Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

High-Level Physical Modeling Description and Symbolic Computing

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Abstract: We present a high-level modeling formulation based on a conserved quantities approach, with the goal of making the physical modeling process reliable and repeatable. The system of equations generated as a result of this formulation will, in general, be non-linear differential algebraic equations (DAEs). We make use of symbolic reduction techniques in order to eliminate spurious, non-physical solutions as well as to reduce to a system of ordinary differential equations, if possible.

1. INTRODUCTION

The complexity of physical models is rapidly increasing, while development cycles are getting shorter. While dealing with these constraints, we need to ensure quality and accuracy of the models that are produced. A modeling process is needed that formalizes a reliable and repeatable path towards high quality models.

We present a modeling approach that is based on conserved quantities. Since the conservation of these quantities is strictly enforced, the resulting model quality increases, because the model engineer is lead to consider or explicitly ignore physical phenomena that are often neglected or forgotten when using other modeling techniques. For example, when modeling a resistor, the thermal energy it dissipates is often neglected. However, using the conserved quantities approach compels the user to consider the conversion of electrical energy into other energy forms such as thermal energy. This leads to a formalized, predictable, and repeatable modeling process that makes the modeling assumptions visible, facilitating a rigorous model review process to ensure quality.

2. MODELING WITH CONSERVED QUANTITIES

The method presented here is based on a conserved quantities approach (Ohata *et al.*, 2004), which describes multi-domain systems using a single domain-neutral methodology. The overall system is described by a nested hierarchy of sub-systems made out of a collection of components and other sub-systems as shown in Fig. 1. At each level the state of each sub-system is described by conserved quantities, such as charge, energy, momentum, mass, which must be conserved. The application of conservation laws at each component or sub-system ensures that no conserved laws are violated. The individual components and sub-systems interact with each other by exchanging conserved quantities. Other conservative approaches, such as bond graphs, are more restrictive w.r.t. the types of conservation equations that can be defined (Ohata *et al.*, 2004). The systems modeled with this methodology can be exported to traditional tools such as Simulink or Modelica.

To illustrate the modeling technique using the conserved quantity approach, a simple model of a capacitor is shown in Fig. 2.

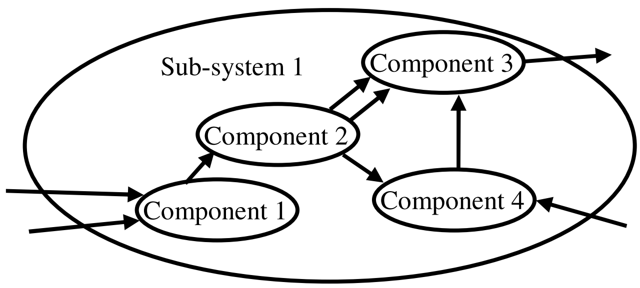

Fig. 1. The modeling of a sub-system using the conserved quantities approach

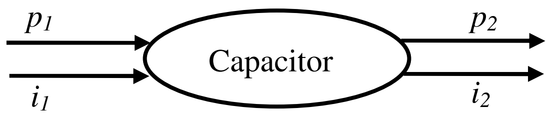

Fig. 2. The model of a capacitor using conserved quantity approach

The corresponding equations used to model the capacitor are:

$$E_C = \frac{{Q_C}^2}{2 \cdot C} \tag{1}$$

$$i_1 = \frac{d}{dt} Q_C \tag{2}$$

$$\frac{d}{dt} E_C = p_1 - p_2 \tag{3}$$

$$0 = i_1 - i_2 \tag{4}$$

where E_C and Q_C represent the energy stored in the capacitor and the charge on one of the capacitor's plates respectively, C

is the capacitance value, p_1 and p_2 represent the power (energy flow) entering and exiting the capacitor respectively, i_1 and i_2 represent the current (flow of charge) entering and exiting the capacitor respectively.

3. SYMBOLIC MANIPULATION

While having many advantages as described above, the conserved quantity approach leads to a number of difficulties in the simulation stage; the models typically generate systems of differential algebraic equations (DAEs) that require sophisticated solvers to get accurate solutions; the conserved quantities approach also results in systems of equations that typically have spurious, non-physical solutions that need to be identified and removed in order to get useful simulation results; finally, the systems of equations are complex and need to be simplified in order to be able to run simulations in a reasonable amount of time. We propose to take advantage of symbolic techniques, as for example found in Maple (Maplesoft, 2007), to address these difficulties.

3.1 DAE Solutions

The systems of DAEs are intrinsically more difficult to solve than a system of ordinary differential equations (ODE), and therefore require special purpose symbolic and numeric approaches to make the solution tractable.

A popular way to deal with a system of DAEs is index reduction (Pantelides, 1998) that converts it to a system of ODEs. One way to do this is by repeatedly differentiating and manipulating the algebraic equations until they essentially become a system of differential equations. The resulting system of ODEs can then be solved by standard techniques. The original algebraic constraints are no longer present in this system, and as the system is integrated, the solution may drift and the algebraic constraints may no longer be satisfied. Some of the techniques to prevent this drift include: manifold projection (Ascher *et al.*, 1998) and constraint stabilization (Baumgarte, 1972).

With the Pantelides method of index reduction, an additional problem is with the initial condition selection. As the algebraic equations are differentiated, additional states are introduced. These artificially created states require initial conditions for the ODE solver to function, however the initial conditions are not defined in the original formulation. A combination of symbolic and numeric techniques is used to derive the missing initial conditions.

3.2 Spurious Solutions

The conserved quantities modeling approach inherently leads to non-linear systems, which may lead to multiple solutions. The additional solutions manifest themselves as spurious solutions, which are non-realizable solutions that still satisfy the system of equations. These solutions would not be present by using more traditional formulation. The spurious solutions arise in the form:

$$p \cdot q = 0 \tag{5}$$

where p and q are both sub-expressions containing differential terms. This leads to two different cases: $p=0$ and $q=0$, and a decision must be made which case to select. Given a system of equations, we use a symbolic technique called differential elimination (Wittkopf, 2004) to both identify different cases and eliminate the cases that correspond to spurious solutions.

3.3 Complex Systems

Systems of equations that are automatically generated from a high level description tend to have a high degree of redundancy. For example, there will be many trivial equations of the form $x_i(t)=x_j(t)$. Symbolic manipulation techniques can remove the redundancy and, therefore significantly reduce the size of the system.

Additionally, through symbolic analysis, the system can be rewritten in terms of a different set of state variables. The resulting formulation is more compact, and therefore the time required to solve the overall system is also decreased. In some cases the appropriate co-ordinate selection can reduce a DAE system to an ODE system (Arczewski *et al.*, 1996).

4. CONCLUSIONS

We presented a high level physical modeling approach that uses conserved quantities to increase the reliability and predictability of the modeling process. The complications that arise in the presented formulation are being resolved through the application of symbolic methods. The symbolic manipulation includes DAE model simplification, complexity reduction, and the elimination of spurious solutions.

REFERENCES

Arczewski, K. and W. Blajer (1996). A Unified Approach to the Modelling of Holonomic and Nonholonomic Mechanical Systems. *Mathematical and Computer Modelling of Dynamical Systems*, **Vol. 2**, No. 3, pp. 157-174.

Ascher, U.M. and L.R. Petzold (1998). *Computer Methods for Ordinary Differential Equations and Differential-Algebraic Equations*. Society for Applied and Industrial Mathematics. Philadelphia, PA.

Baumgarte, J. (1972). Stabilization of constraints and integrals of motions in dynamical systems, *Computer methods in applied mechanics and engineering 1*, pp. 1-16.

Maplesoft (2007). *Maple 11 User Guide*, Maplesoft.

Ohata, A., Sugiki, A., Ohata, Y., Suzuki, S., and K. Furuta (2004). Engine Modeling Based on Projection Method and Conservation Laws. *Proceedings of the 2004 IEEE International Conference on Control Applications*, Taipei, Taiwan, **Vol. 2**, pp. 1420-1424.

Pantelides, C.C. (1998). The Consistent Initialization of Differential Algebraic Systems. *SIAM J. Sci. Stat. Comput.*, **Vol. 9**, No. 2.

Wittkopf, A. (2004). *Algorithms and Implementations for Differential Elimination*. Ph.D. Thesis. Simon Fraser University.