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Averaging and stability of time-varying discrete-time linear systems ⋆

Alessandro N. Vargas * **João B. R. do Val** *

* *Universidade Estadual de Campinas, UNICAMP, Fac. de Eng. Elétrica e de Computação, Depto. de Telemática, C.P. 6101, 13081-970 Campinas-SP, Brazil. (E-mail: {vargas,jbosco}@ dt.fee.unicamp.br).*

Abstract: Stability results for time-varying systems with output averaged using the norm are established. The discrete-time linear system is considered under a bound on time-average on the mean-value of the output norm. It is shown that under controllability, observability and a uniform bound on the matrix norm, the time-varying system is asymptotically and uniformly stable. If, in addition, the solution of the autonomous part of the system state is almost periodic, then under an additional limiting condition the system is uniformly asymptotically stable.

Keywords: Time-varying systems; linear systems; difference equations; average values; periodic motion.

1. INTRODUCTION

We denote respectively the real and natural numbers by $\mathbb{R}$ and $\mathbb{N}$, and the n–dimensional real Euclidean space by $\mathbb{R}^n$. The normed linear space of all $n \times m$ real matrices is denoted by $\mathbb{R}^{n,m}$. The superscript $'$ indicates the transpose of a matrix.

Consider a time-varying discrete linear system modeled by the following evolution difference equation

$$\mathscr{S}_0 : \begin{cases} x(k+1) = A_k x(k) + B_k w(k), \\ y(k) = C_k x(k) + D_k w(k),\ k \geq 0,\ x(0) = x_0 \in \mathbb{R}^n, \end{cases}$$

The sequences $\{x(k); k \geq 0\}$, $\{w(k); k \geq 0\}$ and $\{y(k); k \geq 0\}$ evolve respectively, in $\mathbb{R}^n$, $\mathbb{R}^m$ and $\mathbb{R}^p$ and they represent the system state, input and output, in this order. The sequence of matrices $\{A_k; k \geq 0\}$, $\{B_k; k \geq 0\}$ and $\{C_k; k \geq 0\}$ are composed by elements of the corresponding dimensions $\mathbb{R}^{n,n}$, $\mathbb{R}^{n,m}$ and $\mathbb{R}^{p,n}$.

Many important properties about the system $\mathscr{S}_0$ can be inferred by analysing the behavior of the following matrix system:

$$X(k+1) = A_k X(k) A_k' + B_k B_k', \tag{1}$$

$$Y(k) = C_k X(k) C_k' + D_k D_k',\ k \geq 0, \tag{2}$$

where $X(0) = x_0 x_0'$. The difference equation (1) is known as the state correlation evolution according to Davis and Vinter [1985], and (2) is the corresponding output. Recurrences (1) and (2) may represent, for instance, the system $\mathscr{S}_0$ evolving in a stochastic environment, as a result of letting $w(k)$ be a second order noisy process driving it.

If the sequence $\{X(k)\}$ generated by (1) is bounded and the pair (A_k, B_k) is stabilizable, Anderson and Moore [1981] showed that the autonomous system $x(k+1) = A_k x(k)$ is uniformly asymptotically stable, provided that the sequences $\{A_k\}$ and $\{B_k\}$ are uniformly bounded. Dragan and Morozan [2006] and Kubrusly [1988] generalized this result to the context of positive operators and infinite-dimensional systems, respectively; however, in both papers $B_k = I$ for all $k \in \mathbb{N}$. The boundedness of $\{X(k)\}$ to assure the uniform exponential stability of the autonomous system is a critical assumption in these works.

We wish to generalize the study of stability in the following way. If $\{\|Y(k)\|\}$ is bounded, then it is straightforward to show that there exists $c > 0$ such that

$$\frac{\|Y(0)\| + \ldots + \|Y(n)\|}{n+1} < c, \quad \text{for all } n \in \mathbb{N}. \tag{3}$$

Conversely, suppose that the elements of $\{\|Y(k)\|\}$ can be averaged in norm, in the sense that that there exists $c > 0$ such that (3) holds. In this situation, an intriguing question is whether the sequence $\{\|Y(k)\|\}$ is bounded or not. The answer would be "no" since it is possible to create a subsequence of $\{\|Y(k)\|\}$ that diverges whereas (3) still holds, e.g. see [Goldberg, 1964, p.55]. However, in the above argument we are not taking into account the particular structure of the dynamical system (1). We shall demonstrate that this feature with some additional conditions leads to a positive answer to the above question.

Another important question is whether the system $\mathscr{S}_0$ would be stable in some sense. Based on the assumption that there exists a real number $c > 0$ such that (3) holds for all $n \in \mathbb{N}$, conditions are derived to ensure that the null solution of the autonomous system $x(k+1) = A_k x(k)$ is

i) asymptotically stable;
ii) uniformly asymptotically stable.

The conditions we employ to obtain (i) are quite common but the reinforcement to guarantee (ii) is purposeful, since property (ii) may be useful specially in applications.

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The average relation (3) arises in certain situations of interest. From observations of the sequence $\{\|Y(k)\|\}, k \leq n$ it is simpler to find a number c from averaging as in (3) than to establish an uniform bound to the sequence. Another situation of interest is when (3) is known to hold beforehand. Suppose that the choice of the sequences $\{A_k, B_k, C_k\}$ are set to minimize the "quadratic cost" $\sum_{k=0}^{n} \|Y(k)\|$ for increasing n. If it is known that these problems have solution, then the minimizing sequence should observe $\sum_{k=0}^{n} \|Y(k)\| < c \cdot n + d$, for some d depending on the initial value $\|X(0)\|$ and n sufficiently large.

The results obtained in this paper extend the results given by Vargas et al. [2007] in two directions. First, we study the averaged-output instead of the averaged-state, and the results are then proved assuming observability and controllability of the time-varying system. Second, the uniform asymptotic stability is established for almost periodic matrices, a weaker condition than the periodic one adopted by Vargas et al. [2007].

Section 2 presents some preliminary results. Section 3 presents the main results.

2. PRELIMINARY RESULTS

Let $\mathbb{R}_+^{n,n}$ be the closed convex cone $\{U \in \mathbb{R}^{n,n} : U = U' \geq 0\}$; $U \geq V$ signifies that $U - V \in \mathbb{R}_+^{n,n}$. Let $\mathrm{tr}\{\cdot\}$ be the trace operator. If U is a matrix in $\mathbb{R}^{m,n}$, we let $\|U\|_2 := \mathrm{tr}\{U'U\}^{\frac{1}{2}}$ be the Euclidean (Frobenius) norm for matrices. With this notation, $U \in \mathbb{R}_+^{n,n}$ if and only if $\|U^{\frac{1}{2}}y\|_2^2 \geq 0$ for all $y \in \mathbb{R}^n$ (where $U^{\frac{1}{2}} \in \mathbb{R}_+^{n,n}$ is such that $U^{\frac{1}{2}}U^{\frac{1}{2}} = U$). Recall that $\|U\|_2 = \|U'\|_2$, and if $U \in \mathbb{R}^{n,m}$ and $V \in \mathbb{R}^{n,p}$ are such that $UU' \geq VV'$, then $\|U\|_2^2 \geq \|V\|_2^2$.

In association with the sequence of matrices $\{A_k; k \geq 0\}$ in $\mathscr{S}_0$, we can define a family of discrete evolution operators, namely,

$$\Phi(k, s) = A_{k-1}A_{k-2}\cdots A_s, \quad \text{for each } k > s \geq 0, \tag{4}$$

with $\Phi(s, s) = I$. By induction on (1) and (4) one can readily verify as in [Halanay and Ionescu, 1994, p. 17] that

$$X(k) = \Phi(k, s)X(s)\Phi(k, s)' + \sum_{j=s}^{k-1} \Phi(k, j+1)B_jB_j'\Phi(k, j+1)', \tag{5}$$

holds for every $k \geq s$, for $X(0) \in \mathbb{R}_+^{n,n}$. For ease of notation, we represent $X(0) \in \mathbb{R}_+^{n,n}$ without loss of generality by $X(0) = B_{-1}B_{-1}'$, where $B_{-1} = X(0)^{1/2}$ is a matrix in $\mathbb{R}^{n,n}$. Thus, (5) yields

$$X(k) = \sum_{j=0}^{k} \Phi(k, j)B_{j-1}B_{j-1}'\Phi(k, j)', \quad \forall k \in \mathbb{N}. \tag{6}$$

In the following sections some properties of (1) and (2) will be exploited.

Observability, controllability and some results for state correlation evolution

The uniform observability and controllability concepts for discrete time-varying linear systems were defined in 1960 by Kalman Kalman [1960]. Among others, we can cite as important contributions in the theoretical setting of time-varying controllability and observability concepts, the papers of Silverman and Anderson [1968], Kern and Przyluski [1990] and Benzaid [1999] for stability; and Moore and Anderson [1980] for estimation and control of time-varying linear systems.

The definition of observability and controllability are given below (see Halanay and Ionescu [1994] and Moore and Anderson [1980] for further details).

Definition 1. The pair (A_k, C_k) is uniformly observable (or simply observable) if there exists $T_o \geq 1$ and a real number $\beta > 0$ such that, for all $k \geq 0$,

$$\sum_{i=0}^{T_o-1} \Phi(k+i, k)'C_{k+i}'C_{k+i}\Phi(k+i, k) \geq \beta I.$$

Definition 2. The pair (A_k, B_k) is uniformly controllable (or simply controllable) if there exists $T_c \geq 1$ and a real number $\sigma > 0$ such that, for all $k \geq T_c$,

$$\sum_{i=0}^{T_c-1} \Phi(k, k-i)B_{k-i-1}B_{k-i-1}'\Phi(k, k-i)' \geq \sigma I.$$

The next two lemmas yield two inequalities related to system (1),(2) that shall be useful in the sequence.

Lemma 3. If the pair (A_k, C_k) is observable, then there exist $T \geq 1$ and $\delta = \delta(T) > 0$ such that, for all $n \geq T - 1$,

$$\sum_{i=0}^{n} \|C_{k+i}\Phi(k+i, k)y\|_2^2 \geq \sum_{i=0}^{n-T+1} \delta\|\Phi(k+i, k)y\|_2^2, \quad \forall k \in \mathbb{N}, \quad \forall y \in \mathbb{R}^n. \tag{7}$$

Proof. Notice that for $n = T - 1$, the relation in the lemma holds straightforwardly from the observability definition, by setting $T = T_o$ and $\delta = \beta$. For $n > T - 1$, a more elaborated argument to show the result shall be developed.

First note from the observability definition that there exist $T = T_o \in \mathbb{N}$ and $\beta > 0$ such that

$$\sum_{\ell=rT+i}^{(r+1)T+i-1} \|C_{k+\ell}\Phi(k+\ell, k+rT+i)y\|_2^2 \geq \beta\|y\|_2^2, \quad \forall r, i, k \in \mathbb{N}, \quad \forall y \in \mathbb{R}^n. \tag{8}$$

The above inequality shall be of use in the sequel.

Choose $N \in \mathbb{N}$. Two cases then arises.

Case 1. ($N \geq T-1$) For each $i = 0, \ldots, T-1$, let us define

$$r^*(i) = \max\{r \in \mathbb{N} : i + rT \leq N\}. \tag{9}$$

From this definition, we get the following identity:

$$\sum_{i=0}^{T-1}\sum_{r=0}^{r^*(i)} \|\Phi(k+rT+i, k)y\|_2^2 = \sum_{\ell=0}^{N} \|\Phi(k+\ell, k)y\|_2^2, \quad \forall k \in \mathbb{N},\ \forall y \in \mathbb{R}^n. \tag{10}$$

In addition, by (9), the following inequality is valid for all $k \in \mathbb{N}$ and $y \in \mathbb{R}^n$ and for each $i = 0, \ldots, T-1$:

$$\sum_{\ell=0}^{N+T-1} \|C_{k+\ell}\Phi(k+\ell,k)y\|_2^2 \geq \sum_{r=0}^{r^*(i)} \sum_{\ell=rT+i}^{(r+1)T+i-1} \|C_{k+\ell}\Phi(k+\ell,k)y\|_2^2, \quad (11)$$

But

$$\|C_{k+\ell}\Phi(k+\ell,k)y\|_2^2 = \|C_{k+\ell}\Phi(k+\ell,k+rT+i)\Phi(k+rT+i,k)y\|_2^2,$$

hence from (8), the right-hand side of (11) is bounded below by

$$\sum_{r=0}^{r^*(i)} \beta\|\Phi(k+rT+i,k)y\|_2^2.$$

Now, summing up on $i = 0, 1, \ldots, T-1$, one obtains that

$$T \cdot \sum_{\ell=0}^{N+T-1} \|C_{k+\ell}\Phi(k+\ell,k)y\|_2^2 \geq \sum_{i=0}^{T-1}\sum_{r=0}^{r^*(i)} \beta\|\Phi(k+rT+i,k)y\|_2^2 = \sum_{\ell=0}^{N} \beta\|\Phi(k+\ell,k)y\|, \quad (12)$$

where the last identity follows from (10). Take $\delta = \beta/T$, and the result then follows from (12) for $N \geq T-1$.

Case 2. ($0 \leq N < T-1$) In this case, define $r^*(i) = 0$ for each $i = 0, \ldots, N$, and the result can be easily shown by repeating the arguments used in *Case 1* only for $i = 0, \ldots, N$. □

The next result is the dual of Lemma 3 and the proof is omited. See Halanay and Ionescu [1994], Moore and Anderson [1980] for further details on duality for time-varying linear systems.

Lemma 4. If the pair (A_k, B_k) is controllable, then there exist $T \geq 1$ and $\rho = \rho(T) > 0$ such that, for each $k \geq T-1$,

$$\sum_{j=0}^{k} \|B'_{j-1}\Phi(k,j)'y\|_2^2 \geq \sum_{j=T-1}^{k} \rho\|\Phi(k,j)'y\|_2^2, \quad \forall y \in \mathbb{R}^n.$$

The next lemma is an auxiliary step to the main result of this section. For the sake of space, we omit its proof.

Lemma 5. Let $\{X(k)\}$ in $\mathbb{R}_+^{n,n}$ and $\{Y(k)\}$ in $\mathbb{R}_+^{p,p}$ be the sequences that satisfy (1) and (2), respectively.

i) If the pair (A_k, C_k) is observable, then there exist $T \in \mathbb{N}$ and $\delta > 0$ such that
$$\sum_{i=k}^{n+T} \|Y(i)^{\frac{1}{2}}\|_2^2 \geq \sum_{i=k}^{n} \delta\|X(i)^{\frac{1}{2}}\|_2^2, \quad \forall n \geq k \in \mathbb{N}. \quad (13)$$

ii) If the pair (A_k, B_k) is controllable, then there exist $T \in \mathbb{N}$ and $\rho > 0$ such that
$$\|X(k)^{\frac{1}{2}}\|_2^2 \geq \sum_{j=T}^{k} \rho\|\Phi(k,j)\|_2^2, \quad \forall k \geq T. \quad (14)$$

The next result provide an important relation between the output $Y(\cdot)$ and the evolution operator $\Phi(\cdot)$.

Lemma 6. Let $\{X(k)\}$ in $\mathbb{R}_+^{r,r}$ and $\{Y(k)\}$ in $\mathbb{R}_+^{p,p}$ be the sequences that satisfy (1) and (2), respectively. If the pairs (A_k, C_k) and (A_k, B_k) are observable and controllable, respectively, then there exist $T \in \mathbb{N}$ and $\alpha = \alpha(T) > 0$ such that, for all $n \geq k$ and all $k \geq T$,

$$\sum_{i=k}^{n+T} \|Y(i)^{\frac{1}{2}}\|_2^2 \geq \sum_{i=k}^{n}\sum_{j=T}^{i} \alpha\|\Phi(i,j)\|_2^2. \quad (15)$$

Proof. It follows from Lemma 5 (i) (resp., (ii)) that there exist $T' \in \mathbb{N}$ and $\delta > 0$ (resp., $T'' \in \mathbb{N}$ and $\rho > 0$) such that (13) (resp., (14)) holds. Take $T = \max\{T', T''\}$, and so (13) and (14) imply

$$\sum_{i=k}^{n+T} \|Y(i)^{\frac{1}{2}}\|_2^2 \geq \sum_{i=k}^{n} \delta\|X(i)^{\frac{1}{2}}\|_2^2 \geq \sum_{i=k}^{n}\sum_{j=T}^{i} \delta\rho\|\Phi(i,j)\|_2^2,$$

for all $n \geq k$ and all $k \geq T$. This shows the result. □

The next assumption plays a key role in the development of stability conditions.

Assumption 7. There exists $J > 0$ such that, for all $N \in \mathbb{N}$,

$$\sum_{k=0}^{N} \|Y(k)^{\frac{1}{2}}\|_2^2 \leq (N+1)J,$$

where $\{Y(k)\}$ in $\mathbb{R}_+^{p,p}$ satisfies (2).

Remark 8. It follows from Assumption 7 and the difference equations (1) and (2) that there exists $J > 0$ such that

$$(N+1)J \geq \sum_{k=0}^{N}\sum_{j=0}^{k} \|C_k\Phi(k,j)B_{j-1}\|_2^2, \quad \forall N \in \mathbb{N}. \quad (16)$$

Note that if all the elements in the right-hand side of (16) would be equal to a positive constant value, namely $c > 0$, then we would have $(N+1)J \geq c(N+1)(N+2)/2$ for all $N \in \mathbb{N}$, which is absurd. This indicates that some elements in the right-hand side of (16) should vanish. Indeed, we prove in Section 3 that observability, controllability and uniform bounds on $\{A_k\}$ are enough to ensure a contraction property of the evolution operator $\Phi(\cdot)$.

The main idea in the following analysis is to consider only matrix sequences $\{A_0, \ldots, A_{N-1}\}$ that satisfy both (1), (2) and Assumption 7. The next definition subsumes this discussion.

Definition 9. For all $N \in \mathbb{N}$,

$$\mathscr{A}_N = \big\{(A_0, \ldots, A_{N-1}) \;:\; A_k \in \mathbb{R}^{n,n},\ k = 0, \ldots, N-1, \text{ and Assumption 7 holds true}\big\}. \quad (17)$$

Now we are able to present the main result of this section.

Theorem 10. Let the pairs (A_k, C_k) and (A_k, B_k) be observable and controllable, respectively. If $(A_0, \ldots, A_{N-1})$ is in $\mathscr{A}_N$, then there exist a subsequence $\{n_i\}$ of $\mathbb{N}$ and a real number $L > 0$ (which does not depend on $\{n_i\}$) such that

$$\sum_{k=0}^{N} \|\Phi(k+n_i, n_i)\|_2^2 \leq L, \quad \forall N, i \in \mathbb{N}. \quad (18)$$

Proof. The proof follows by contradiction. The logical negation of the thesis is as follows. For all $L > 0$, there exist $N_0 \in \mathbb{N}$ and $i_0 \in \mathbb{N}$ such that, for all $i \geq i_0$, we have

$$\sum_{k=0}^{N_0} \|\Phi(k+i,i)\|_2^2 > L. \tag{19}$$

In addition, by Assumption 7, there exists $J > 0$ such that

$$\sum_{k=0}^{N} \|Y(k)^{\frac{1}{2}}\|_2^2 \leq (N+1)J, \quad \forall N \in \mathbb{N}. \tag{20}$$

After some algebraic manipulations in the inequality of Lemma 6, it is possible to show that there exist $T \in \mathbb{N}$ and $\alpha > 0$ so that

$$\sum_{k=0}^{2(N+T)+m} \|Y(k)^{\frac{1}{2}}\|_2^2 \geq \sum_{i=m}^{N+m} \sum_{n=0}^{N} \alpha \|\Phi(n+i+T, i+T)\|_2^2, \tag{21}$$

for all $N, m \in \mathbb{N}$. Now choose $L > \frac{5J}{\alpha}$, then by (19),

$$\sum_{k=0}^{N} \|\Phi(k+i,i)\|_2^2 > \frac{5J}{\alpha}, \quad \forall N \geq N_0, \quad \forall i \geq i_0. \tag{22}$$

Hence, if $\tilde{N} = \max\{N_0, i_0, T\}$, then by (20), (21) and (22),

$$(5\tilde{N}+1)J \geq (2(\tilde{N}+T)+i_0+1)J \geq \sum_{k=0}^{2(\tilde{N}+T)+i_0} \|Y(k)^{\frac{1}{2}}\|_2^2$$

$$\geq \sum_{i=i_0}^{\tilde{N}+i_0} \sum_{n=0}^{\tilde{N}} \alpha \|\Phi(n+i+T, i+T)\|_2^2$$

$$> \sum_{i=i_0}^{\tilde{N}+i_0} \alpha \frac{5J}{\alpha} = 5(\tilde{N}+1)J,$$

which is absurd. This argument completes the proof. □

Note that Theorem 10 is not able to guarantee that the value $L > 0$ is uniform with respect to all n in $\mathbb{N}$ such that (18) holds. If this were true, then the autonomous system $x(k+1) = A_k x(k)$ would be uniformly asymptotically stable (we shall formalize this property latter). Despite the non-uniformity, the result of Theorem 10 enable us to affirm that $L > 0$ is uniform for a subsequence $\{n_i\}$ of $\mathbb{N}$. Moreover, a theoretically interesting point is that the conditions we use in Theorem 10 are quite general and do not require, for instance, bounds on the matrix sequence $\{A_k\}$. In the next section we will let the horizon N in (18) tend to infinity, and so we study under what circumstances should $L > 0$ be uniform not only for a subsequence $\{n_i\}$ of $\mathbb{N}$ but also for all $n \in \mathbb{N}$. One difficulty in this approach is the unboundedness property that may appear in the step $n_i - n_{i-1}$, namely, we may have a subsequence from $\{n_i - n_{i-1}\}$ on which diverges. We shall develop conditions in order to remedy this situation.

3. STABILITY CONDITIONS

It is possible now to obtain some useful results concerning the stability of the system $\mathscr{S}_0$. Attention is focused on the class $\mathscr{A}_N$ defined in (17) when $N \to \infty$. In connection, it will be denoted by $\mathscr{A}_\infty$.

Next, we investigate the stability of the zero-input response of $\mathscr{S}_0$. More specifically, we study the response of

$$x(k+1) = A_k x(k) \tag{23}$$

due to any initial state $x(k_0) = x_0$. In connection, we recall some stability definitions from Willems [1970] and Sastry [1999].

Definition 11. The null solution of the autonomous system (23) (or simply the system (23)) is called:

i) stable, if for any given $k_0 \geq 0$ and $\epsilon > 0$, there exists a positive number $\delta(\epsilon, k_0)$ such that if $\|x_0\| < \delta$, then $\|x(k)\| \leq \epsilon$ holds for all $k \geq k_0$. If δ does not depend on k_0 we say the system is uniformly stable.
ii) asymptotically stable, if for any $k_0 \geq 0$, there exists a positive number $\delta(k_0)$ such that if $\|x_0\| < \delta$, then for any $\epsilon > 0$, there exists a natural number $T(\epsilon, \delta, k_0)$ such that $\|x(k)\| < \epsilon$ for all $k > k_0 + T$.
iii) uniformly asymptotically stable, if the numbers δ and T, introduced in the item (ii) can be taken independent of k_0.

Remark 12. It may be shown [Willems, 1970, Ch.4] that the system (23) is stable (uniformly stable) if and only if there exists $c > 0$ which depends on k_0 (does not depend on k_0) such that $\|\Phi(k, k_0)\| \leq c$ for all $k \geq k_0$. Asymptotic stability then follows if, in addition, $\|\Phi(k, k_0)\| \to 0$ as $k \to \infty$. Some equivalences for uniform asymptotic stability are provided later.

3.1 *Asymptotic stability results*

The following theorem provides arguments to guarantee that any sequence $\{A_k\}$ generated according to (17) (i.e., $\{A_k\} \in \mathscr{A}_\infty$) is asymptotically stable and uniformly stable. The proof follows similarly as in [Vargas et al., 2007, Th.2].

Theorem 13. Let the pairs (A_k, C_k) and (A_k, B_k) be observable and controllable, respectively. If the matrix sequence $\{A_k\} \in \mathscr{A}_\infty$ is bounded, then the following holds:
(a) For all $k \geq 0$ and all $\ell > 0$, there exists $c > 0$ (which does not depend on k nor ℓ) such that

$$\|\Phi(k+\ell, k)\| < c.$$

(b) There exist a subsequence $\{n_m\}$ of $\mathbb{N}$ and a real number $L > 0$ (which does not depend on $\{n_m\}$) such that

$$\sum_{k=0}^{\infty} \|\Phi(k+n_m, \ell)\| \leq L, \quad \forall \ell = 0, \ldots, n_m, \quad \forall m \in \mathbb{N}.$$

The above theorem leads to the following result.

Corollary 14. Suppose that the conditions of Theorem 13 hold. Then the system (23) is uniformly stable and asymptotically stable.

An advantage of the above approach is that it enables us to derive some conditions in order to assure the uniform asymptotic stability of the system (23). The study of this important context of stability will be carried out in the next section.

3.2 *Uniform asymptotic stability results*

The aim of this section is to investigate how the conditions of Theorem 13 (and Corollary 14), which ensure the asymptotic stability, can be strengthened to yield to the "uniform" asymptotic stability.

The following equivalence shall be used in the sequel (see [Willems, 1970, Th.1.5, p.101] and [Kubrusly, 1988, Th.1]).

Proposition 15. The following conditions are equivalent:

i) The system (23) is uniformly asymptotically stable.

ii) For some $p > 0$, there exists a positive number σ_p such that
$$\sum_{i=0}^{\infty} \|\Phi(k+i,k)\|_2^p \leq \sigma_p, \quad \forall k \in \mathbb{N}.$$

It was shown by Vargas et al. [2007] that if the matrices A_k are periodic, then the system (23) is uniformly asymptotically stable provided that the conditions of Theorem 13 are satisfied. Here it is demonstrated that the periodic condition can be replaced by a weaker one.

Almost periodic systems The theory of almost periodic functions has been a subject of intensive research in pure and applied mathematics. There are applications to the theory of differential and difference equations, statistics and celestial mechanics (see Corduneanu [1961], Fink [1972] and the references therein).

Our contribution is to show that, if $\xi_{k_0}(k)$ is an almost periodic solution of (23) for each $k_0 \geq 0$, and if the hypotheses of Theorem 13 are satisfied, then (23) is uniformly asymptotically stable.

Before passing to the study of almost periodic solutions of systems, let us recall the Bohr's definition of almost periodic sequences. Such definition can be found, for instance, in [Halanay and Ionescu, 1994, p.205],[Corduneanu, 1961, p.45].

Definition 16. Let X be a Banach space. An X-valued sequence $\{f(k)\}$ is called almost periodic, if to any $\varepsilon > 0$ there corresponds a natural number $N(\varepsilon)$, such that among any N consecutive integers there exists a natural number p with the property
$$\|f(k+p) - f(k)\| < \varepsilon, \quad k = 0, 1, \ldots. \tag{24}$$

Remark 17. It follows from the above definition that, for any given $\varepsilon > 0$, one can extract a subsequence $\{p_i\}$ from $\mathbb{N}$ such that $\|f(k+p_i) - f(k)\| < \varepsilon$, for all $k, i \in \mathbb{N}$.

Lemma 18. Suppose that $\{\xi_{k_0}(k)\}_{k \geq k_0}$ is an almost periodic solution of (23). If δ is a finite natural number, then for all $\varepsilon > 0$ there exists $\tau = \tau(\varepsilon, \delta) \in \mathbb{N}$ (with $\tau > \delta$) such that, for all $k \geq k_0$,
$$\varepsilon > \left| \sum_{i=0}^{\delta} \|\Phi(\tau+k+i,k_0)x_0\|_2^2 - \|\Phi(k+i,k_0)x_0\|_2^2 \right|.$$

Proof. Let $\{F(k)\}_{k \geq k_0}$ be the nonnegative real sequence defined as
$$F(k) = \sum_{i=0}^{\delta} \|\xi_{k_0}(k+i)\|_2^2, \quad \forall k \geq k_0. \tag{25}$$
It is clear that $\{F(k)\}_{k \geq k_0}$ is almost periodic, since $\{\xi_{k_0}(k)\}_{k \geq k_0}$ is almost periodic and so is any finite sum of almost periodic sequences [Corduneanu, 1961, p.46]. Hence, by Definition 16 and Remark 17, for any $\varepsilon > 0$ there exists $\tau \in \mathbb{N}$ with $\tau > \delta$ such that $\varepsilon > |F(k+\tau) - F(k)|$ for all $k \in \mathbb{N}$. The result then follows from this inequality and (25). □

Now it is possible to provide the main result of this section.

Theorem 19. Suppose that $\{\xi_{k_0}(k)\}_{k \geq k_0}$ is an almost periodic solution of (23) for any $k_0 \geq 0$. Then, under the assumptions of Theorem 13, the autonomous system (23) is uniformly asymptotically stable.

Proof. Notice first that, by Theorem 13, there exist a subsequence $\{n_m\}$ of $\mathbb{N}$ and a real number $L > 0$ such that
$$\sum_{k=0}^{\infty} \|\Phi(k+n_m, j)\| \leq L, \quad \forall j : n_m \geq j > n_{m-1}, \quad \forall m \geq 1. \tag{26}$$
Assume that there exists $M > 0$ such that
$$\sum_{k=j}^{n_m - 1} \|\Phi(k,j)\| \leq M, \quad \forall j : n_m > j > n_{m-1}, \quad \forall m \geq 1. \tag{27}$$
Then (26) and (27) imply that
$$\sum_{k=0}^{\infty} \|\Phi(j+k, j)\| \leq L+M, \quad \forall j : n_m \geq j \geq n_{m-1}, \quad \forall m \geq 1. \tag{28}$$
Evidently, (28) holds for all $j \in \mathbb{N}$, and so the result follows from Proposition 15.

It remains to show that the claim (27) is valid. To do so we employ a contradiction argument. If (27) were not true, then there would exist a subsequence $\{j_i\}$ of $\mathbb{N}$ and a subsequence $\{n_{m_i}\}$ of $\{n_m\}$ with $n_{m_i} > j_i > n_{m_i - 1}$ for which
$$\sum_{k=j_i}^{n_{m_i}-1} \|\Phi(k,j)\|_2^2 \to \infty \text{ as } i \to \infty. \tag{29}$$
If we let $\varepsilon_0 > 0$ be any given number, then (29) assures that there are natural numbers n_{m_0} and j_0 (with $n_{m_0} > j_0 > n_{m_0 - 1}$) such that
$$\sum_{k=j_0}^{n_{m_0}-1} \|\Phi(k, j_0)\|_2^2 > \varepsilon_0 + L.$$
Set $\delta_0 = n_{m_0} - j_0 - 1$, and one can find a suitable $x_0 \in \mathbb{R}^n$ with $\|x_0\|_2^2 = 1$ such that
$$\sum_{k=0}^{\delta_0} \|\Phi(j_0+k, j_0)x_0\|_2^2 > \varepsilon_0 + L. \tag{30}$$
Now, in Lemma 18, set $\delta = \delta_0$, $\varepsilon = \varepsilon_0$, and $k_0 = j_0$. Then there exists $\tau \in \mathbb{N}$ (with $\tau > \delta_0$) such that
$$\varepsilon_0 > \sum_{k=0}^{\delta_0} \|\Phi(j_0+k, j_0)x_0\|_2^2 - \|\Phi(j_0+\tau+k, j_0)x_0\|_2^2. \tag{31}$$
However, since $\tau > \delta_0 = n_{m_0} - j_0 - 1$, it follows from (26) that
$$\sum_{k=0}^{\delta_0} \|\Phi(j_0+\tau+k, j_0)x_0\|_2^2 \leq \sum_{k=0}^{\infty} \|\Phi(k+n_{m_0}, j_0)x_0\|_2^2 \leq L. \tag{32}$$
Thus, by (30), (31) and (32),
$$\begin{aligned}\varepsilon_0 + L &< \sum_{k=0}^{\delta_0} \|\Phi(j_0+k, j_0)x_0\|_2^2 \\ &= \sum_{k=0}^{\delta_0} \|\Phi(j_0+k, j_0)x_0\|_2^2 - \|\Phi(j_0+\tau+k, j_0)x_0\|_2^2 \\ &\quad + \sum_{k=0}^{\delta_0} \|\Phi(j_0+\tau+k, j_0)x_0\|_2^2 < \varepsilon_0 + L,\end{aligned}$$

which is absurd. This establishes the contradiction and so the result is proven. □

Intuitively one expects that any bounded solution of (23), with almost periodic matrices A_k, is almost periodic. However, it may be shown that this statement is false (e.g. Conley and Miller [1965], Mingarelli et al. [1995] and [Fink, 1974, p.97]). Many authors have considered the problem of how to obtain almost periodic solutions for difference and differential equations (see Corduneanu [1961], Fink [1972], Sell [1966], and the references therein). In particular, the Amerio's condition is the only one used here because of its simplicity.

Let $g = \{g(k)\}$ be an almost periodic sequence. Then there exist subsequences $\{t_n\}$ of $\mathbb{N}$ for which $g(k + t_n)$ converges to $\hat{g}(k)$, where $\hat{g}(k)$ may be different from $g(k)$ (see Fink [1974] for a detailed proof). Let one considers all possible subsequences $\{t_n\}$ of $\mathbb{N}$ for which there is a uniform convergence

$$g(k+t_n) \to \hat{g}(k) \quad (\text{as} \quad n \to \infty),$$

and denote the set of limit sequences $\{\hat{g}(k)\}$ by $\mathscr{H}(g)$.

The following statement follows from the Amerio's theorem (see [Corduneanu, 1961, p.109] and [Pennequin, 2001, Cor.2.7] for further details). Suppose that the following two conditions hold:

(A) The autonomous system (23), with almost periodic matrices A_k, is uniformly stable;

(B) For each $\{\hat{A}_k\} \in \mathscr{H}(A)$, the system $y(k+1) = \hat{A}_k y(k)$, $y(k_0) = y_0 \in \mathbb{R}^n$, has just the null solution as bounded solution.

Then any bounded solution of (23) is almost periodic. An immediate consequence from the above fact and Theorem 19 is the following.

Corollary 20. Assume that the conditions of Theorem 13 are satisfied. If $\{A_k\} \in \mathscr{A}_\infty$ is almost periodic and the condition (B) holds true, then the autonomous system (23) is uniformly asymptotically stable.

It is noteworthy that, if $\{A_k\} \in \mathscr{A}_\infty$ is periodic, there is no need of condition (B) in Corollary 20, since (23) should have an almost periodic solution [Fink, 1974, Cor.6.5].

4. CONCLUDING REMARKS

The main results of this paper rely on the assumption that $\{A_k\} \in \mathscr{A}_\infty$. Asymptotic stability and uniform stability are established for the null solution of the autonomous system $x(k+1) = A_k x(k)$, but this holds only if the pairs (A_k, C_k) and (A_k, B_k) are observable and controllable, respectively Future research can investigate if the above statement is true if observability and controllability are replaced by detectability and stabilizability, in this order.

It is shown that the existence of an almost periodic solution of $x(k+1) = A_k x(k)$ is sufficient to the uniform asymptotic stability (see Theorem 19). In contrast with almost periodic systems, the development of results for general time-varying systems has additional difficulties because uniformity is, indeed, a somewhat restrictive characteristic imposed on the system. Thus, another point of future investigation could be the search for conditions to guarantee the uniform asymptotic stability when $\{A_k\} \in \mathscr{A}_\infty$ does not have almost periodic matrices.

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