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On ZOH Discretization of Higher-Order Sliding Mode Control Systems ⋆

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Abstract: In this paper, the zero-order-hold (ZOH) discretization of higher-order sliding mode control (SMC) systems is studied. The equivalent control based SMC systems with relative degree higher than one is first formulated into a canonical form which is easy for control design. Theoretical results for the ZOH-discretized SMC systems with relative degree higher than one are given, including accurate estimates of the bounds of steady states and higher order sliding mode functions. Simulation results are presented to show the effectiveness of the analysis.

Keywords: Sliding mode control; discretization; implementation; stability; convergence.

1. INTRODUCTION

Sliding mode control (SMC) has been successfully applied to solving many practical control problems (Utkin (1992); Utkin et al. (1999)) due to its attractive features such as invariance to matched uncertainties (Drazenovic (1969)). When a sliding mode is realized, the system exhibits robustness properties with respect to external matched uncertainties.

Despite the claimed robustness properties, high frequency oscillations of the state trajectories around the sliding manifold known as chattering phenomenon are the major obstacle for the implementation of SMC in a wide range of applications. A number of methods have been proposed to reliably overcome chattering, for example, the boundary layer solution (Slotine (1984)); the observer-based solution (Utkin et al. (1999)) and the higher-order SMC (Levant (1993); Fridman and Levant (1996); Levant (2003); Bartolini et al. (1998)), which has attracted an increasing attention due to its effectiveness of reducing chattering (Levant and Fridman (2004)). Its main idea can be described as follows (Laghrouche et al. (2007)). Let $s(x,t)$ be the sliding variable and $r \in N$ ($r > 1$) the sliding order. The control drives s and its $(r-1)$th order time derivatives to zero in finite time by acting discontinuously on the rth order time derivative of s. As a result, the chattering effect is reduced and higher-order precision is provided.

Discretization effect study has attracted a lot of attention in the digital control of dynamical systems and digital implementation. A primary reason is that there are some intrinsic dynamic properties within the discretized systems which do not appear in their continuous-time counterpart systems. Periodic phenomenon is common in the discrete switching systems (Yu and Galias (2001); Yu and Chen (2003)). So far, there are several key discretization methods that are used in industrial applications, such as the Zero-Order-Hold (ZOH), Euler method and Tustin method. ZOH has been most frequently used in practice, especially in feedback control implementation. When ZOH is applied to digital implementation, the control signal is 'frozen' as a constant during the time interval. ZOH effect on SMC systems has been studied recently in Yu and Chen (2003); Wang et al. (June 2006); Yu et al. (February 2008). Nevertheless, in those cases we mentioned, the sliding mode systems are only with relative degree one.

Higher-order SMC systems play an important role in SMC application due to its desirable chattering reduction. However, sampled higher-order SMC systems are not easy to implement (Bartolini et al. (2001); Yu et al. (2007)). In this paper, the ZOH discretization of higher-order SMC systems is studied. Boundary conditions for the steady states are derived. Some intriguing periodic behaviors are depicted. Finally, simulation examples are presented to verify the theoretical results.

2. THE CONCEPT OF HIGHER-ORDER SLIDING MODES

First, let us introduce the higher-order SMC systems. Consider a smooth dynamic affine system $\dot{x} = v(x) + g(x)u$ where $x \in R^n$ is the system state and $u \in R^1$ the scalar control, and $v(x)$ and $g(x)$ are smooth functions. For a smooth output function σ, which is considered as the sliding variable, provided that successive time derivatives $\sigma, \dot{\sigma}, \cdots, \sigma^{(r-1)}$ are continuous functions, and the r-sliding point set ($r > 1$)

$$\sigma = \dot{\sigma} = \ddot{\sigma} = \cdots = \sigma^{(r-1)} = 0 \tag{1}$$

is non-empty and consists locally of Filippov trajectories, the motion on set (1) is called r-sliding mode (rth- order sliding mode (Levant (1993); Fridman and Levant (1996)). The sliding order characterizes the dynamics smoothness degree in some vicinity of the sliding mode. If $s^{(r)}$ is steered to zero, we call this the r-sliding mode.

Suppose that $\sigma, \dot{\sigma}, \ddot{\sigma}, \cdots, \sigma^{(r-1)}$ are differentiable functions of x and that

$$rank[\nabla\sigma, \nabla\dot{\sigma}, \cdots, \nabla\sigma^{(r-1)}] = r \tag{2}$$

Equality (2) together with the requirement for the corresponding derivatives of σ to be differentiable functions of x is referred to as r-sliding regularity condition. If regularity condition (2) holds, then the r-sliding set is a differentiable manifold and $\sigma, \dot{\sigma}, \cdots, \sigma^{(r-1)}$ may be supplemented up to new local coordinates.

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In a simplified way the equality of the relative degree to r means that u first appears explicitly only in the rth order time-derivative of σ. In that case regularity condition (2) is satisfied (Isidori (1995)).

3. THE 1-SLIDING MODE SYSTEM MODEL AND ZOH MODEL

Before studying ZOH of higher-order SMC systems, let us first recall the results for ZOH discretization of the 1-sliding mode systems. Consider the general controllable system with canonical form

$$\dot{x} = Ax + bu \tag{3}$$

where $x \in R^n$ is a state vector, $u \in R$ and

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \\ -a_1 & -a_2 & -a_3 & \cdots & -a_n \end{bmatrix}, b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{4}$$

The switching manifold is defined as $\sigma(x) = cx = c_1 x_1 + \underline{c} z$, where $\underline{c} = [c_2, c_3, \cdots, 1]$ and $z = (x_2, x_3, \cdots, x_n)^\top$. The coefficients $c_1, c_2, \cdots, c_{n-1}$ constitute a Hurwitz polynomial. Here, the relative degree of $\sigma(x)$ with respect to the control u is one. The equivalent control based sliding mode control is

$$u = u_{eq} + u_s \tag{5}$$

where $u_{eq} = -(cb)^{-1}cAx$ and $u_s = -\alpha(cb)^{-1}\text{sgn}(s)$ with $\alpha > 0$. With this control, one has $\sigma\dot{\sigma} < -\alpha|s|$. Therefore, the ideal 1-sliding mode is guaranteed to be reached in finite time.

Substituting (5) into (3) and taking into account that $cb = 1$ yield

$$\dot{x} = A_c x - \alpha \text{sgn}(cx) b \tag{6}$$

where

$$A_c = (I - bc)A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & -c_1 & \cdots & -c_{n-2} & -c_{n-1} \end{bmatrix} = \begin{bmatrix} 0 & A_{c1} \\ 0 & A_{c2} \end{bmatrix} \tag{7}$$

with $A_{c1} = [1, 0, \cdots, 0] \in R^{n-1}$ and

$$A_{c2} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \\ -c_1 & \cdots & -c_{n-2} & -c_{n-1} \end{bmatrix} \tag{8}$$

The expression of A_c is illustrative to demonstrate the dynamics of system states. With the equivalent control based sliding mode control (5), the sliding mode $\sigma = 0$ can be reached in finite time. The subsystem z is asymptotically stable because the eigenvalues of A_{c2} are zeros of the characteristic equation

$$\lambda^{n-1} + c_{n-1}\lambda^{n-2} + \cdots + c_2\lambda + c_1 = 0$$

which is Hurwitz.

Through the ZOH dicretization, $u(t) = u(k)$ over the time interval $[kh, (k+1)h]$, where h is the sampling period. The continuous-time system (3) with the ZOH can be converted into the discrete form as

$$x(k+1) = e^{Ah}x(k) + \int_0^h e^{A\tau} d\tau b u(k) \tag{9}$$

The control law in discrete-time is

$$u(k) = -(cb)^{-1}cAx(k) - \alpha(cb)^{-1}\text{sgn}(\sigma(x(k))) \tag{10}$$

In the following, for simplicity, we denote $\text{sgn}(\sigma(x(k)))$ as s_k, the aforementioned discrete system can then be rewritten as

$$x(k+1) = \Phi x(k) - \alpha\Gamma s_k \tag{11}$$

where $\Phi = e^{Ah} - \int_0^h e^{A\tau}d\tau bcA = \begin{bmatrix} 1 & v^\top(h) \\ \bar{0} & D(h) \end{bmatrix}$ and $\Gamma = \int_0^h e^{A\tau}d\tau b = \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix}$. Here, $v(h)$ is an $(n-1)$-dimensional vector, $\bar{0}$ is an $(n-1)$-dimensional zero vector, $D(h)$ is an $(n-1) \times (n-1)$ matrix. Γ_1 is a scalar and Γ_2 is an $(n-1)$-dimensional vector. Then, the discretized SMC system can be represented by (Yu and Chen, 2003)

$$x_1(k+1) = x_1(k) + v^\top z(k) - \alpha\Gamma_1 s_k \tag{12}$$

$$z(k+1) = Dz(k) - \alpha\Gamma_2 s_k \tag{13}$$

In Yu and Chen (2003), the properties of the discretized behaviors of SMC systems have been analyzed and a bound for steady states was given. However, deriving that bound needs a quite strict condition which is hard to satisfy in practice. In Yu et al. (February 2008), a new bound under a relatively loose condition for steady states has been derived:

Theorem 1. The discretized SMC system (12) and (13) is eventually bounded if $\sup_k \|c\Phi(h) - c\| < \alpha c\Gamma$ and $\|D\| < 1$. Furthermore, the system states are bounded by

$$\|x_1(\infty)\| \leq \alpha\|c_1^{-1}\underline{c}\|\|\Gamma_2\|(1 - \|D\|)^{-1} + 2\alpha\|c_1\|^{-1}c\Gamma$$
$$\|z(\infty)\| \leq \alpha\|\Gamma_2\|(1 - \|D\|)^{-1} \tag{14}$$

Here, $\|\cdot\|$ is the spectral norm.

4. ZOH DISCRETIZATION OF HIGHER-ORDER SLIDING MODE SYSTEMS

We now consider the single-input linear system with relative degree $r > 1$. In classical SMC systems, one has $\sigma = 0$. In higher-order SMC systems, that is, r-sliding mode systems, $\sigma = \dot{\sigma} = \ddot{\sigma} = \cdots = \sigma^{(r-1)} = 0$. The single-input higher-order SMC systems in the controllable canonical from is the same as in (3) except for the switching function, which is now defined as

$$\sigma(x) = c_1 x_1 + c_2 x_2 + \cdots + c_{n-r-1}x_{n-r-1} + x_{n-r} \tag{15}$$

Here, $c_1, c_2, \cdots, c_{n-r-1}$ are assumed to be coefficients of a Hurwitz polynomial. To apply the equivalent control based sliding mode control to the system, we need to construct a new switching function,

$$\bar{\sigma}(w) = \bar{c}w = \bar{c}_1\bar{\sigma}_1 + \bar{c}_2\bar{\sigma}_2 + \cdots + \bar{c}_r\bar{\sigma}_r + \bar{\sigma}_{r+1} \tag{16}$$

where $\bar{\sigma}_1 = \sigma, \bar{\sigma}_2 = \dot{\sigma}, \cdots, \bar{\sigma}_{r+1} = d^r\sigma/dt^r$ and $w = (\bar{\sigma}_1, \bar{\sigma}_2, \cdots, \bar{\sigma}_{r+1})^T \in R^{r+1}$. It is assumed that the coefficients $\bar{c}_1, \bar{c}_2, \cdots, \bar{c}_r$ also constitute a Hurwitz polynomial. Then, a new system state y is built as $y = (\bar{x}, w) \in R^n$ with $\bar{x} = (x_1, x_2, \cdots, x_{n-r-1})^T \in R^{n-r-1}$. Let's denote $\underline{x} = (x_{n-r}, x_{n-r+1}, \cdots, x_n)^T \in R^{r+1}$. We can reformulate the system with form (3) to the new state y through a state transformation matrix P,

$$\begin{bmatrix} \bar{x} \\ w \end{bmatrix} = P \begin{bmatrix} \bar{x} \\ \underline{x} \end{bmatrix} \tag{17}$$

where

$$P = \begin{bmatrix} I_{n-r-1} & 0 \\ P_1 & P_2 \end{bmatrix} \tag{18}$$

with

$$P_1 = \begin{bmatrix} c_1 & c_2 & \cdots & c_{n-r-1} \\ 0 & c_1 & \cdots & c_{n-r-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots \end{bmatrix}$$

$$P_2 = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ c_{n-r-1} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \cdots & \cdots & c_{n-r-1} & 1 & 0 \\ \cdots & \cdots & c_{n-r-2} & c_{n-r-1} & 1 \end{bmatrix}$$

One has

$$\begin{bmatrix} \bar{x} \\ w \end{bmatrix}' = PAP^{-1} \begin{bmatrix} \bar{x} \\ w \end{bmatrix} + Pbu$$

If we denote $\bar{A} = PAP^{-1}$, then the new system is

$$\dot{y} = \bar{A}y + bu \tag{19}$$

where $\bar{A}$ is shown in (20 on the next page. The new sliding mode is $\bar{\sigma}(y) = \bar{\sigma}(w)$. Then it can be easily derived that

$$[\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_n] = \\ [a_1, \cdots, a_{r+1}, a_{r+2} - c_1, \cdots, a_n - c_{n-r-1}]\, P^{-1} \tag{21}$$

Now consider the new system (19) with the new coordinates of y, the equivalent control can be derived from $\dot{\bar{\sigma}} = 0$. Since

$$\begin{aligned} \dot{\bar{\sigma}} &= \sum_{i=1}^{r} \bar{c}_i \bar{\sigma}_{i+1} + \dot{\bar{\sigma}}_{r+1} \\ &= \sum_{i=1}^{r} \bar{c}_i y_{n-r+i} + \dot{y}_n \\ &= \sum_{i=1}^{r} \bar{c}_i y_{n-r+i} - \sum_{i=1}^{n} \bar{a}_i y_i + u \end{aligned}$$

The equivalent control based SMC is then

$$u(y) = -\sum_{i=1}^{r} \bar{c}_i y_{n-r+i} + \sum_{i=1}^{n} \bar{a}_i y_i - \alpha \mathrm{sgn}(\bar{\sigma}(y)) \tag{22}$$

This control results in $\bar{\sigma}\dot{\bar{\sigma}} = -\alpha|\bar{\sigma}|$, which ensures the finite time convergence. Substituting (22) into (19) yields

$$\dot{y} = \bar{A}_c x - \alpha \mathrm{sgn}(\bar{\sigma}(y)) b \tag{23}$$

where $\bar{A}_c$ is shown in (24)on the next page. In the following, we will show that all the system states are asymptotically stable.

Denote

$$\bar{A}_c = \begin{bmatrix} \bar{A}_{c1} & \bar{A}_{c2} \\ 0 & \bar{A}_{c3} \end{bmatrix} \tag{25}$$

where $\bar{A}_{c1} \in R^{(n-r-1)\times(n-r-1)}$, $\bar{A}_{c2} \in R^{(n-r-1)\times(r+1)}$ has all zero rows except the last row, which is $[1, 0, \cdots, 0] \in R^{r+1}$ and $\bar{A}_{c3} \in R^{(r+1)\times(r+1)}$,

$$\bar{A}_{c1} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -c_1 & -c_2 & \cdots & -c_{n-r-2} & -c_{n-r-1} \end{bmatrix} \tag{26}$$

$$\bar{A}_{c3} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & -\bar{c}_1 & \cdots & -\bar{c}_{r-1} & -\bar{c}_r \end{bmatrix} \tag{27}$$

The control law (22) ensures that the sliding mode $\bar{\sigma} = 0$ can be reached in finite time. Taking into account that $\bar{\sigma}(y)$ is an explicit function of w, i.e. $\bar{\sigma}(y) = \bar{\sigma}(w)$, we can decompose (23) as

$$\dot{\bar{x}} = \bar{A}_{c1}\bar{x} + \bar{A}_{c2} w \tag{28}$$

$$\dot{w} = \bar{A}_{c3} w - \alpha \mathrm{sgn}(\bar{\sigma}(w))\bar{b} \tag{29}$$

where $\bar{b} = [0, 0, \cdots, 0, 1]^\top \in R^{r+1}$.

First we consider the subsystem (29). Notice that it is equivalent to (6), which was analyzed in Section 3. It follows that the states of the subsystem w are asymptotically stable.

Now we turn to the other subsystem $\bar{x}$. Here, $y_{n-r} = 0$ since y_{n-r} is the first state of subsystem w. Then, the subsystem $\bar{x}$ can be rewritten as

$$\dot{\bar{x}} = \bar{A}_{c1}\bar{x} \tag{30}$$

Note that $\bar{A}_{c1}$ is just analogous to (8), then from the same analysis procedure as shown in Section 3. We can draw the conclusion that subsystem $\bar{x}$ is also asymptotically stable. Thus, all the system states are asymptotically stable. Since the state transformation matrix P is invertible, we have the conclusion that the original state $x = [\bar{x}, \underline{x}]^\top$ is asymptotically stable.

In the following, the ZOH discretization will be applied to the new system model (28) and (29). For simplicity to study to reveal intriguing characteristics of the ZOH effect on higher-order SMC systems, the ZOH discretization is applied to (23), which leads to

$$y(k+1) = \Phi y(k) - \alpha \Gamma \mathrm{sgn}(\bar{\sigma}(y(k))) \tag{31}$$

where $\Phi = e^{\bar{A}_c h}$ and $\Gamma = \int_0^h e^{\bar{A}_c \tau} d\tau b$. Furthermore, $\Phi = I + \bar{A}_c h + \bar{A}_c^2 h^2/2! + \cdots$ and $\int_0^h e^{\bar{A}_c \tau} d\tau = hI + h^2/2!\bar{A}_c + h^3/3!\bar{A}_c^2 + \cdots + h^{i+1}/(i+1)!\bar{A}_c^i + \cdots$. With the expression of $\bar{A}_c$ (25), one can derive

$$\Phi = \begin{bmatrix} \Phi_1 & \Phi_2 \\ 0 & \Phi_3 \end{bmatrix} \tag{32}$$

$$\Phi_1 = e^{\bar{A}_{c1} h}, \ \Phi_3 = \begin{bmatrix} 1 & \bar{v}^T \\ \bar{0} & \bar{D} \end{bmatrix}$$

$$\Phi_2 = \bar{A}_{c2}(\bar{A}_{c1} - \bar{A}_{c3})^{-1}(\Phi_1 - \Phi_3) \tag{33}$$

Here, the dimension of each block in Φ is as same as the corresponding block of $\bar{A}_c$ in (25). In Φ_3, $\bar{v}^T$ is an r-dimensional row vector, $\bar{D}$ is a $r \times r$ matrix and $\bar{0}$ denotes an r-dimensional zero column vector. Hence, the ZOH discretization of system (23) gives rise to the following discrete-time dynamical system:

$$\bar{x}(k+1) = \Phi_1 \bar{x}(k) + \Phi_2 w(k) - \alpha \Gamma_1 \bar{s}_k \tag{34}$$

$$w(k+1) = \Phi_3 w(k) - \alpha \Gamma_2 \bar{s}_k \tag{35}$$

where $\bar{s}_k = \mathrm{sgn}(\bar{\sigma}(y(k)))$ and $\Gamma = [\Gamma_1^\top, \Gamma_2^\top]^\top$, $\Gamma_1^\top \in R^{n-r-1}$, $\Gamma_2^\top \in R^{r+1}$.

Note that (35) is equivalent to system (11) except that its dimension is $r+1$, which was analyzed in the previous section. We rewrite $\bar{\sigma}$ as

$$\bar{\sigma}(y) = \bar{\sigma}(w) = \bar{c}_1 w_1 + dv$$

where $d = [\bar{c}_2, \bar{c}_3, \cdots, \bar{c}_r, 1]$ and $v = [w_2, w_3, \cdots, w_{r+1}]^\top$. Let's assume eigenvalues of $\bar{D}$ are within the unit circle. Then, the bounds for the steady states of subsystem w can be given by (14):

$$\bar{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ -c_1 & -c_2 & \cdots & -c_{n-r-2} & -c_{n-r-1} & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ -\bar{a}_1 & -\bar{a}_2 & \cdots & -\bar{a}_{n-r-2} & \bar{a}_{n-r-1} & -\bar{a}_{n-r} & -\bar{a}_{n-r+1} & \cdots & -\bar{a}_{n-1} & -\bar{a}_n \end{bmatrix} \tag{20}$$

$$\bar{A}_c = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ -c_1 & -c_2 & \cdots & -c_{n-r-2} & -c_{n-r-1} & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 & 0 & -d_1 & \cdots & -d_{r-1} & -d_r \end{bmatrix} \tag{24}$$

$$\|w_1(\infty)\| \le \alpha\|\bar{c}_1^{-1}d\|\|\Gamma_{22}\|(1-\|\bar{D}\|)^{-1} + 2\alpha\|\bar{c}_1\|^{-1}\bar{c}\Gamma_2$$
$$\|v(\infty)\| \le \alpha\|\Gamma_{22}\|(1-\|\bar{D}\|)^{-1}$$

where $\Gamma_2 = [\Gamma_{21}, \Gamma_{22}^{\top}]^{\top}$, Γ_{21} is a scalar and Γ_{22} is a r-dimensional vector. According to the definition of spectral norm, $\|w\| \le \sqrt{2}\max\{\|w_1\|, \|v\|\}$.

Let us now consider the subsystem (34). Assume the eigenvalues of Φ_1 lie within unit circle. It follows from (34) that

$$\|\bar{x}(k+1)\| \le \|\Phi_1\|\|\bar{x}(k)\| + \|\Phi_2\|\|w(k)\| + \alpha\|\Gamma_1\|$$

Iterating n times on the above inequality yields

$$\|\bar{x}\| \le \|\Phi_1\|^n\|\bar{x}(0)\| + (\|\Phi_2\|\|w(k)\| + \alpha\|\Gamma_1\|)\sum_{0}^{n-1}\|\Phi_1\|^{n-1-i}$$
$$= \|\Phi_1\|^n\|\bar{x}(0)\| + (\|\Phi_2\|\|w(k)\| + \alpha\|\Gamma_1\|)$$
$$(1-\|\Phi_1\|^n)(1-\|\Phi_1\|)^{-1}$$

Since $\|\Phi_1\| < 1$, as $n \to \infty$, one has

$$\|\bar{x}\| \le (\|\Phi_2\|\|w\| + \alpha\|\Gamma_1\|)(1-\|\Phi_1\|)^{-1}$$

Using the state transformation (17), one has

$$\underline{x} = P_2^{-1}w - P_2^{-1}P_1\bar{x}$$

Then, the bound for the subsystem state $\underline{x}$ can be derived as

$$\|\underline{x}\| \le \|P_2^{-1}\|(\|w\| + \|\bar{x}\|\|P_1\|)$$

The above analysis and inference result in the following theorem:

Theorem 2. If eigenvalues of Φ_1 and $\bar{D}$ are all within the unit circle, then the states of system (34) and (35) converge to the steady state solutions of the discretized higher-order SMC system (28) and (29) bounded by

$$\|w_1\| \le \alpha\|\bar{c}_1^{-1}d\|\|\Gamma_{22}\|(1-\|\bar{D}\|)^{-1} + 2\alpha\|\bar{c}_1\|^{-1}\bar{c}\Gamma_2$$
$$\|v\| \le \alpha\|\Gamma_{22}\|(1-\|\bar{D}\|)^{-1}$$
$$\|\bar{x}\| \le (\beta\|\Phi_2\| + \alpha\|\Gamma_1\|)(1-\|\Phi_1\|)^{-1}$$
$$\|\underline{x}\| \le \|P_2^{-1}\|(\beta + (\beta\|\Phi_2\| + \alpha\|\Gamma_1\|)(1-\|\Phi_1\|)^{-1}\|P_1\|)$$

where $\beta = \sqrt{2}\max\{\|w_1\|, \|v\|\}$.

Note that it is the first time the specific bounds of ZOH-discretized higher-order SMC systems states have been derived. In Theorem 2, Φ and Γ are all functions of h. It is difficult to derive their general analytic expressions. But we could still draw some convergence properties from the above analysis. It can be easily seen that if we rewrite (31) as

$$y(k+1) - y(k)/h = (\Phi - I)y(k)/h - \alpha\Gamma\bar{s}_k/h$$

From definitions of Φ and Γ, one has $\lim_{h\to 0}(\Phi - I)/h = \bar{A}_c$ and $\lim_{h\to 0}\Gamma/h = b$. That means when $h \to 0$, the solution of discretized system converges to the corresponding continuous-time higher-order SMC system solution. Thus, systems states converge to 0 as $h \to 0$.

We now specify the convergence accuracy of the discretized higher-order SMC systems, which would be helpful in engineering applications. The concept of "boundary layer" was first introduced for eliminating the chattering phenomenon in SMC systems (Slotine (1984)). In discrete SMC systems, the convergence accuracy is usually measured by the width of boundary layers. It has been known that higher-order SMC systems can achieve higher-order convergence precision (Levant (1993)). By the big O notation, we have known that if h is the sampling period, the width is $O(h)$ in standard SMC systems (Furuta (1990)) whereas it is $O(h^r)$ in the rth order SMC systems (Levant (1993)).

Consider the subsystem w given in (35),

$$w(k+1) = \Phi_3 w(k) - \alpha\Gamma_2\bar{s}_k$$

Denote the width of boundary layer w_i as ε_i, $i = 1, 2, \cdots, r+1$. According to the definitions of w_i, we have $w_1 = s$ and $w_{j+1} = dw_j/dt, j = 1, 2, \cdots, r$. Then, when $h \to 0$, the convergence accuracy can be inferred as $\varepsilon_1 = O(h^r)$. It means that the steady state motion lie within an $O(h^r)$ vicinity of sliding manifolds. An illustrative simulation will be presented in the next section to verify the above conclusion.

5. SIMULATIONS

For illustration, consider a sixth order system

$$A = \begin{bmatrix} 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 1&0&0&0&0&0 \end{bmatrix}, \quad b = \begin{bmatrix} 0\\0\\0\\0\\0\\1 \end{bmatrix}$$

Let the designated sliding mode be defined by

$$s(x) = x_1 + 2x_2 + x_3 + x_4$$

Here, the relative degree r is 2. Make the new switching function as

$$\bar{s}(y) = \bar{s}_1 + \bar{s}_2 + \bar{s}_3 = y_4 + y_5 + y_6$$

Whereas the state transformation matrix P is

$$P = \begin{bmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 1&2&1&1&0&0\\ 0&1&2&1&1&0\\ 0&0&1&2&1&1 \end{bmatrix}$$

Here, $d = (1, 1)$. Therefore, according to (24) $\bar{A}_c$ is defined by

$$\bar{A}_c = \begin{bmatrix} 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ -1&-2&-1&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 0&0&0&0&-1&-1 \end{bmatrix}$$

In the simulation, the gain $\alpha = 1$ and the sampling period is selected as $h = 0.1$. As an example, a system trajectory with the initial state x_0=$(1, 1, 0, -3, 1, 4)^\top$ is shown in Fig. 1 and Fig. 2. It finally converges to a period-2 orbits (Fig. 3 and Fig. 4) which are $(0.05, 4.1667 \cdot 10^{-8}, -8.3502 \cdot 10^{-8}, -4.1667 \cdot 10^{-5}, 8.3500 \cdot 10^{-5}, 0.05)$ and $(0.05, -4.1667 \cdot 10^{-8}, 8.3502 \cdot 10^{-8}, 4.1667 \cdot 10^{-5}, -8.3500 \cdot 10^{-5}, -0.05)$ within the bounds derived from Theorem 2 as $\|\bar{x}\| \le 3.8871$ and $\|\underline{x}\| \le 4.0879$. To verify the convergence accuracy, several simulation results with selected sampling periods 0.0001 and 0.001 are presented in Table 1, which clearly shows the reduction of chattering amplitude for different order sliding functions $\varepsilon_1, \varepsilon_2, \varepsilon_3$.

Notice that all the simulation examples in this paper are with period-2. However, periodic sequences with period $4, 6, 8, \cdots$ can be found by assigning different initial conditions under the same sampling time. For example, if the system trajectory starts from the original point with the sampling time $h = 0.1$, it will converge to a period-4 orbits. (Fig. 5 and Fig. 6). Despite the fact that the amplitude of periodic orbits decreases with h, it is possible that periodic orbits with complex switching patterns exist for arbitrary small sampling period.

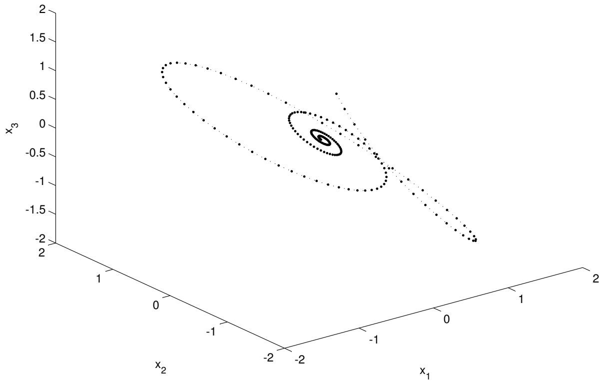

Fig. 1. Trajectory under coordinates x_1, x_2, x_3

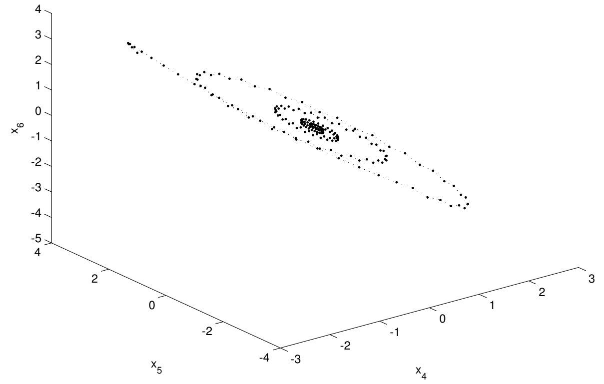

Fig. 2. Trajectory under coordinates x_4, x_5, x_6

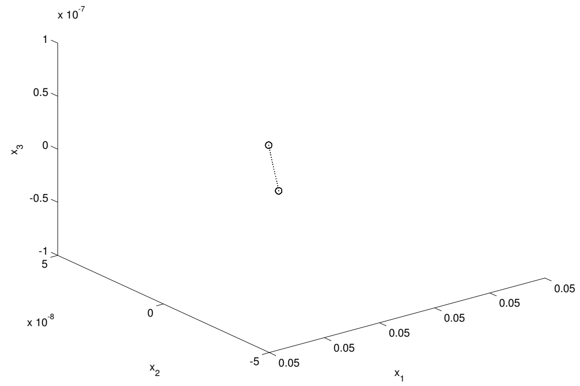

Fig. 3. Periodic orbits under coordinates x_1, x_2, x_3

6. CONCLUSION

In this paper, the ZOH discretization of higher-order SMC systems has been studied. The boundary layer issue has been considered as well, which confirms the higher-order accuracy performance. This work allows us to estimate the maximum chattering amplitude when the ZOH is applied to SMC systems.

It should be noted that after discretization, there are some commonalities between classical 1-sliding mode control systems and r-sliding mode systems such as periodic phenomenon. However, due to the page limit, this topic has not been discussed. Further work will be focused on the ZOH sampling of the higher-order multi-input SMC systems.

REFERENCES

G. Bartolini, A. Ferrara, and E. Usai. Chattering avoidance by second-order sliding mode control. *IEEE Transactions on Automatic Control*, 43:241–246, 1998.

G. Bartolini, A. Pisano, and E. Usai. Digital second-order sliding mode control for uncertain nonlinear systems. *Au-*

h	$\bar{\sigma}_1(w_1)$	$\bar{\sigma}_2(w_2)$	$\bar{\sigma}_3(w_3)$	ε_1	ε_2	ε_3
0.0001	$(1.7012 \cdot 10^{-5}, 1.7015 \cdot 10^{-5})$	$\pm 3.1553 \cdot 10^{-8}$	$\pm 5.0000 \cdot 10^{-5}$	$2.6676 \cdot 10^{-9}$	$6.3105 \cdot 10^{-8}$	$1.0000 \cdot 10^{-4}$
0.001	$(5.2249 \cdot 10^{-6}, 5.2272 \cdot 10^{-6})$	$\pm 2.5379 \cdot 10^{-7}$	$\pm 5.0025 \cdot 10^{-4}$	$2.3443 \cdot 10^{-9}$	$5.0757 \cdot 10^{-7}$	$1.0005 \cdot 10^{-3}$

Table 1: Convergence accuracy with selected sampling periods 0.0001 and 0.001.

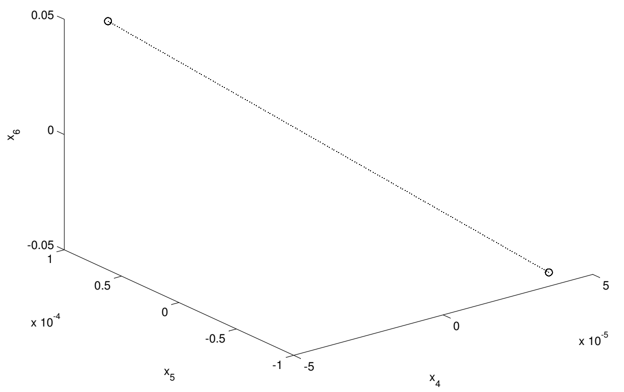

Fig. 4. Periodic orbits under coordinates x_4, x_5, x_6

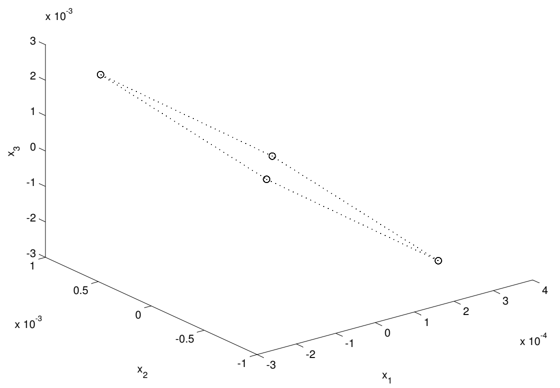

Fig. 5. Period-4 orbits under coordinates x_1, x_2, x_3

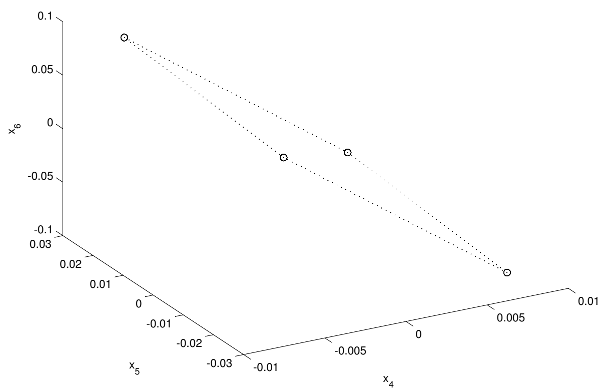

Fig. 6. Period-4 orbits under coordinates x_4, x_5, x_6

tomatica, 37:1371–1377, 2001.

B. Drazenovic. The invariance conditions in variable structure systems. *Automatica*, 5:287–295, 1969.

L. Fridman and A. Levant. Higher order sliding modes as a natural phenomenon in control theory. In F. Garofalo and L. Glielmo, editors, *Robust Control via Variable Structure and Lyapunov Techniques*, chapter 6, pages 107–133. Springer-Verlag, London, 1996.

K. Furuta. Sliding mode control of a discrete system. *Systems and Control Letters*, 14:145–152, 1990.

A. Isidori. *Nonlinear Control Systems: An Introduction, 3rd edition*. Springer, Berlin, 1995.

S. Laghrouche, F. Plestan, and A. Glumineau. Higher order sliding mode control based on intergral sliding mode. *Automatica*, 43:531–537, 2007.

A. Levant. Higher order sliding modes, differetiation and output-feedback control. *International Journal of Control*, 76:924–941, 2003.

A. Levant. Sliding order and sliding accuracy in sliding mode control. *International Journal of Control*, 58:1247–1263, 1993.

A. Levant and L. Fridman. Robustness issues of 2-sliding mode control. In A. Sabanovic, L.M. Fridman, and S. Spurgeon, editors, *Variable Structure Systems: from Principles to Implementation*, chapter 6, pages 131–156. IEE, London, 2004.

J. J. Slotine. Sliding mode controller design for non-linear systems. *International Journal of Control*, 40:421–434, 1984.

V. I. Utkin, J. Guldner, and J. Shi. *Sliding mode control in electromechanical systems*. Taylor & Francis, London, 1999.

V.I. Utkin. *Sliding Modes in Control Optimization*. Springer-Verlag, Berlin, 1992.

B. Wang, X. Yu, and G. Chen. Discretization behaviors of sliding mode control systems with matched uncertainties. *Proceedings of WCICA*, June 2006.

X. Yu and G. Chen. Discretization behaviors of equivalent control based variable structure systems. *IEEE Transactions on Automatic Control*, 48:1641–1646, 2003.

X. Yu and Z. Galias. Periodic behaviors in a digital filter with two's complement arithmetic. *IEEE Transactions on Circuits and Systems–I*, 48:1177–1190, 2001.

X. Yu, J. Xu, Y. Hong, and S. Yu. Analysis of a class of discrete-time systems with power rule. *Automatica*, 43:562–566, 2007.

X. Yu, B. Wang, Z. Galias, and G. Chen. Discretization effect on equivalent control based multi-input sliding mode control systems. *IEEE Transactions on Automatic Control, accepted for publication*, February 2008.