) can be synchronized even when not all connection configurations are synchronizable, (b) the coupling strength can be chosen greater than or equal to 2.66 for the network (33) to synchronize, and (c) the second largest eigenvalues of each configuration based on Lemma 3 may not always give a sufficient condition for synchronizability of a time-varying network.

5. CONCLUSION

In this paper, we achieved synchronization of a complex network with switched coupling via a switched system approach. The significance of the results is as follows:

- (i) We formulated a complex network with switched coupling in terms of a switched system with a reduced number of subsystems via partitioning the family of all possible configurations generated from the switching topology in (2) by topological isomorphism. In fact, each subsystem is a static complex network with a non isomorphic configuration matrix. Then, we obtained the finite number of coupling strength candidates from each subsystem.
- (ii) We applied the average dwell time approach (Zhai et al., 2001) for stability analysis to the obtained switched system from (i) by choosing one among

the candidates for the coupling strength. Then, we achieved synchronization without requiring synchronizability of all possible network configurations. This gives a new criterion on how to choose the coupling strength for a time-varying complex network.

Due to the switching nature of a network with switched coupling, the second largest eigenvalues of all frozen configurations, namely for any fixed t , may not give a sufficient condition for synchronization. From Figure 1 and 2, synchronization is indeed achieved even when some configurations are not synchronizable with the chosen coupling strength. Also, as shown in Figure 4, although the coupling strength is chosen as $c = c_o = 5.3$ based on Lemma 3, synchronization was not achieved due to the violation of the average dwell time condition. Hence, our proposal in Sec. 3.2 leads to a relatively simple but systematic method to obtain a desired coupling strength and synchronization.

These results might be useful in areas such as ad hoc networks, sensor networks and power systems. One important thing to mention is that each choice of a coupling strength c_o requires a revised switching scheme. Hence, for future work, we may need to consider an optimal strategy on how to choose a coupling strength among the candidates. Moreover, we would like to include disconnected network configurations. In this case, the appearance of such configurations could be considered as weighted impulses over time. Hence, we can regard such networks as switched impulsive systems.

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