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Robust One-step Model Predictive Control for Discrete Time-delay Systems ⋆

Yujing Shi * **Tianyou Chai** *,** **Heng Yue** *

* *Key Laboratory of Integrated Automation of Process Industry, Ministry of Education, Northeastern University, Shenyang, CO 110004 China (e-mail: yjshi168@ 126.com).*
** *Research Center of Automation, Northeastern University, Shenyang, CO 110004 China (e-mail: tychai@ mail.neu.edu.cn)*

Abstract: In this paper a robust one-step model predictive control (MPC) scheme is developed for discrete time-delay systems with polytopic-type uncertainty. The proposed MPC is obtained by minimizing a new cost function that includes multi-terminal weighting terms, subject to constraints on input. This MPC scheme allows the first move $u(k|k)$ to be separated from the control moves governed by a state feedback law, which can reduce conservatism and improve feasibility and optimality. A linear matrix inequality (LMI) approach is applied to the controller synthesis. It can be shown that the proposed model predictive controller guarantees closed-loop stability. Simulation results are given to illustrate the performance of the proposed algorithm.

1. INTRODUCTION

Time delay commonly exists in dynamic systems due to measurement, transmission and transport lags, which has generally been regarded as a main source of instability and poor performance. In addition, it is well-known that parameter uncertainties are unavoidable in practice. Therefore, considerable research has been devoted to the robust control of uncertain time-delayed systems(see Moon et al.[2001], Fridman and Shaked[2002], Chen et al.[2003]).

For delay-free systems, MPC has been extensively studied and successfully employed in industrial fields(Mayne et al.[2000], Qin and Badgwell[2003]). It has been recognized that MPC is able to handle the constraints and possesses good robustness(Kothare et al.[1996], Lu and Arkun[2000], Park and Kwon[2002]).

For time-delay systems, however, only a few MPC algorithms have been published. A simple MPC method for delayed systems has appeared in Kwon et al.[2003]. However, closed-loop stability cannot be guaranteed by the suggested design. Then a general MPC for time-delay systems has been proposed in Kwon, Lee and Han[2004] by applying the generalized Riccati method. It is noticed that the plant is in the continuous-time domain, and the model uncertainty and constraints are not taken into account in both aforementioned papers. For discrete-time systems with time-delay, robust MPC has been addressed. In Kothare et al.[1996], the robust constrained MPC scheme for delay-free systems has been extended to a delayed system by simply employing equivalent augmented systems without delay. However, this is not an effective alternative for general time-delay systems, especially for systems with unknown delays or systems with time-varying delays. It could lead to a high degree of complexity in the control design. In Jeong and Park[2005], an MPC algorithm for uncertain systems with input constraints and unknown state-delay has been presented. Due to unknown delay indices, the authors reduced the optimization problem that minimizes a cost function to two other optimization problems and check the closed-loop stability under an assumption that the weighting matrix is fixed to a constant matrix at all time. This assumption is very restricted, which may lead to conservatism. On all accounts, all these design methods with regard to MPC for the delayed systems are delay-independent. In general, delay-independent results are conservative because the time delay is not taken into account in the process of designing controllers. It is also worth mentioning that a common feature of the existing approaches is that they are based on single linear state-feedback gain. This facilitates the treatment of the MPC design. However, this limits the performance of the controller.

This paper is concerned with robust MPC of discrete-time linear systems with polytopic-type uncertainty and state-delay. We introduce a new cost function that includes multi-terminal weighting terms, which are crucial to guarantee the closed-loop stability. The proposed robust MPC scheme allows first move $u(k|k)$ to separated from the control moves governed by a linear state feedback law. Next the delay-independent robust one-step MPC algorithm is presented. In addition, by using the descriptor system approach(Fridman and Shaked[2002]) and a new bounding technique(Moon et al[2001]), the delay-dependent one is developed.

2. PROBLEM DESCRIPTION

Consider the following discrete-time uncertain time-varying systems with state-delay.

$$x(k+1) = A(k)x(k) + \bar{A}(k)x(k-d) + B(k)u(k) \quad (1)$$

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subject to input constraints

$$-\bar{u} \le u(k) \le \bar{u}, \quad \bar{u} \ge 0, \quad k \in [0, \infty) \tag{2}$$

where $x(k) \in \mathbf{R}^n$ is the system state, $u(k) \in \mathbf{R}^m$ is the control input,d is the delay. $\bar{u} = [\bar{u}_1, \cdots, \bar{u}_m]$where $\bar{u}_i > 0$,$i = 1, \cdots, m$ and the initial condition is $x(k) = \phi(k), k \in [-d, 0]$. Furthermore, we assume that system matrices $[A(k)\, \bar{A}(k)\, B(k)]$ are unknown but belong to a polytope $\Omega = Co[A_1(k)\, \bar{A}_1(k)\, B_1(k)], \cdots, [A_p(k)\, \bar{A}_p(k)\, B_p(k)]$.

It is assumed that the state $x(k)$ is fully measured at each time k. In robust MPC, our goal is to design a robust model predictive controller for system (1) and achieve the following robust performance index at each time k. We consider the following min-max optimization problem:

$$\min_{u(k+j|k), j \ge 0} \max_{[A(k+j)\bar{A}(k+j)B(k+j)] \in \Omega} J(k) \tag{3}$$

subject to

$$\begin{aligned} J(k) &= \sum_{j=0}^{\infty} [\|x(k+j|k)\|_Q^2 + \|u(k+j|k)\|_R^2] \\ &= \|x(k|k)\|_Q^2 + \|u(k|k)\|_R^2 + J_1^{\infty}(k) \end{aligned} \tag{4}$$

$$\begin{aligned} x(k+1+j|k) = A(k+j)x(k+j|k) + \\ \bar{A}(k+j)x(k+j-d|k) + B(k+j)u(k+j|k) \end{aligned} \tag{5}$$

$$-\bar{u} \le u(k+j|k) \le \bar{u},\ j \ge 0 \tag{6}$$

where $J_1^{\infty}(k) = \sum_{j=1}^{\infty} [\|x(k+j|k)\|_Q^2 + \|u(k+j|k)\|_R^2]$, Q and R are positive-definite weighting matrices, $x(k+j|k)$ and $u(k+j|k)$ denote the predicted state of the plant at time $k+j$ and the future control move at time $k+j$, respectively, with $x(k|k) = x(k)$. The following lemma plays an important role in our later development.

Lemma 1. Assume that $\alpha \in \mathbf{R}^{n_a}, \beta \in \mathbf{R}^{n_b}$ and $N \in \mathbf{R}^{n_a \times n_b}$. Then for any matrices $X \in \mathbf{R}^{n_a \times n_a}, Y \in \mathbf{R}^{n_a \times n_b}, Z \in \mathbf{R}^{n_b \times n_b}$ satisfying $\begin{bmatrix} X & Y \\ Y^{\mathrm{T}} & Z \end{bmatrix} \ge 0$, the following inequality holds:

$$-2\alpha^{\mathrm{T}} N \beta \le \begin{bmatrix} \alpha \\ \beta \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} X & Y - N \\ Y^{\mathrm{T}} - N^{\mathrm{T}} & Z \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

3. DELAY-INDEPENDENT ROBUST ONE-STEP MPC

The exact solution to this min-max optimization problem (3)-(6) is not in general tractable. To obtain a practical optimization problem, we found the upper bound of $J_1^{\infty}(k)$, based on the worst case scenario for all the possible state matrices in the prescribed polytope Ω. Firstly, we introduce the following quadratic function

$$V(X(k+j|k)) = \|x(k+j|k)\|_{P_1(k)}^2 + \sum_{l=1}^{d} \|x(k+j-l|k)\|_{S_1(k)}^2 \tag{7}$$

where $X(k+j|k) = [x^{\mathrm{T}}(k+j|k), x^{\mathrm{T}}(k+j-1|k), \cdots, x^{\mathrm{T}}(k+j-d|k)]^{\mathrm{T}}$,$P_1(k) > 0, S_1(k) > 0$. Suppose that the following inequality holds for all$[A(k+j)\bar{A}(k+j)B(k+j)] \in \Omega, j \ge 1$

$$\begin{aligned} V(X(k+j+1|k)) - V(X(k+j|k)) \\ \le -[\|x(k+j|k)\|_Q^2 + \|u(k+j|k)\|_R^2], \quad j \ge 1 \end{aligned} \tag{8}$$

Summing both sides of the inequality (8) from $j = 1$ to $j = \infty$ and assuming that $V(X(k+\infty)) = 0$, it is easy to obtain

$$\begin{aligned} \max_{[A(k+j)\bar{A}(k+j)B(k+j)] \in \Omega, j \ge 1} J_1^{\infty}(k) \le V(X(k+1|k)) \\ = \|x(k+1|k)\|_{P_1(k)}^2 + \sum_{l=1}^{d} \|x(k+1-l|k)\|_{S_1(k)}^2 \end{aligned} \tag{9}$$

Thus, from (9) the original min-max optimization problem (3)-(6) is transformed into the following constrained optimization problem for system (1) with the actual state $x(k)$ in the moving horizon fashion:

$$\begin{aligned} \min_{u(k|k), U_1^{\infty}(k), P_1(k), S_1(k)} \bar{J}(k) = \|x(k|k)\|_Q^2 + \|u(k|k)\|_R^2 \\ + \|x(k+1|k)\|_{P_1(k)}^2 + \sum_{l=1}^{d} \|x(k+1-l|k)\|_{S_1(k)}^2 \end{aligned} \tag{10}$$

subject to (5),(6) and (8), where the first control move $u(k|k)$ is a free decision variable and the rest of the future control moves are given by a following state feedback

$$U_1^{\infty}(k) : \{u(k+j|k) = K(k)x(k+j|k),\ j \ge 1\}$$

Note that the cost function $\bar{J}(k)$ has two terminal weighting terms. Moreover, the stability of the proposed MPC depends on the choice of terminal weighting matrices $P_1(k)$ and $S_1(k)$. The optimization problem (10) can be solved by converting all the constraints into the form of LMIs. The solution is provided by the following theorem.

Theorem 1. Consider the discrete time-varying state delayed system(1)and the system matrices$[A(k)\bar{A}(k)B(k)] \in \Omega$. The optimization problem (10) subject to (5),(6)and(8) can be solved by the following semi-definite programming:

$$\min_{\gamma, u(k|k), \bar{K}(k), X_1(k), U_1(k), E(k)} \gamma \tag{11}$$

subject to

$$\begin{bmatrix} 1 & [A_\sigma x(k|k) + \bar{A}_\sigma x(k-d|k) + B_\sigma u(k|k)]^{\mathrm{T}} & x^{\mathrm{T}}(k|k) & \cdots & x^{\mathrm{T}}(k+1-d|k) & x^{\mathrm{T}}(k|k)Q^{1/2} & u^{\mathrm{T}}(k|k)R^{1/2} \\ * & X_1(k) & 0 & \cdots & 0 & 0 & 0 \\ * & * & U_1(k) & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ * & * & * & \cdots & U_1(k) & 0 & 0 \\ * & * & * & \cdots & * & \gamma I & 0 \\ * & * & * & \cdots & * & * & \gamma I \end{bmatrix} \ge 0 \tag{12}$$

$$\begin{bmatrix} X_1(k) & 0 & \Omega_1 & X_1(k) & X_1(k)Q^{\frac{1}{2}} & \bar{K}^{\mathrm{T}}(k)R^{\frac{1}{2}} \\ * & U_1(k) & U_1(k)\bar{A}_\sigma^{\mathrm{T}} & 0 & 0 & 0 \\ * & * & X_1(k) & 0 & 0 & 0 \\ * & * & * & U_1(k) & 0 & 0 \\ * & * & * & * & \gamma I & 0 \\ * & * & * & * & * & \gamma I \end{bmatrix} \ge 0 \tag{13}$$

$$|u_i(k|k)| \le \bar{u}_i,\ i = 1, \cdots, m \tag{14}$$

$$\begin{bmatrix} E(k) & \bar{K}(k) \\ \bar{K}^{\mathrm{T}}(k) & X_1(k) \end{bmatrix} \geq 0,\ E_{ii}(k) \leq \bar{u}_i^2,\ i=1,\cdots,m \quad (15)$$

where $\Omega_1 = X_1(k)A_\sigma^{\mathrm{T}} + \bar{K}^{\mathrm{T}}(k)B_\sigma^{\mathrm{T}}$, $X_1(k) = \gamma P_1^{-1}(k)$, $U_1(k) = \gamma S_1^{-1}(k)$,$\gamma > 0$,$\bar{K}(k) = K(k)X_1(k)$, $\sigma = 1\cdots p$ and $E_{ii}(k)$ is the ith diagonal entry of $E(k)$.

For simplicity, in symmetric block matrices, we use $*$ to represent a term that is induced by symmetry.

Proof. Minimization of $\bar{J}(k)$ is equivalent to the following optimization problem

$$\min_{\gamma,u(k|k),P_1(k),S_1(k)} \gamma \quad (16)$$

subject to

$$\bar{J}(k) = \|x(k|k)\|_Q^2 + \|u(k|k)\|_R^2 + \|x(k+1|k)\|_{P_1(k)}^2 + \sum_{l=1}^{d} \|x(k+1-l|k)\|_{S_1(k)}^2 \leq \gamma$$

By the Schur complement, we can easily deduce(11)and(12).

Substituting state-feedback control law $u(k+j|k) = K(k)x(k+j|k), j\geq 1$ into (8), it is easy to see that inequality (8) holds for all $[A(k)\bar{A}(k)B(k)]\in\Omega$ and for all $j \geq 1$ if

$$\begin{bmatrix} \Omega_2 & [A(k+j)+B(k+j)K(k)]^{\mathrm{T}}P_1(k)\bar{A}(k+j) \\ * & \bar{A}^{\mathrm{T}}(k+j)P_1(k)\bar{A}(k+j) - S_1(k) \end{bmatrix} \leq 0 \quad (17)$$

where $\Omega_2 = \|A(k+j)+B(k+j)K(k)\|_{P_1(k)}^2 - P_1(k) + S_1(k) + Q + K^{\mathrm{T}}(k)RK(k)$

We pre- and postmultiply (17) by diag$[\gamma^{-\frac{1}{2}}X_1(k), \gamma^{-\frac{1}{2}}U_1(k)]$. Applying the Schur complement and $[A(k)\bar{A}(k)B(k)] \in \Omega$, we can ensure that inequality (13) holds.

Considering the input constraint (6) is transform into LMI, it can be split into two parts that the constraint on $u(k|k)$ and on $U_1^\infty(k)$. Since $u(k|k)$ is a free decision variable, (14) can be obtained directly. The constraint on $U_1^\infty(k)$ can easily be turned into LMI (15) by using similar method of Kothare et al.[1996]. Hence, we omit the detailed procedure.

Robust asymptotic stability of the closed-loop system is guaranteed in the following theorem.

Theorem 2. Once a feasible solution of the optimization problem (11)-(15) is found, the MPC law obtained from Theorem 1 robustly asymptotically stabilizes the closed-loop system.

Proof. Feasibility: Assume that a feasible solution exists at time k, denoted by $u^*(k|k), K^*(k), P_1^*(k), S_1^*(k)$. Then, at the next time $k+1$ the following solution can be determined as feasible.

$$u(k+j|k+1) = u^*(k+j|k) = K^*(k)x(k+j|k),$$
$$P_1(k+1) = P_1^*(k),\ S_1(k+1) = S_1^*(k) \quad (18)$$

Inequalities (13) and (15) can easily be proved by choice of decision variables (18) because the parameters are independent of time k.Specifically, we only need to prove that inequalities (12) and (14) are feasible as they explicitly depends on the measured state and implemented input.

When (15) is satisfied at time k, then $u^*(k+1|k)$ is feasible. It is clear that (14) is satisfied at time $k+1$ with $u(k+1|k+1) = u^*(k+1|k)$. Now we need to verify (12). Since $x(k+1-i|k+1) = x(k+1-i|k), i = 0,1,\cdots,d$ and $V(X(k+1|k)) \leq \gamma$, after substituting (18) into (12), it is known that (12) is equivalent to

$$\zeta^{\mathrm{T}}\begin{bmatrix} \Omega_3 & [A(k+1)+B(k+1)K^*(k)]^{\mathrm{T}}P_1^*(k)\bar{A}(k+1) \\ * & \bar{A}^{\mathrm{T}}(k+1)P_1^*(k)\bar{A}(k+1)-S_1^*(k) \end{bmatrix}\zeta \leq 0 \quad (19)$$

where $\Omega_3 = \|A(k+1)+B(k+1)K^*(k)\|_{P_1^*(k)}^2 - P_1^*(k) + S_1^*(k) + Q + K^{*\mathrm{T}}(k)RK^*(k)$, $\zeta = [\, x^{\mathrm{T}}(k+1|k)\ x^{\mathrm{T}}(k+1-d|k)\,]^{\mathrm{T}}$

The above inequality (19) holds since (13) is satisfied at time k.Thus the feasible solution of the optimization problem (11)-(15) at time k is also feasible at time $k+1$. This argument can be continued for all future instants.

Stability: Denote the optimal solutions at time k and $k+1$ respectively by $u^*(k+j|k), j \geq 0$,$P_1^*(k)$,$S_1^*(k)$ and $u^*(k+j|k+1), j \geq 0$,$P_1^*(k+1)$,$S_1^*(k+1)$.Then we have

$$\begin{aligned}
\bar{J}^*(k+1) &= \|x(k+1|k+1)\|_Q^2 + \|u^*(k+1|k+1)\|_R^2 \\
&+ \|x(k+2|k+1)\|_{P_1^*(k+1)}^2 + \sum_{l=1}^{d}\|x(k+2-l|k+1)\|_{S_1^*(k+1)}^2 \\
&\leq \|x(k+1|k)\|_Q^2 + \|u^*(k+1|k)\|_R^2 + \|x(k+2|k)\|_{P_1^*(k)}^2 \\
&+ \sum_{l=1}^{d}\|x(k+2-l|k)\|_{S_1^*(k)}^2 \qquad (20)
\end{aligned}$$

This is because $u^*(k+j|k+1), P_1^*(k+1)$ and$S_1^*(k+1)$ are optimal, while $u^*(k+j|k), P_1^*(k)$ and$S_1^*(k)$ are feasible only at time $k+1$. Moreover, it follows from (8) when $j = 1$ that

$$\begin{aligned}
&\|x(k+2|k)\|_{P_1^*(k)}^2 + \sum_{l=1}^{d}\|x(k+2-l|k)\|_{S_1^*(k)}^2 \\
&-\|x(k+1|k)\|_{P_1^*(k)}^2 + \sum_{l=1}^{d}\|x(k+1-l|k)\|_{S_1^*(k)}^2 \\
&\leq -\|x(k+1|k)\|_Q^2 - \|u^*(k+1|k)\|_R^2
\end{aligned} \quad (21)$$

Combining (20) and (21), we obtain

$$\begin{aligned}
\bar{J}^*(k+1) &\leq \|x(k+1|k)\|_{P_1^*(k)}^2 + \sum_{l=1}^{d}\|x(k+1-l|k)\|_{S_1^*(k)}^2 \\
&\leq \|x(k|k)\|_Q^2 + \|u^*(k|k)\|_R^2 + \|x(k+1|k)\|_{P_1^*(k)}^2 \\
&+ \sum_{l=1}^{d}\|x(k+1-l|k)\|_{S_1^*(k)}^2 = \bar{J}^*(k) \qquad (22)
\end{aligned}$$

Therefore, $\bar{J}^*(k)$ is a monotonically non-increasing and bounded Lyapunov function. We have the conclusion that $x(k)$ converge to zero. Hence, the closed-loop stability is obtained.

Note that Theorem 1 is a delay-independent approach, which provides feasible solutions irrespective of the size of delay. Since the time delay is not taken into account in the design process of MPC, the delay-independent approach is generally regarded as being more conservative than the delay-dependent one, especially in situations where delays are small. In the next section, a delay-dependent robust one-step MPC scheme shall be given.

4. DELAY-DEPENDENT ROBUST ONE-STEP MPC

We consider original min-max optimization problem (3)-(6) again. Set $x(k+j+1|k) = x(k+j|k)+y(k+j|k)$. Then we have $x(k+j-d|k) = x(k+j|k) - \sum_{l=1}^{d} y(k+j-l|k)$, As in Chen et al.[2003], both of the above equations were substituted into(5), then the prediction equation(5)can be transformed into the following equivalent descriptor form

$$\begin{aligned} 0 = [A(k+j) + \bar{A}(k+j) - I]x(k+j|k) - y(k+j|k) \\ -\bar{A}(k+j)\sum_{l=1}^{d} y(k+j-l|k) + B(k+j)u(k+j|k) \end{aligned} \tag{23}$$

Similarly, we choose a quadratic function as follows:

$$\begin{aligned} \tilde{V}(\tilde{X}(k+j|k)) = \|x(k+j|k)\|^2_{\tilde{P}_1(k)} + \sum_{l=1}^{d} \|x(k+j-l|k)\|^2_{\tilde{S}_1(k)} \\ + \sum_{\theta=1}^{d} \sum_{l=1}^{\theta} \|y(k+j-l|k)\|^2_{\tilde{S}_2(k)} \end{aligned} \tag{24}$$

where $\tilde{X}(k+j|k) = [x^{\mathrm{T}}(k+j|k), \cdots, x^{\mathrm{T}}(k+j-d|k), y^{\mathrm{T}}(k+j-1|k), \cdots, y^{\mathrm{T}}(k+j-d|k)]^{\mathrm{T}}$, $\tilde{P}_1(k)$, $\tilde{S}_1(k)$ and $\tilde{S}_2(k)$ are positive-definite weighting matrices. At sampling time k, for all $[A(k+j)\bar{A}(k+j)B(k+j)] \in \Omega, j \geq 1$, we suppose that $\tilde{V}(\tilde{X}(k+j|k))$ satisfies the following inequality:

$$\begin{aligned} \tilde{V}(\tilde{X}(k+j+1|k)) - \tilde{V}(\tilde{X}(k+j|k)) \\ \leq -[\|x(k+j|k)\|^2_Q + \|u(k+j|k)\|^2_R] \end{aligned} \tag{25}$$

Summing both sides of the inequality (25) from $j = 1$ to $j = \infty$ yields

$$\max_{[A(k+j)\bar{A}(k+j)B(k+j)] \in \Omega, j\geq 1} J_1^{\infty}(k) \leq \tilde{V}(\tilde{X}(k+1|k)) \tag{26}$$

Thus, the original min-max optimization problem (3)-(6) can be turned into the following optimization problem that minimizes the sum of the first stage cost and the terminal cost corresponding to the upper bound on J_1^{∞}

$$\begin{aligned} \min_{u(k|k), U_1^{\infty}(k), \tilde{P}_1(k), \tilde{S}_1(k), \tilde{S}_2(k)} \tilde{J}(k) = \|x(k|k)\|^2_Q + \|u(k|k)\|^2_R \\ + \|x(k+1|k)\|^2_{\tilde{P}_1(k)} + \sum_{l=1}^{d} \|x(k+1-l|k)\|^2_{\tilde{S}_1(k)} \\ + \sum_{\theta=1}^{d} \sum_{l=1}^{\theta} \|y(k+1-l|k)\|^2_{\tilde{S}_2(k)} \end{aligned} \tag{27}$$

subject to (6),(23) and (25), where the function $\tilde{J}$ includes three terminal weighting terms which are closely related to the closed-loop stability.

Theorem 3. Consider the system (1), where the system matrices $[A(k)\bar{A}(k)B(k)]$ belong to a polytope Ω. The optimization problem(27), subject to (6), (23) and (25) can be solved by the following semi-definite programming:

$$\min_{\eta, u(k|k), \tilde{K}, \tilde{X}_1, \tilde{Y}, \tilde{Z}, \tilde{U}_1, \tilde{U}_2, \tilde{W}_1, \tilde{W}_2, \tilde{W}_3, \tilde{E}} \eta \tag{28}$$

subject to

$$\begin{bmatrix} 1 & \Omega_4 & x^{\mathrm{T}}(k|k) & \cdots & x^{\mathrm{T}}(k+1-d|k) & d\Omega_5 & (d-1)y^{\mathrm{T}}(k-1|k) & \cdots & y^{\mathrm{T}}(k+1-d|k) & x^{\mathrm{T}}(k|k)Q^{1/2} & u^{\mathrm{T}}(k|k)R^{1/2} \\ * & \tilde{X}_1(k) & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ * & * & \tilde{U}_1(k) & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & \cdots & \tilde{U}_1(k) & 0 & 0 & \cdots & 0 & 0 & 0 \\ * & * & * & \cdots & * & d\tilde{U}_2(k) & 0 & \cdots & 0 & 0 & 0 \\ * & * & * & \cdots & * & * & (d-1)\tilde{U}_2(k) & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ * & * & * & \cdots & * & * & * & \cdots & \tilde{U}_2(k) & 0 & 0 \\ * & * & * & \cdots & * & * & * & \cdots & * & \eta I & 0 \\ * & * & * & \cdots & * & * & * & \cdots & * & * & \eta I \end{bmatrix} \geq 0 \tag{29}$$

$$\begin{bmatrix} \tilde{Z}(k) + \tilde{Z}^{\mathrm{T}}(k) + d\tilde{W}_1(k) & \Omega_6 & 0 & \tilde{Z}^{\mathrm{T}}(k) & d\tilde{Z}^{\mathrm{T}}(k) & \tilde{X}_1(k) & \tilde{X}_1(k)Q^{1/2} & \tilde{K}^{\mathrm{T}}(k)R^{1/2} \\ * & \Omega_7 & (1-\varepsilon)\bar{A}_\sigma \tilde{U}_1(k) & \tilde{Y}^{\mathrm{T}}(k) & d\tilde{Y}^{\mathrm{T}}(k) & 0 & 0 & 0 \\ * & * & -\tilde{U}_1(k) & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -\tilde{X}_1(k) & 0 & 0 & 0 & 0 \\ * & * & * & * & -d\tilde{U}_2(k) & 0 & 0 & 0 \\ * & * & * & * & * & -\tilde{U}_1(k) & 0 & 0 \\ * & * & * & * & * & * & -\eta I & 0 \\ * & * & * & * & * & * & * & -\eta I \end{bmatrix} \leq 0 \tag{30}$$

$$\begin{bmatrix} \tilde{W}_1(k) & \tilde{W}_2(k) & 0 \\ * & \tilde{W}_3(k) & \varepsilon \bar{A}_\sigma \tilde{U}_2(k) \\ * & * & \tilde{U}_2(k) \end{bmatrix} \geq 0 \tag{31}$$

$$|u_i(k|k)| \leq \bar{u}_i,\ i = 1, \cdots, m \tag{32}$$

$$\begin{bmatrix} \tilde{E}(k) & \tilde{K}(k) \\ * & \tilde{X}_1(k) \end{bmatrix} \geq 0,\ \tilde{E}_{ii}(k) \leq \bar{u}_i^2,\ i = 1, \cdots, m \tag{33}$$

where $\Omega_4 = [A_\sigma x(k|k) + \bar{A}_\sigma x(k-d|k) + B_\sigma u(k|k)]^{\mathrm{T}}$, $\Omega_5 = [A_\sigma - I]x(k|k) + \bar{A}_\sigma x(k-d|k) + B_\sigma u(k|k)$, $\Omega_6 = \tilde{X}_1(k)(A_\sigma^{\mathrm{T}} + \varepsilon \bar{A}_\sigma^{\mathrm{T}} - I) + \tilde{Y}(k) + \tilde{K}^{\mathrm{T}}(k)B_\sigma^{\mathrm{T}} - \tilde{Z}^{\mathrm{T}}(k) + d\tilde{W}_2(k)$, $\Omega_7 = -\tilde{Y}(k) - \tilde{Y}^{\mathrm{T}}(k) + d\tilde{W}_3(k)$, $\sigma = 1 \cdots p$. $\tilde{E}_{ii}(k)$ is the ith diagonal entry of $\tilde{E}(k)$, ε is a prescribed scalar. $\tilde{X}_1(k) = \eta \tilde{P}_1^{-1}(k)$, $\tilde{U}_1(k) = \eta \tilde{S}_1^{-1}(k)$, $\tilde{U}_2(k) = \eta \tilde{S}_2^{-1}(k)$, $\eta > 0$ and $\tilde{K}(k) = K(k)\tilde{X}_1(k)$.

Proof. Taking into account condition (25), we have

$$\begin{aligned} \tilde{V}(\tilde{X}(k+j+1|k)) - \tilde{V}(\tilde{X}(k+j|k)) \\ = 2x^{\mathrm{T}}(k+j|k)\tilde{P}_1(k)y(k+j|k) + x^{\mathrm{T}}(k+j|k)\tilde{S}_1(k)x(k+j|k) \end{aligned}$$

$$+y^{\mathrm{T}}(k+j|k)(\tilde{P}_1(k)+d\tilde{S}_2(k))y(k+j|k)$$
$$-x^{\mathrm{T}}(k+j-d|k)\tilde{S}_1(k)x(k+j-d|k)$$
$$-\sum_{l=1}^{d}y^{\mathrm{T}}(k+j-l|k)\tilde{S}_2(k)y(k+j-l|k) \quad (34)$$

By the descriptor equation (23) and Lemma 1, we obtain

$$2x^{\mathrm{T}}(k+j|k)\tilde{P}_1(k)y(k+j|k)=2\varsigma^{\mathrm{T}}(k+j|k)\tilde{P}^{\mathrm{T}}\begin{bmatrix} y(k+j|k) \\ 0 \end{bmatrix}$$
$$\leq 2\varsigma^{\mathrm{T}}(k+j|k)\tilde{P}^{\mathrm{T}}\begin{bmatrix} 0 & I \\ A(k+j)+B(k+j)K(k)-I & -I \end{bmatrix}\varsigma(k+j|k)$$
$$+2\varsigma^{\mathrm{T}}(k+j|k)\tilde{P}^{\mathrm{T}}\begin{bmatrix} 0 \\ \bar{A}(k+j) \end{bmatrix}x(k+j|k)+d\varsigma^{\mathrm{T}}(k+j|k)W \quad (35)$$
$$\varsigma(k+j|k)+2\varsigma^{\mathrm{T}}(k+j|k)\left(M-\tilde{P}^{\mathrm{T}}\begin{bmatrix} 0 \\ \bar{A}(k+j) \end{bmatrix}\right)(x(k+j|k)$$
$$-x(k+j-d|k))+\sum_{l=1}^{d}y^{\mathrm{T}}(k+j-l|k)\tilde{S}_2(k)y(k+j-l|k)$$

where $\varsigma(k+j|k)=\begin{bmatrix} x(k+j|k) \\ y(k+j|k) \end{bmatrix}$, $\tilde{P}=\begin{bmatrix} \tilde{P}_1(k) & 0 \\ \tilde{P}_2(k) & \tilde{P}_3(k) \end{bmatrix}$ and W, M are matrices with appropriate dimensions satisfying that

$$\begin{bmatrix} W & M \\ M^{\mathrm{T}} & \tilde{S}_2 \end{bmatrix} \geq 0 \quad (36)$$

Substituting (35) into (34), we obtain

$$\tilde{V}(\tilde{X}(k+j+1|k))-\tilde{V}(\tilde{X}(k+j|k))$$
$$\leq \xi^{\mathrm{T}}\begin{bmatrix} \Phi & \tilde{P}^{\mathrm{T}}\begin{bmatrix} 0 \\ \bar{A}(k+j) \end{bmatrix}-M \\ * & -\tilde{S}_1(k) \end{bmatrix}\xi \quad (37)$$

where $\xi = \begin{bmatrix} x^{\mathrm{T}}(k+j|k) & y^{\mathrm{T}}(k+j|k) & x^{\mathrm{T}}(k+j-d|k) \end{bmatrix}^{\mathrm{T}}$ $\Phi = \tilde{P}^{\mathrm{T}}\begin{bmatrix} 0 & I \\ A(k+j)+B(k+j)K(k)-I & -I \end{bmatrix}$ $+\begin{bmatrix} 0 & I \\ A(k+j)+B(k+j)K(k)-I & -I \end{bmatrix}^{\mathrm{T}}\tilde{P}+dW+\begin{bmatrix} M & 0 \end{bmatrix}+$ $\begin{bmatrix} M^{\mathrm{T}} \\ 0 \end{bmatrix}+\begin{bmatrix} \tilde{S}_1(k) & 0 \\ 0 & \tilde{P}_1(k)+d\tilde{S}_2(k) \end{bmatrix}$. Replacing (25) with (37), the inequality (25) can be written as:

$$\begin{bmatrix} \Phi+\begin{bmatrix} Q+K^{\mathrm{T}}(k)RK(k) & 0 \\ 0 & 0 \end{bmatrix} & \tilde{P}^{\mathrm{T}}\begin{bmatrix} 0 \\ \bar{A}(k+j) \end{bmatrix}-M \\ * & -\tilde{S}_1(k) \end{bmatrix} \leq 0 \quad (38)$$

In order to obtain LMI, we define

$M = \varepsilon\tilde{P}^{\mathrm{T}}\begin{bmatrix} 0 \\ \bar{A}(k+j) \end{bmatrix}$, $\tilde{K}(k) = K(k)\tilde{X}_1(k)$, $\eta\tilde{P}^{-1} = \begin{bmatrix} \tilde{X}_1(k) & 0 \\ \tilde{Z}(k) & \tilde{Y}(k) \end{bmatrix}$, $\tilde{W} = \eta(\tilde{P}^{-1})^{\mathrm{T}}W\tilde{P}^{-1} = \begin{bmatrix} \tilde{W}_1 & \tilde{W}_2 \\ \tilde{W}_2^{\mathrm{T}} & \tilde{W}_3 \end{bmatrix}$

Then we pre- and postmultiply(38)by diag$[\eta^{1/2}(\tilde{P}^{-1})^{\mathrm{T}}, \eta^{-1/2}\tilde{U}_1]$ and diag$[\eta^{1/2}\tilde{P}^{-1}, \eta^{-1/2}\tilde{U}_1]$, respectively. Next we pre- and postmultiply (36) by diag$[\eta^{1/2}(\tilde{P}^{-1})^{\mathrm{T}}, \eta^{-1/2}\tilde{U}_2]$ and diag$[\eta^{1/2}\tilde{P}^{-1}, \eta^{-1/2}\tilde{U}_2]$, respectively. Using the Schur complement and system matrices $[A(k)\bar{A}(k)B(k)] \in \Omega$, inequalities (30) and (31) can be obtained. In terms of the proof method of Theorem 1, (28) and (29) can be established easily and the input constraint (6) can easily be turned into the LMIs (32) and (33). The proof is complete.

Theorem 4. Once a feasible solution of the optimization problem (28)-(33) is found, the MPC law obtained from Theorem 3 robustly asymptotically stabilizes the closed-loop system.

Proof. This proof is similar to that of Theorem 2, and is not included here.

5. ILLUSTRATIVE EXAMPLES

In this section, two examples are provided to illustrate the effectiveness of the proposed robust MPC algorithms.

Example 1: The first example is a backing up control of a computer simulated truck-trailer adapted from Jeong and Park [2005]. The model of truck-trailer is obtained by discretizing with sampling time $T = 0.1s$ the continuous-time equations of the system (see Jeong and Park [2005]).

$$x(k+1) = \begin{bmatrix} 1.0509 & 0 & 0 \\ -0.0509 & 1 & 0 \\ 0.0509\alpha(k) & -0.4\alpha(k) & 1 \end{bmatrix} x(k)$$
$$+\begin{bmatrix} 0.0218 & 0 & 0 \\ -0.0218 & 0 & 0 \\ 0.0218\alpha(k) & 0 & 0 \end{bmatrix} x(k-d) + \begin{bmatrix} -0.1429 \\ 0 \\ 0 \end{bmatrix} u(k)$$

where the time-delay $d = 3$ and the uncertain parameter $\alpha(k) \in [1, 1.5915]$ is time-varying. Thus we have $[A(k)\,\bar{A}(k)\,B(k)] \in \Omega = Co\{[A_1\,\bar{A}_1\,B_1], [A_2\,\bar{A}_2\,B_2]\}$

$$A_1 = \begin{bmatrix} 1.0509 & 0 & 0 \\ -0.0509 & 1 & 0 \\ 0.0509 & -0.4 & 1 \end{bmatrix}, \bar{A}_1 = \begin{bmatrix} 0.0218 & 0 & 0 \\ -0.0218 & 0 & 0 \\ 0.0218 & 0 & 0 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} -0.1429 \\ 0 \\ 0 \end{bmatrix}, A_2 = \begin{bmatrix} 1.0509 & 0 & 0 \\ -0.0509 & 1 & 0 \\ 0.0810 & -0.6366 & 1 \end{bmatrix},$$
$$\bar{A}_2 = \begin{bmatrix} 0.0218 & 0 & 0 \\ -0.0218 & 0 & 0 \\ 0.0347 & 0 & 0 \end{bmatrix}, B_2 = \begin{bmatrix} -0.1429 \\ 0 \\ 0 \end{bmatrix}$$

The control objective is to regulate the state variables from the initial value $x(0) = \begin{bmatrix} 0.5\pi & 0.75\pi & -5 \end{bmatrix}^{\mathrm{T}}$ to the origin. Simultaneity, the constraints on input $|u(t)| \leq \pi$ should be satisfied. Choose weighting matrices $Q = \mathrm{diag}[10, 10, 10]$, $R = 1$ and parameters $\varepsilon = 1$. In order to test the advantage of the proposed delay-independent and delay-dependent robust one-step MPC schemes in Theorem 1 and in Theorem 3, respectively, they are compared with the robust MPC technique presented in Jeong and Park [2005]. Fig.1(a) shows the state trajectories of closed-loop systems achieved by the above three MPC techniques. From Fig.1(a), it is obvious that both of our one-step robust MPC algorithms have better performance as the state variables reach the origin faster compared with robust MPC of Jeong and Park [2005]. The corresponding control inputs are given in Fig.1(b). It is clear that no control inputs calculated from three MPC methods violate constraints. Fig.1(c) shows the upper bounds of the cost function obtained by the three MPC techniques. From Fig.1(c), it is observed that both of our proposed MPC algorithms giving smaller coefficient γ (or η).

Example 2: Consider the uncertain discrete-time delayed system as follows:

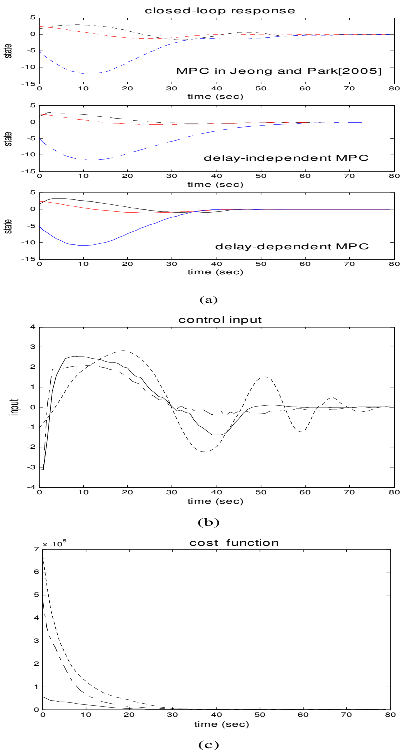

Fig. 1. Performance comparision of three MPC technique: the solid line refers to delay-dependent MPC, the dash-dotted line to delay-independent MPC and the dashed line to MPC in Jeong and Park [2005].

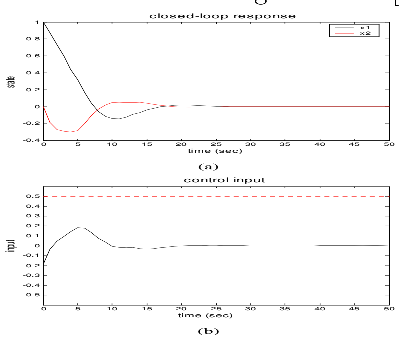

Fig. 2. Closed-loop responses and control input using our delay-dependent MPC technique

$$x(k+1) = \begin{bmatrix} 1.01+0.08\delta(k) & 0 \\ 0 & 1.2+0.08\delta(k) \end{bmatrix} x(k) + \begin{bmatrix} -0.25+0.08\delta(k) & 0.1 \\ 0 & 0.1+0.08\delta(k) \end{bmatrix} x(k-d) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

where the time-delay $d = 2$ and the uncertain parameter $0 \leq \delta(k) \leq 1$ is time-varying. We have $[A(k)\,\bar{A}(k)\,B(k)] \in \Omega = Co\{[A_1\,\bar{A}_1\,B_1], [A_2\,\bar{A}_2\,B_2]\}$

$$A_1 = \begin{bmatrix} 1.01 & 0 \\ 0 & 1.2 \end{bmatrix}, \bar{A}_1 = \begin{bmatrix} -0.25 & 0.1 \\ 0 & 0.1 \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 1.09 & 0 \\ 0 & 1.28 \end{bmatrix}, \bar{A}_2 = \begin{bmatrix} -0.17 & 0.1 \\ 0 & 0.18 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The input variable is constrained by $|u(k)| \leq 0.5$. Note that the system is not delay-independently stabilisable. Therefore the robust MPC scheme in Jeong and Park [2005] and one-step robust MPC in Theorem 1 cannot both find feasible solutions. Using the delay-dependent one-step robust MPC scheme of Theorem 3 and choosing parameters $Q = \text{diag}[1,1]$, $R = 0.01$ and $\varepsilon = 1$, we found the feasible solutions. Setting the initial state $x(0) = [\,1\ 0\,]^{\mathrm{T}}$, we obtain the corresponding results, which are illustrated in Fig. 2. It is seen from Fig.2 that the closed-loop system is robust asymptotically stable. Hence, the delay-dependent MPC presented in this note is much less conservative than the delay-independent MPC .

6. CONCLUSIONS

A robust one-step MPC scheme for the uncertain discrete-time systems with state-delays is proposed. By applying a new type of cost function that includes multi-terminal weighting terms, some sufficient conditions for the one-step MPC synthesis problem are derived in terms of LMIs. It is proved that receding horizon implementation of the feasible solutions guarantees closed-loop stability. Numerical results show that the proposed MPC technique provides better performance.

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