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Robust Stability of Systems with Fuzzy Parametric Uncertainty ⋆

Petr Hušek *

* *Department of Control Engineering*
Faculty of Electrical Engineering
Czech Technical University
Technická 2, 166 27 Prague 6, Czech Republic
(e-mail: husek@fel.cvut.cz).

Abstract: The paper deals with the problem of determining stability margin of a linear continuous-time systems with fuzzy parametric uncertainty. The coefficients of characteristic polynomial with linear affine dependency on system parameters are considered. The system parameters are described by fuzzy numbers with nonsymmetric triangular membership functions. An elegant solution, graphical in nature, based on generalization of Tsypkin-Polyak plot is presented.

1. INTRODUCTION

Very often the dynamic system works almost all the time in one operating point. The system is designed in such a way that it performs optimally in some sense in this point. Nevertheless, sometimes it has to work in different conditions (at least for a short period). In such a case the system is usually not required to preserve the optimal behaviour. It often suffices if the system remains stable, i.e. it is robustly stable.

When parametric uncertainty is considered the following problem can be stated. Let a system be (asymptotically) stable for some nominal values of its parameters. The question is, within what boundary the stability remains preserved. Such a problem is called *stability margin* determination. Since the celebrated Kharitonov's theorem (Kharitonov [1978]) was published big attention is devoted to solving both problems – checking stability of the uncertain system and determining its stability margin. Kharitonov's theorem provides very efficient tool for stability analysis of interval systems, i.e. linear systems whose coefficients are supposed to lie in the prescribed mutually independent intervals. To check stability of a system with linear parameter dependency the Edge theorem (Bartlett et al. [1988]) provides an elegant solution. More complicated coefficient structures such multilinear or polynomic dependency on an interval vector parameter are also considered, however the corresponding algorithms are rather complicated.

All the problems mentioned above and solved by classical robust analysis approach assume that the uncertainty remains the same independently on the working conditions. It means that the worst case has to be considered and conservative results are obtained. However, in many practical situations the uncertainty varies, e.g. depending on operation conditions. In such a case the uncertainty interval can be often parameterized by a confidence level. This parameter is usually tough to measure but it can be estimated by a human operator. If each parameter of a system is described in this way the system corresponds to a family of interval linear time-invariant systems parameterized by the confidence level.

Naturally, as in classical analysis of systems with structured uncertainty the parameterized uncertainty intervals can enter into the coefficients linearly, multilinearly, polynomically or even in more complicated manner. To handle such type of uncertain systems a mathematical framework is desired. Such a framework was proposed by Bondia and Picó [1999]. They adopted the concept of *fuzzy numbers* and *fuzzy functions*, see Dubois and Prade [1980]. The approach interprets a set of intervals parameterized by a confidence level as a fuzzy number with its membership degree given by this confidence level. It means that all the coefficients c_i are characterized by means of fuzzy numbers with membership functions $\alpha_i = \mu_{\tilde{c}_i}(c_i)$. When a confidence level α_i is specified then the coefficient interval is determined by the α_i-cut $[c_i]_{\alpha_i}$. If $\alpha_i = 1$ (the maximum confidence level – the system works in normal operating conditions) the coefficient c_i can take any value (crisp or interval) within the cores of $\tilde{c}_i$'s ($c_i = ker(\tilde{c}_i)$). If $\alpha_i = 0$ (the minimum confidence level) the coefficient c_i is the interval equal to the support of $\tilde{c}_i$ ($c_i \in supp(\tilde{c}_i)$). It is supposed that $supp(\tilde{c}_i)$ are finite sets, e.g. sigmoidal membership functions cannot be applied.

The question is what minimum confidence level α_{min} guarantees stability of the system under the assumption that the nominal system (i.e. for $\alpha = 1$) is Hurwitz stable. Different approach for definition of fuzzy system stability was proposed in (Nguyen and Kreinovich [1994]). In the sequel stable means Hurwitz stable.

Let us consider characteristic polynomial

$$\tilde{C}(s) = \tilde{q}_0 + \tilde{q}_1 s + \cdots + \tilde{q}_n s^n \qquad (1)$$

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where the coefficients $\tilde{q}_k, k = 0, \dots, n$ are described by triangular membership functions (generally nonsymmetric). More precisely, considering common confidence level α, if triangular membership function with $ker\{\tilde{q}_k\} = q_k^0, supp\{\tilde{q}_k\} = [q_k^-, q_k^+]$ characterizes the coefficient $\tilde{q}_k$ then the functions

$$q_k^-(\alpha) = (q_k^0 - q_k^-)\alpha + q_k^- \,, \quad q_k^+(\alpha) = (q_k^0 - q_k^+)\alpha + q_k^+ \tag{2}$$

determine the α-cut representation of polynomial (1) defined as an interval polynomial

$$\tilde{C}^\alpha(s) = C(s, \alpha) = \sum_{k=0}^{n} q_k(\alpha) s^k \,, \quad q_k(\alpha) \in [q_k^-(\alpha), q_k^+(\alpha)] \,. \tag{3}$$

Let us suppose that the nominal (1-cut) polynomial $C(s, 1) = \sum_{i=0}^{n} q_i^0 s^i$ is stable. The task is to find stability margin of the polynomial (1), i.e. confidence level $\alpha_{\min} \in [0, 1]$ such that interval polynomial (3) is stable for $\alpha > \alpha_{\min}$ and unstable for $\alpha \le \alpha_{\min}$.

The solution of this problem was stated by Bondia and Picó [2003] with the help of Argoun stability test (Argoun [1987]), which is graphical in nature, or by Lan [2005] using Kharitonov theorem and by Hušek [2004] using the generalization of Tsypkin-Polyak loci (Tsypkin and Polyak [1991]). Unfortunately, there are only very few systems where the parameters coincide with the coefficients of the characteristic polynomial as in (1). Much more typically the parameters of a system enter in the characteristic polynomial in linear, multilinear or polynomic fashion. In this paper stability margin of the linear affine dependency of the coefficients of characteristic polynomial on parameters described by nonsymmetric triangular membership functions is studied.

2. LINEAR AFFINE FUZZY PARAMETRIC UNCERTAINTY

In the sequel we will consider polynomial

$$\tilde{D}(s) = \tilde{d}_0 + \tilde{d}_1 s + \cdots + \tilde{d}_n s^n \tag{4}$$

where the coefficients $\tilde{d}_i, i = 0, \dots, n$ are supposed to be linear affine functions of the parameters $\tilde{q}_k, k = 1, \dots, m$, i.e.

$$\tilde{d}_i = \beta_i + \sum_{k=1}^{m} \gamma_{ik} \tilde{q}_k, \quad \beta_i, \gamma_{ik} \in \Re \,. \tag{5}$$

The parameters $\tilde{q}_k, k = 1, \dots, m$ are described by nonsymmetric triangular membership functions sharing common confidence level α. If the triangular membership function with $ker\{\tilde{q}_k\} = q_k^0, supp\{\tilde{q}_k\} = [q_k^-, q_k^+]$ describes the coefficient $\tilde{q}_k$ then the linear functions

$$q_k^-(\alpha) = (q_k^0 - q_k^-)\alpha + q_k^- \,, \quad q_k^+(\alpha) = (q_k^0 - q_k^+)\alpha + q_k^+ \tag{6}$$

characterize the linear interval polynomial

$$D(s, \alpha) = d_0(\alpha) + d_1(\alpha) s + \cdots + d_n(\alpha) s^n \tag{7}$$

where

$$d_i(\alpha) = \beta_i + \sum_{k=1}^{m} \gamma_{ik} q_k(\alpha), \quad i = 0, \dots, n \,, \quad q_k(\alpha) \in [q_k^-(\alpha), q_k^+(\alpha)] \,. \tag{8}$$

Let us suppose that the nominal (1-cut) polynomial $D(s, 1) = \sum_{i=0}^{n} d_i^0 s^i, d_i^0 = \beta_i + \sum_{k=1}^{m} \gamma_{ik} q_k^0$ is stable. We are looking for confidence level $\alpha_{\min} \in [0, 1]$ such that linear interval polynomial (7) is stable for $\alpha > \alpha_{\min}$ and unstable for $\alpha \le \alpha_{\min}$.

In order to solve the problem a generalization of the Tsypkin-Polyak plot (Tsypkin and Polyak [1992]) will be used.

3. STABILITY MARGIN DETERMINATION

3.1 Zero exclusion theorem

Let $\mathcal{Q}$ be a connected region in the $(n + 1)$-dimensional space. Let us consider family of polynomials

$$\delta(s, \mathcal{Q}) = p_0 + \cdots + p_n s^n, \mathbf{p} = [p_0, \dots, p_n], \mathbf{p} \in \mathcal{Q}. \tag{9}$$

To derive the main result of this paper well-known boundary crossing theorem will be used.

Theorem 1. (Boundary crossing theorem) (Bhattacharyya et al. [1995]). The family of polynomials $\delta(s, \mathcal{Q})$ (9) of invariant degree is stable if and only if

a) there exists a stable polynomial $\delta(s, \mathbf{p}^*), \mathbf{p}^* \in \mathcal{Q}$,
b) $j\omega \notin roots\{\delta(s, \mathcal{Q})\} \forall \omega \in \Re$.

This intuitive result simply states the fact that the first encounter of polynomial with fixed degree (i.e. coefficient p_n does not include zero) with instability has to be on the boundary of stability domain. Computationally more efficient version of the boundary crossing theorem is formulated by the zero exclusion principle.

Theorem 2. (Zero exclusion principle) (Bhattacharyya et al. [1995]). The family of polynomials $\delta(s, \mathcal{Q})$ (9) of invariant degree is stable if and only if

a) there exists a stable polynomial $\delta(s, \mathbf{p}^*), \mathbf{p}^* \in \mathcal{Q}$,
b) $0 \notin \delta(j\omega, \mathcal{Q}) \forall \omega \in \Re$.

The set $\delta(j\omega, \mathcal{Q}), \omega \in \Re$ is called the value set. Due to symmetry of value sets it suffices to check zero exclusion for $\omega \ge 0$ only.

3.2 Main result

Let us consider the polytope of polynomials of constant degree

$$Q(s, \rho) = A(s) + \rho \sum_{k=1}^{m} r_k B_k(s), r_k^- \le r_k \le r_k^+ \tag{10}$$

where

$$A(s) = d_0^0 + d_1^0 s + \cdots + d_n^0 s^n, d_i^0 = \beta_i + \sum_{k=1}^{m} \gamma_{ik} q_k^0 \,,$$

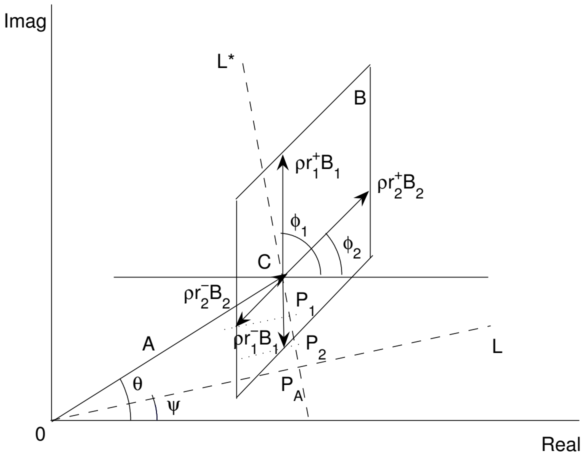

Fig. 1. Projection of the value set onto L*

$$
\begin{aligned}
&i = 0, \ldots, n\,,\\
&B_k(s) = \gamma_{0k} + \gamma_{1k}s + \cdots + \gamma_{nk}s^n\,,\\
&r_k^- = q_k^- - q_k^0, r_k^+ = q_k^+ - q_k^0, k = 1, \ldots, m\,,\\
&\rho > 0\,. \qquad (11)
\end{aligned}
$$

The family of polynomials (10) is usually written as

$$Q(s,\rho) = A(s) + \rho \sum_{k=1}^{m} [r_k^-, r_k^+] B_k(s)\,. \qquad (12)$$

Theorem 3. The minimum confidence level preserving stability of (7)

$$\alpha_{\min} = \max\{0, 1 - \rho_{\max}\} \qquad (13)$$

where $\rho_{\max}$ is maximum value of ρ preserving stability of (12) called stability margin.

Proof. Substituting $\alpha = 1 - \rho$ into (7) one obtains $D(s, 1-\rho) = Q(s,\rho)$ from which (13) immediately follows.

Let us examine the value set of polynomial family (12) in some point $s = j\omega$,

$$Q(j\omega,\rho) = A(j\omega) + \rho \sum_{k=1}^{m} [r_k^-, r_k^+] B_k(j\omega)\,. \qquad (14)$$

Since $r_i, i = 1, \ldots m$, are interval parameters the value set is a polygon, see Fig. 1. In particular, if the complex numbers $A, B_1, \ldots, B_m$ are defined as

$$
\begin{aligned}
A(j\omega) &= |A| e^{j\theta}\,,\\
B_k(j\omega) &= |B_k| e^{j\phi_k}, k = 1, \ldots, m \qquad (15)
\end{aligned}
$$

then the value set (14) equals to the set $A + \rho\mathcal{B}$ where

$$\mathcal{B} = \left\{ \sum_{k=1}^{m} r_k B_k : r_k^- \le r_k \le r_k^+ \right\}\,. \qquad (16)$$

Due to zero exclusion theorem we need to examine when zero is excluded from value set $A + \rho\mathcal{B}$. The following result gives the answer.

Theorem 4. The condition

$$0 \notin A + \rho\mathcal{B}, \rho > 0 \qquad (17)$$

holds if and only if

$$
\begin{aligned}
&\max_{1 \le k \le m} \frac{|A||\sin(\theta - \phi_k)|}{\sum_{i=1}^{m} |r_i^* B_i||\sin(\phi_i - \phi_k)|} > \rho\,,\\
&\text{if } \sin(\phi_i - \phi_k) \ne 0 \text{ for some } i, k \qquad (18)
\end{aligned}
$$

where

$$
\begin{aligned}
r_i^* &= r_i^- \text{ if } sign\,(\sin(\phi_i - \phi_k)) = sign\,(\sin(\phi_k - \theta))\,,\\
r_i^* &= r_i^+ \text{ if } sign\,(\sin(\phi_i - \phi_k)) \ne sign\,(\sin(\phi_k - \theta))
\end{aligned}
$$

and

$$
\begin{aligned}
&\max_{1 \le k \le m} \frac{|A|}{\sum_{i=1}^{m} |r_i^* B_i|} > \rho\,,\\
&\text{if } \sin(\phi_i - \phi_k) = 0 \text{ and } \sin(\phi_k - \theta) = 0 \quad \forall i, k \qquad (19)
\end{aligned}
$$

where

$$
\begin{aligned}
r_i^* &= r_i^- \text{ if } \theta = \phi_k,\\
r_i^* &= r_i^+ \text{ if } \theta = -\phi_k.
\end{aligned}
$$

Proof. Zero is excluded from the set $A + \rho\mathcal{B}$ if and only if there exists a line L which separates the set from the origin in complex plane. We will use the polygonal shape of value set and try to project the set into direction L^* which is orthogonal to the line L passing the origin at an angle ψ with the real axis, see Fig. 1. The length of the projection of the vector A into this direction is $|A||\sin(\theta - \psi)|$ (the line CP_A in Fig. 1). The total length of the projection of the set $\mathcal{B}$ is

$$
\begin{aligned}
&\rho(|r_1^+| + |r_1^-|)|\sin(\phi_1 - \psi)| + \cdots\\
+ \quad &(|r_m^+| + |r_m^-|)|\sin(\phi_m - \psi)|.
\end{aligned}
$$

The line L separates the set $A + \rho\mathcal{B}$ from the origin if and only if the projection of A is greater than the part of the projection of $\mathcal{B}$ (of the projections of each B_i) whose direction is opposite to the direction of the projection of A. In Fig. 1 these directions are $-B_1$ and $-B_2$ and the corresponding projections are the lines CP_1 and CP_2. The total length of this part is

$$\rho(|r_1^*||\sin(\phi_1 - \psi)| + \cdots + |r_m^*||\sin(\phi_m - \psi)|) \qquad (20)$$

where

$$
\begin{aligned}
r_i^* &= r_i^- \text{ if } sign\,(\sin(\phi_i - \psi)) = sign\,(\sin(\phi_k - \theta))\,,\\
r_i^* &= r_i^+ \text{ if } sign\,(\sin(\phi_i - \psi)) \ne sign\,(\sin(\phi_k - \theta))\,. \qquad (21)
\end{aligned}
$$

It means that if and only if there exists an angle $\psi \in [0, 2\pi)$ such that

$$
\begin{aligned}
&|A||\sin(\theta - \psi)| >\\
&\rho(|r_1^*||\sin(\phi_1 - \psi)| + \cdots + |r_m^*||\sin(\phi_m - \psi)|) \qquad (22)
\end{aligned}
$$

with r_i^* defined in (21) then the value set $A + \rho\mathcal{B}$ does not contain the origin. Because of polygonal shape of $\mathcal{B}$ it suffices to test if the inequality (31) holds only for $\psi = \phi_i, i = 1, \ldots, m$, which corresponds to the formula

(18). The formula (19) solves the case when the value set $\mathcal{B}$ degenerates to a line.

In order to determine the stability margin $\rho_{\max}$ of the polytope (12) we will look for maximum $\rho = \rho(\omega)$ for each $\omega \geq 0$ such that the inequalities (18) and (19) are satisfied. Then $\rho_{\max} = \inf_\omega \rho(\omega)$.

4. LINEAR INTERVAL FAMILY WITH FUZZY PARAMETRIC UNCERTAINTY

The obtained result can be applied to a special form of polynomials with linear dependency of its coefficients on the parameters characterized by fuzzy numbers.

Let us consider a family of polynomials

$$\begin{aligned} \tilde{\delta}(s) &= F_1(s)\tilde{P}_1(s) + \cdots + F_m(s)\tilde{P}_m(s)\,, \\ F_i(s) &= f_{i0} + f_{i1}s + \cdots\,, \\ \tilde{P}_i(s) &= \tilde{p}_{i0} + \tilde{p}_{i1}s + \cdots \end{aligned} \tag{23}$$

where the coefficients $\tilde{p}_{ij}$ are characterized by non-symmetric triangular membership functions $ker\{\tilde{p}_{ik}\} = p^0_{ik}, supp\{\tilde{p}_{ik}\} = [p^-_{ik}, p^+_{ik}]$ which are described by the linear functions

$$\begin{aligned} p^-_{ik}(\alpha) &= \left(p^0_{ik} - p^-_{ik}\right)\alpha + p^-_{ik}\,, \\ p^+_{ik}(\alpha) &= \left(p^0_{ik} - p^+_{ik}\right)\alpha + p^+_{ik}\,. \end{aligned}$$

We are looking for minimum confidence level $\alpha_{\min}$ such that the interval polynomial

$$\begin{aligned} \delta(s,\alpha) &= F_1(s)P_1(s,\alpha) + \cdots + F_m(s)P_m(s,\alpha)\,, \\ P_i(s,\alpha) &= \left[p^-_{i0}(\alpha), p^+_{i0}(\alpha)\right] + \left[p^-_{i1}(\alpha), p^+_{i1}(\alpha)\right]s + \cdots\,, \\ i &= 1,\ldots,m \end{aligned} \tag{24}$$

is stable for $\alpha > \alpha_{\min}$ and unstable for $\alpha \leq \alpha_{\min}$ under assumption that the nominal polynomial $\delta^0(s)$,

$$\begin{aligned} \delta^0(s) &= \delta(s,1) = F_1(s)P_1(s,1) + \cdots + F_m(s)P_m(s,1)\,, \\ P^0_i(s) &= P_i(s,1) = p^0_{i0} + p^0_{i1}s + \cdots\,, \end{aligned}$$

is stable.

Let us consider linear interval family of polynomials

$$\begin{aligned} \Delta(s) &= F_1(s)P_1(s) + \cdots + F_m(s)P_m(s)\,, \\ F_i(s) &= f_{i0} + f_{i1}s + \cdots\,, \\ P_i(s) &= p_{i0} + p_{i1}s + \cdots \end{aligned} \tag{25}$$

where

$$\begin{aligned} \left|p_{ik} - p^0_{ik}\right| &\leq \rho\beta^-_{ik} \text{ if } p_{ik} \leq p^0_{ik}\,, \\ \left|p_{ik} - p^0_{ik}\right| &\leq \rho\beta^+_{ik} \text{ if } p_{ik} > p^0_{ik}, \rho > 0\,, \\ \beta^-_{ik} &= p^0_{ik} - p^-_{ik}\,, \\ \beta^+_{ik} &= p^+_{ik} - p^0_{ik}\,. \end{aligned}$$

Theorem 5. The minimum confidence level preserving stability of (24)

$$\alpha_{\min} = \max\{0, 1 - \rho_{\max}\} \tag{26}$$

where $\rho_{\max}$ is maximum value of ρ preserving stability of (25).

Proof. Substituting $\alpha = 1 - \rho$ into (24) one obtains $\delta(s, 1-\rho) = \Delta(s)$ from which (26) immediately follows.

Denote

$$\begin{aligned} A(s) &= F_1(s)P^0_1(s) + \cdots + F_m(s)P^0_m(s)\,, \\ S^-_i(\omega) &= \beta^-_{i0} + \beta^+_{i2}\omega^2 + \beta^-_{i4}\omega^4 + \cdots\,, \\ S^+_i(\omega) &= \beta^+_{i0} + \beta^-_{i2}\omega^2 + \beta^+_{i4}\omega^4 + \cdots\,, \\ T^-_i(\omega) &= \omega(\beta^-_{i1} + \beta^+_{i3}\omega^2 + \beta^-_{i5}\omega^4 + \cdots)\,, \\ T^+_i(\omega) &= \omega(\beta^+_{i1} + \beta^-_{i3}\omega^2 + \beta^+_{i5}\omega^4 + \cdots)\,. \end{aligned}$$

In order to determine stability margin of (25) we will examine the value set $\Delta(j\omega)$ for $0 \leq \omega \leq \infty$. Since $P_i(s)$ are interval polynomials application of Kharitonov theorem yields

$$\begin{aligned} P_i(j\omega) = \big\{ & s_i(\omega) + jt_i(\omega) : -S^-_i(\omega) \leq s_i(\omega) \leq S^+_i(\omega)\,, \\ & -T^-_i(\omega) \leq t_i(\omega) \leq T^+_i(\omega), i = 1,\ldots,m \big\}\,. \end{aligned}$$

Then the value set

$$\Delta(j\omega) = A(j\omega) + \rho\mathcal{B}(\omega) \tag{27}$$

where

$$\begin{aligned} \mathcal{B}(\omega) = \Bigg\{ & \sum_{i=1}^{m} \left(s_i(\omega) + jt_i(\omega)\right) F_i(j\omega) : \\ & -S^-_i(\omega) \leq s_i(\omega) \leq S^+_i(\omega), \\ & -T^-_i(\omega) \leq t_i(\omega) \leq T^+_i(\omega), i = 1,\ldots,m \Bigg\}\,. \end{aligned}$$

that is usually written as

$$\begin{aligned} \mathcal{B}(\omega) = \Bigg\{ & \sum_{i=1}^{m} \left(\left[-S^-_i(\omega), S^+_i(\omega)\right] F_i(j\omega)\right. \\ & \left. + \left[-T^-_i(\omega), T^+_i(\omega)\right] jF_i(j\omega)\right), i = 1,\ldots,m \Bigg\}\,. \end{aligned}$$

Now theorem 4 can be applied for zero exclusion test of the set (27) with

$$\begin{aligned} A(j\omega) &= F_1(j\omega)P^0_1(j\omega) + \cdots + F_m(j\omega)P^0_m(j\omega)\,, \\ r^-_k &= S^-_k(\omega), r^+_k = S^+_k(\omega)\,, \\ r^-_{m+k} &= T^-_k(\omega), r^+_{m+k} = T^+_k(\omega)\,, \\ B_k(j\omega) &= F_k(j\omega), B_{m+k}(j\omega) = jF_k(j\omega), k = 1,\ldots,m\,. \end{aligned}$$

The stability margin $\rho_{\max}$ will be determined from the frequency plot of $\rho(\omega)$ as $\rho_{\max} = \inf_{0\leq\omega\leq\infty} \rho(\omega)$. However, there could be discontinuities of the function $\rho(\omega)$ in the points 0 and ∞. For $\omega = 0$ we have

$$\rho(0) = \rho_0 = \frac{\left|\sum_{i=1}^{m} p^0_{i0} f_{i0}\right|}{\sum_{i=1}^{m} \beta_{i0}|f_{i0}|}\,, \tag{28}$$

$$\beta_{i0} = \beta^-_{i0} \text{ if } \left(\sum_{j=1}^{m} p^0_{j0} f_{j0}\right) f_{i0} \geq 0\,,$$

$$\beta_{i0} = \beta_{i0}^{+} \text{ if } \left(\sum_{j=1}^{m} p_{j0}^{0} f_{j0} \right) f_{i0} < 0 \,.$$

For $\omega \to \infty$ we have

$$\rho(\infty) = \rho_n = \frac{\left| \sum_{i=1}^{m} \sum_{k+l=n} p_{ik}^{0} f_{il} \right|}{\sum_{i=1}^{m} \sum_{k+l=n} \beta_{ik} |f_{il}|} \,, \tag{29}$$

$$\beta_{ik} = \beta_{ik}^{-} \text{ if } \left(\sum_{j=1}^{m} \sum_{k+r=n} p_{jk}^{0} f_{jr} \right) f_{il} \geq 0 \,,$$

$$\beta_{ik} = \beta_{ik}^{+} \text{ if } \left(\sum_{j=1}^{m} \sum_{k+r=n} p_{jk}^{0} f_{jr} \right) f_{il} < 0$$

where n is degree of polynomial $\Delta(s)$. The value ρ_n corresponds to degree drop of $\Delta(s)$.

Then if $\rho_{\min} := \inf_{0<\omega<\infty} \rho(\omega)$

$$\rho_{\max} = \min\{\rho_0, \rho_n, \rho_{\min}\} \,.$$

5. EXAMPLE

In Safari-Shad and Takabe [1997] the characteristic polynomial of the Fiat Dedra engine model was obtained as a fourth-order polynomial with seven uncertain parameters

$$p(s, \mathbf{q}) = a_0(\mathbf{q}) + a_1(\mathbf{q})s + a_2(\mathbf{q})s^2 + a_3(\mathbf{q})s^3 + s^4 \tag{30}$$

where

$$\begin{aligned}
a_0(\mathbf{q}) &= (k_{11}(k_{24} + 0.05) - k_{14}k_{21})q_1q_4q_7, \\
a_1(\mathbf{q}) &= (k_{11} - k_{14}k_{23} + k_{13}(k_{24} + 0.05))q_1q_4q_7 \\
&\quad +(k_{12}(k_{24} + 0.05) - k_{14}k_{22})q_1q_5q_7 \\
&\quad +(k_{12}k_{21} - k_{11}k_{22})q_1q_6q_7 + (k_{24} + 0.05)q_2q_5q_7 \\
&\quad +k_{21}q_2q_6q_7 + (k_{24} + 0.05)q_3q_4q_7, \\
a_2(\mathbf{q}) &= k_{13}q_1q_4q_7 + k_{12}q_1q_5q_7 + (k_{12}k_{23} - k_{13}k_{22})q_1q_6q_7 \\
&\quad +q_2q_5q_7 + k_{23}q_2q_6q_7 + q_3q_4q_7 \\
&\quad +(k_{24} + 0.05)q_5q_7 - k_{22}q_3q_6q_7 + k_{21}q_6q_7 + \\
&\quad (k_{24} + 0.05)q_2 + ((k_{24} + 0.05)k_{12} - k_{22}k_{14})q_1, \\
a_3(\mathbf{q}) &= k_{12}q_1 + q_2 + k_{23}q_6q_7 + q_5q_7 + k_{24} + 0.05.
\end{aligned}$$

k_{ij} denotes the elements of the controller gain matrix (Barmish [1994])

$$K = \begin{bmatrix} 0.0081 & 0.1586 & 0.8072 & -0.1202 \\ 0.0187 & 0.0848 & 0.1826 & -0.0224 \end{bmatrix} . \tag{31}$$

The coefficients of the characteristic polynomial (30) depend multilinearly on the uncertain parameters defined as a box

$$Q = \left\{ \mathbf{q} = \left[q_i, q_i \in [q_i^-, q_i^+], i = 1, \ldots, 7 \right] \right\} \tag{32}$$

where the vector of lower and upper bounds is given as

$$\begin{aligned}
\mathbf{q}^- &= [q_i^-, i = 1, \ldots, 7] \\
&= [0.3261 \;\; -0.2073 \;\; 0.0357 \;\; 0.2539 \;\; 0.0100 \;\; 2.0247 \;\; 0.1000] \\
\mathbf{q}^+ &= [q_i^+, i = 1, \ldots, 7] \\
&= [3.4329 \;\; 0.1627 \;\; 0.1139 \;\; 0.5607 \;\; 0.0208 \;\; 4.4962 \;\; 1.0000]
\end{aligned}$$

respectively.

In Barmish [1994] the nominal parameter values corresponding to most operating point representing slightly loaded engine at idle speed are considered as

$$\begin{aligned}
& q_1^0 = q_1^+ = 3.4329 \,, q_2^0 = q_2^+ = 0.1627 \,, q_3^0 = q_3^+ = 0.1139 \,, \\
& q_4^0 = q_4^- = 0.2539 \,, q_5^0 = q_5^+ = 0.0208 \,, q_6^0 = q_6^- = 2.0247 \,, \\
& q_7^0 = q_7^+ = 1.0000 \,.
\end{aligned} \tag{33}$$

It should be noted that the nominal parameter values do not lie in the middle of the admissible intervals. The question is how far we can get away from the nominal point to preserve stability of (30).

The characteristic polynomial (30) has multilinear uncertainty structure. In Kanno and Yang [2002] affine linearization is carried out – by fixing some parameters the original polynomial is changed to an affine linear interval polynomial. Such transformation leads to a necessary stability condition only, however, this information can still provide very useful insight to the original problem using effective methods.

In particular, an inspection of the coefficients of the characteristic polynomial (30) reveals that if q_4, q_5, q_6 and q_7 are fixed then the coefficients depend affine linearly on q_1, q_2 and q_3. In Kanno and Yang [2002] the parameters q_4, q_5, q_6 and q_7 are fixed at their nominal values but for the parameters q_1, q_2 and q_3 the midpoints of the admissible intervals are chosen as "nominal" values and do not correspond to the real nominal (operating) point (33). In fact it means that if the stability margin of (30) (or the corresponding polytope (10)) will be less than 1 then the corresponding maximum admissible intervals, in which the parameters can lie to preserve stability of the characteristic polynomial, do not cover the real operating point. This is a serious drawback of that procedure.

In order to overcome the drawback mentioned above we will characterize the uncertain parameters q_1, q_2 and q_3 by fuzzy numbers described by nonsymmetric triangular membership functions $\tilde{q}_1, \tilde{q}_2$ and $\tilde{q}_3$ with

$$supp\{\tilde{q}_i\} = [q_i^-, q_i^+] \,, ker\{\tilde{q}_i\} = q_i^0 \,, i = 1, 2, 3 \,. \tag{34}$$

Affine linearization of (30) by fixing q_4, q_5, q_6 and q_7 at the nominal point and characterization of q_1, q_2 and q_3 by $\tilde{q}_1, \tilde{q}_2$ and $\tilde{q}_3$ (34) form polynomial

$$\tilde{D}(s) = \tilde{d}_0 + \tilde{d}_1 s + \tilde{d}_2 s^2 + \tilde{d}_3 s^3 + s^4 \tag{35}$$

where

$$\begin{aligned}
\tilde{d}_0 &= 0.0006\tilde{q}_1 \,, \\
\tilde{d}_1 &= 0.0182\tilde{q}_1 + 0.0384\tilde{q}_2 + 0.0070\tilde{q}_3 \,, \\
\tilde{d}_2 &= 0.0384 + 0.1429\tilde{q}_1 + 0.4181\tilde{q}_2 + 0.0822\tilde{q}_3 \,, \\
\tilde{d}_3 &= 0.4181 + 0.1586\tilde{q}_1 + \tilde{q}_2 + 0.0822\tilde{q}_3 \,,
\end{aligned}$$

with $\tilde{q}_i, i = 1, 2, 3$ given by (34).

Transformation of polynomial (35) into the polytope (10)

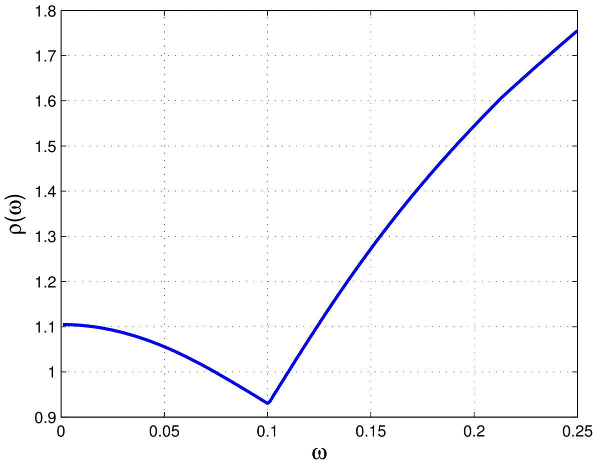

Fig. 2. Frequency plot of $\rho(\omega)$ of Fiat Dedra engine

$$Q(s,\rho) = A(s) + \rho \sum_{i=1}^{m} r_i B_i(s), r_i^- \leq r_i \leq r_i^+ \quad (36)$$

yields

$$\begin{aligned} A(s) &= s^4 + 1.1253s^3 + 0.6063s^2 + 0.0695s + 0.0022\,, \\ B_1(s) &= 0.1586s^3 + 0.1429s^2 + 0.0182s + 0.0006\,, \\ B_2(s) &= s^3 + 0.4181s^2 + 0.0384s\,, \\ B_3(s) &= 0.0822s^2 + 0.0070s\,, \\ r_1^- &= -3.1068, r_2^- = -0.3600, r_3^- = -0.0782, \\ r_1^+ &= 0, r_2^+ = 0, r_3^+ = 0\,. \end{aligned}$$

Since the polytope of polynomials (36) is of constant degree and the polynomial $A(s)$ is Hurwitz stable we can apply the result from theorem 4. The plot of $\rho(\omega)$ against frequency is depicted in Fig. 2. From this plot $\rho_{\min} = \inf_{0<\omega<\infty} \rho(\omega) = 0.9304$ and using (28) and (29) $\rho_0 = 1.1050$ and $\rho_n = \infty$. The stability margin $\rho_{\max}$ of (36) is

$$\rho_{\max} = \min\{\rho_0, \rho_n, \rho_{\min}\} = 0.9304\,.$$

Let us note that this result differs from the value obtained by the method presented in Kanno and Yang [2002] ($\rho^*_{\max} = 0.8608$). Also the maximum admissible intervals of the parameters preserving stability ($q_1 \in [0.5423, 3.4329], q_2 \in [-0.1816, 0.1627], q_3 \in [0.0411, 0.1139]$) reflects better the operating conditions than those obtained in Kanno and Yang [2002], ($q_1^* \in [0.5423, 3.2167]$, $q_2^* \in [-0.1816, 0.1370], q_3^* \in [0.0802, 0.1085]$).

The corresponding minimum confidence level preserving stability of (35) is $\alpha_{\min} = \max\{0, 1 - \rho_{\max}\} = 0.0696$.

6. CONCLUSION

The paper extends the known results about systems with fuzzy parametric uncertainty. A more realistic case is considered when the coefficients of characteristic polynomial are linear affine functions of parameters described by fuzzy numbers. This is for example the case when a plant with the coefficients of transfer function described by fuzzy numbers is controlled with a fixed controller. Since nonsymmetric membership functions are involved the presented approach can deal with the systems whose operating point does not lie in the middle of admissible parameter intervals that is a common situation.

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