Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

Solvability of the Regulator Equation: L^2-space Approach

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Abstract: An alternative proof of solvability of the differential equation that is a part of the Regulator Equation which arises from the solution of the Output Regulation Problem is presented. The proof uses the standard Hilbert-space based theory of solutions of elliptic partial differential equations for the case of the linear Output Regulation Problem. In the nonlinear case, a sequence of linear equations is defined so that their solution converges to the solution of the nonlinear problem. This is proved using the Banach Contraction Theorem.

Keywords: Output Regulation Problem, Nonlinear Systems, Partial Differential Equations.

1. INTRODUCTION

The Output Regulation Problem belongs to central problems of the recent control theory, especially in its nonlinear version. The aim is to find a control of a plant so that its output tracks the reference signal. The characteristic features are the following: the reference signal (and, in some cases, the disturbances) is generated by the so-called exosystem, asymptotic tracking is achieved and, lastly, the method of solution provides the solution for a certain class of reference signals.

The crucial part is finding a solution of the so-called Regulator Equation. Its solution describes a "zero-error" manifold. To be more precise, assume the state of the exosystem v and the state of the plant x are given. If x equals to the solution of the Regulator Equation evaluated at the point v: $x = \mathrm{x}(v)$ then the tracking error equals zero.

The first attempt to the solution of the Output Regulation Problem can be found in Isidori and Byrnes [1990]. The Regulator Equation turns out to be a matrix equation in the linear case, the authors give conditions of existence of its solution. (For details, see also Isidori [1995]). The Regulator Equation arising from the nonlinear Output Regulation Problem is formulated and studied in many articles, both for minimum as well as for nonminimum phase systems, see Huang [2000, 2003] and others. (A thorough presentation of these methods can be found in Huang [2004].) To solve the Regulator Equation, Taylor series are used. All the functions involved are decomposed into a Taylor series. Then, first terms of the solution can be computed. The results of this method can be verified only experimentally, no convergence analysis is provided. Another interesting approach involves use of neural networks for the solution of the Regulator Equation Wang and Huang [2001].

There were no attempts to apply the results of the partial differential equations (PDE) theory, especially those concerning the Finite Element Method (FEM), to the solution of the Output Regulation Problem. This equation is rather nonstandard - it is a first-order differential equation that are not so frequently studied. A greater challenge seems to be imposed through presence of the algebraic condition (which will be referred to as the "algebraic part" in contrast with the "differential part" that is composed of partial differential equations) that is a natural part of the Regulator Equation. These obstacles might have prevented use of FEM to the output regulation problem.

The goal of this paper is to fill this gap - to join the classical theory of partial differential equations with the Output Regulation Problem. The related approach was presented in Čelikovský and Rehák [2004]. The solution of the Regulator Equation was split into two parts: the solution of the differential part and finding the control so that the algebraic condition is satisfied. An iterative approach was chosen: each iteration consisted of a choice of the feedforward control, then the differential part of the Regulator Equation was evaluated. At the end, a penalty functional - the integral of the square of the difference made in the algebraic condition - is evaluated. This value is used to change the value of the feedforward in the next iteration so that the value of the penalty functional decreases. A stabilization of the system is done before the start of the iterative algorithm. This is a crucial fact as this allows to apply the theory described here to the Regulator Equation obtained by this method. An illustrative example was added: the controlled system was a mathematical model of an inverted pendulum on a chart with an added chart (see Devasia [1996]). A more detailed description of this method is also contained in Rehák and Čelikovský [2008].

There are basically two facts that need to be proved:

- Convergence of the iterative method

[1] Supported by MSMT CR through the grant MSMT LA 274 and by the Czech Science Foundation through the grant GA CR 102/07/P413.

- Existence of a solution for each partial differential equation that is solved in every iteration

While attention is paid to the first point in Rehák and Čelikovský [2008], namely, certain conditions guaranteeing convergence of the iterative scheme are introduced, the second point has not been studied yet. It is the topic of the presented article.

The main result is that the conditions usually posed on the controlled system and the exosystem guarantee existence of the solution of the differential part of the Regulator Equation. This opens avenue to the alternative solution of the Regulator Equation with use of numerical methods that were successfully applied to various kinds of problems arising from physics, engineering and other fields of science and technology. One among the most important methods is the Finite Element Method (FEM). These methods also provide various kinds of error estimates. The advantage of this is twofold: firstly, one can prove convergence of the numerical method, secondly, having numerically solved a PDE, one can get evaluation of the error that is done which is in the case of the Output Regulation Problem closely related to the tracking error.

The article is purely theoretically oriented as the use of the FEM to the solution of the Regulator Equation has already been shown before. This paper can thus be considered as an ex-post justification of the results already obtained.

Let us mention a different approach which was experimentally verified in Rehák *et al.* [2005]. The algebraic condition is replaced by a singularly perturbed equation containing a derivative of the feedforward control. The system of (purely) differential equations allows to construct the Regulator Equation so that it is composed of partial differential equations only, without presence of any algebraic condition. However, the system of equations obtained by this approach does not seem to possess the properties required here. Thus, a further analysis is necessary.

For the sake of clarity and simplicity, the described method is presented for the case of linear systems first. Then, a generalization to a class of nonlinear systems is made. In the second case, a sequence of linear problems is constructed with the property that the sequence of the solutions converge to the solution of the Regulator Equation of the nonlinear problem. The proof of this convergence is given by the Banach Fixed Point Theorem.

Let us remark that all equalities are considered in the "almost everywhere" sense.

2. REGULATOR EQUATION

let $n, \mu \in N$ and let the smooth functions $F, G : R^n \to R^n$, $h : R^n \to R$ be given. Moreover, let $S \in R^{\mu\times\mu}, Q \in R^{1\times\mu}$.

Define the controlled system (the plant) by

$$\dot{x} = F(x) + G(x)u,\ x(0) = x(0),\ y = h(x) \tag{1}$$

where $x(t) \in R^n; u(t), y(t) \in R$. Define also the reference generator (the exosystem) by

$$\dot{v} = Sv,\ v(0) = v_0,\ reference = Qv. \tag{2}$$

(Disturbances are not considered in this paper.)

The main goal of the Output Regulation problem is to find a control u (defined as a function of the state of the plant and the exosystem) such that, if this control is applied, the following holds $\lim_{t\to+\infty} |y(t) - reference(t)| = 0$. (For a more general setting see Huang [2004].)

In the linear case, it is assumed that the plant is described by the equation

$$\dot{x} = Ax + Bu,\ y = Hx \tag{3}$$

with matrices A, B, H having suitable dimensions.

A crucial role in the theory plays the so-called zero-error manifold. It is a μ-dimensional manifold in the $n+1$-dimensional Euclidean space. It can be described as the graph of functions that solve the so-called **Regulator Equation**:

$$\frac{\partial \mathrm{x}(v)}{\partial v} Sv = F(\mathrm{x}(v)) + G(\mathrm{x}(v))c(v) \tag{4}$$

$$0 = h(\mathrm{x}(v)) - Qv. \tag{5}$$

The main characteristic of the zero-error manifold is the following: if the state of the exosystem at the time t equals $v(t)$ then the tracking error is zero provided the equalities $x(t) = \mathrm{x}(v(t)), u(t) = c(v(t))$ hold.

The Regulator Equation consists of two parts: a system of partial differential equations (PDE) and the output condition which is an algebraic equation (5). The differential equation (4) is called *the differential part of RE* in this text and will be denoted by DRE. This equation exhibits some features that can be considered "nonstandard" in the common theory of PDE:

- It is a first-order PDE,
- It is being solved on the whole domain R^μ,
- This implies no boundary condition is given while a condition is required. $$\mathbf{x}(0) = 0 \tag{6}$$

As stated before the aim of this article is to investigate solvability of the differential part via the finite-element method.

2.1 Linear Case

In this case, the Regulator Equation (4,5) attains the form

$$\frac{\partial \mathbf{x}(\mathrm{v})}{\partial v} Sv = A\mathbf{x}(v) + Bc(v), \tag{7}$$

$$0 = H\mathbf{x}(v) - Qv. \tag{8}$$

The following assumptions are crucial:
Assumption L1: the matrix A is Hurwitz and eigenvalues of S are simple and lie on the imaginary axis.

Assumption L2: there exist a diagonal matrix D and a regular matrix T such that $A = T^{-1}DT$. Define also the scalars d_{ii} as the diagonal elements of the matrix D.

Assumption L3: there exists a smooth function $V : R^\mu \to [0, +\infty)$ such that $V(0) = 0$, $V(v) > 0$ for $v \neq 0$, $\nabla V(v).Sv \leq 0$, $V(v) \to +\infty$ if $\|v\| \to +\infty$.

Using **L2**, the differential part attains the form

$$T\frac{\partial \mathbf{x}}{\partial v} Sv = DT\mathbf{x}(v) + TBc(v).$$

The following change of variables is introduced:

$$\xi(v) = T\mathbf{x}(v). \tag{9}$$

Observe that $T\frac{\partial \mathbf{x}}{\partial v} = T\frac{\partial T^{-1}\xi(v)}{\partial v} = \frac{\partial \xi(v)}{\partial v}$.

Then the differential part is converted into the form:

$$\frac{\partial \xi}{\partial v} Sv = D\xi(v) + TBc(v). \tag{10}$$

This equation is solved iteratively. One equation is solved while the others are kept fixed. Hence the i-th equation to be solved can be rewritten into the form

$$\frac{\partial \xi_i}{\partial v} Sv = d_{ii}\xi_i(v) + (TBc(v))_i \tag{11}$$

A theorem guaranteeing existence of a solution of a PDE of this type can be found in Roos *et al.* [1996], Lemma 1.6. Before the theorem is cited one has to deal with boundary conditions for this type of equations.

First, one has to avoid the necessity of solving the Regulator Equation on the whole space R^μ. Rather, one has to seek the solution on a bounded domain $\Omega \subset R^\mu$ with Lipschitz boundary such that $0 \in \Omega$.

Next, denote S_{ij} the element of the matrix S on the i, j position. Moreover, let $b : R^\mu \to R^\mu$ be defined as follows

$$b(v) = (\sum_{k=1}^{n} S_{1k}v_k, \ldots, \sum_{k=1}^{n} S_{\mu k}v_k).$$

Then note that for every $i \in \{1, \ldots, n\}$:

$$\frac{\partial \xi_i}{\partial v} Sv = b(v)\nabla_v \xi_i.$$

For each $v \in \partial\Omega$ let $n(v)$ denote the outward normal to the domain Ω at the point v. As in Roos *et al.* [1996], denote also

$$\Gamma_-(\Omega) = \{v \in \partial\Omega | b(v).n(v) < 0\}$$

The lemma guaranteeing existence of a solution of (11) is cited here without proof:

Lemma 1. Let the functions $b : \bar{\Omega} \to R^n$, $\beta : \bar{\Omega} \to R$ are continuously differentiable, let $\varphi \in L^2(\Omega)$. Assume there exists a constant $\omega > 0$ such that

$$\beta(v) - \frac{1}{2}\text{div } b(v) \geq \omega \ \forall v \in \Omega. \tag{12}$$

Then the equation

$$b(v)\nabla_v \xi(v) + \beta(v)\xi(v) = \varphi(v) \text{ in } \Omega,$$
$$\xi(v) = \xi_0 \text{ on } \bar{\Gamma}_-(\Omega).$$

has a solution $\xi \in L^2(\Omega)$.

Proof: see Roos *et al.* [1996], Lemma 1.6. □

Now let us look how to apply this theorem to the Regulator Equation. First, note that this theorem implies existence of a solution of (11) on the domain Ω for every function $c(v)$ and for every i, thus there is a solution of the system (10). The function x defined by $\mathrm{x}(v) = T^{-1}\xi(v)$ then solves the original (7).

It remains to clarify the meaning of the conditions posed on the functions as well as to define the domain Ω precisely.

Remark 2. Under the previous assumptions,
div $b(v) = 0$, $\beta(v) = -d_{ii} > 0$. In particular, the condition (12) is satisfied.

Proof: From the definition of b one has

$$\text{div } b(v) = \sum_{j=1}^{\mu} \frac{\partial}{\partial v_j}(\sum_{k=1}^{\mu} S_{jk}v_k) = \sum_{j=1}^{\mu} S_{jj} = \text{Trace } S.$$

As all eigenvalues of the matrix S have zero real part and all of them are conjugated one has Trace $S = 0$. The second statement is obvious from the definition of d_{ii} and the fact that the matrix A is supposed to be stable. □

Remark 3. Let $\alpha > 0$, $\Gamma_\alpha = \{v \in R^\mu | V(v) = \alpha\}$ and $\Omega_\alpha = \text{Int } \Gamma_\alpha = \{v \in R^\mu : V(v) < \alpha\}$. (Here, Int Γ denotes the interior of the closed curve Γ.) Then, according to the Assumption L3, there following holds: $0 \in \Omega_\alpha \forall \alpha > 0$, $\Omega_\alpha \subset \Omega_\beta$ if $\alpha < \beta$, $n(v).b(v) = 0 \forall v \in \partial\Omega_\alpha$. The last relation implies also that $\bar{\Gamma}_-(\Omega_\alpha)$ is empty, thus no boundary conditions are to be defined.

The previous results can be summarized in the following theorem.

Theorem 4. Assume the system (7) is given such that Assumptions **L1, L2, L3** are satisfied and the eigenvalues of S have zero real parts. Then for every $\alpha > 0$ and every $c \in L^2(\Omega_\alpha)$ there exists a function $\mathbf{x} \in L^2(\Omega_\alpha)$ satisfying (4) in Ω_α while no boundary condition is defined.

Finally a useful estimate of the solution will be proved.

Lemma 5. Let the domain Ω be defined as in the previous theorem. Then there exists a constant C (independent of the function c) such that for the solution of the equation (7) holds (the symbol $\|.\|_2$ denotes the $L^2(\Omega)$-norm):

$$\|\mathrm{x}\|_2 \leq C\|c\|_2. \tag{13}$$

Proof: This result will be proved for the transformed equation (10) first. Let $i \in \{1, \ldots, n\}$ be fixed. The i-th equation in the system (10) will be multiplied by ξ_i. Then one can integrate these equations on Ω:

$$\int_\Omega \xi_i \frac{\partial \xi_i}{\partial v} Svdv = \int_\Omega d_{ii}\xi_i^2 + (TB)_i c\xi_i dv.$$

Note that

$$\xi_i \frac{\partial \xi_i}{\partial v} = \frac{1}{2}\frac{\partial \xi_i^2}{\partial v}$$

Then the Stokes' theorem is applied on this term. It yields:

$$\int_\Omega (-d_{ii} - \frac{1}{2}\text{div } b)\xi^2 dv + \int_{\partial\Omega} \xi_i^2 b(v).n(v)dS_v =$$

$$\int_\Omega \xi_i (TB)_i cdv.$$

Due to the assumption, $-d_{ii} > 0$, div $b = 0$. The boundary term equals zero as shown in the previous result. The right-hand side term can be estimated using Hölder inequality. This yields $\|\xi_i\|_2 \leq \frac{1}{-d_{ii}}\|(TB)_i c\|_2$. Thus the inequality

$$\|\xi\|_2 \leq C\|(TB)\|\|c\|_2. \tag{14}$$

holds for the solution of the vector equation (10) (with a suitable constant $C > 0$) and finally, thanks to properties of the matrix T, for the solution $\mathbf{x}$ of the equation (7) as well. □

The next task is to verify the condition $\mathrm{x}(0) = 0$. As all the involved functions are elements of the L^2-spaces one cannot speak about their values. Instead, one can use the expression

$$L(\mathrm{x},0)=\lim_{t\to 0+}\frac{1}{\mathrm{meas}B_t}\int\limits_{B_t}\mathrm{x}(v)dv$$

where the symbol B_t denotes the open ball with radius t and the center at the origin.

The result is formulated as follows:

Lemma 6. Assume $L(c,0)=0$. Then $L(\mathrm{x},0)=0$.

The assumption implies also that $L(TBc,0)=0$. Thus, using (14) there exists a constant $K>0$ so that $\|\xi\|\leq K\|c\|$. Hence also for every $t>0$ such that $B_t\subset\Omega$:

$$(\frac{1}{\mathrm{meas}B_t}\int\limits_{B_t}\|\xi\|^2dv)^{\frac{1}{2}}\leq K(\frac{1}{\mathrm{meas}B_t}\int\limits_{B_t}\|c\|^2dv)^{\frac{1}{2}}$$

On the other hand, using the Jensen's Inequality:

$$(\frac{1}{\mathrm{meas}B_t}\int\limits_{B_t}\|\xi\|^2dv)^{\frac{1}{2}}\geq\frac{1}{\mathrm{meas}B_t}\int\limits_{B_t}|\xi|dv.$$

Remark 7. Let the functions c,x be continuous, moreover let $c(0)=0$. Then $\xi(0)=0$. Consequently $\mathrm{x}(0)=0$, so that the condition 3 is valid.

This follows from the fact that the equation (7) attains the form for $v=0$: $0=D\xi$. Regularity of D and the transformation (9) yield the result. □

2.2 Nonlinear Case

In this subsection, attention is paid to the problem of solvability of the equation (4) if the right-hand side is nonlinear. As a nonlinear counterpart of the Theorem 1 is not known to the authors the nonlinear equation (4) is solved iteratively using the linear case. We assume the equation is solved on the domain Ω.

Originating in (1), denote by A the Jacobi matrix of F (evaluated at the origin) and let $B=G(0)$. Moreover, define the functions $f:R^n\to R^n$ and $g:R^n\to R^n$ by $f(x)=F(x)-Ax,\ g(x)=G(x)-B$.

The following assumption will be useful:

Assumption N1: there exist positive constants K_f,K_g such that for every $x\in R^n$ the inequalities $\|f(x)\|\leq K_f\|x\|,\ \|g(x)\|\leq K_g\|x\|$ hold.

Remark 8. The domain Ω is the domain where the solution is sought. It always contains origin. As the functions f,g are $O(\|x\|^2)$-functions in a neighborhood of the origin this constant diminishes if the domain is smaller.

The equation (4) can be written in form

$$\begin{gathered}\frac{\partial\mathbf{x}(\mathrm{v})}{\partial v}Sv=\\ A\mathbf{x}(v)+Bc(v)+f(\mathbf{x}(v))+g(\mathbf{x}(v))c(v).\end{gathered}\tag{15}$$

The assumptions about the exosystem are the same as in the previous subsection. Moreover it os assumed that the matrices A and B satisfy the same conditions as in the linear case.

Using Theorem 4 one has

Proposition 9. Assume the matrices A,B and S satisfy the assumptions of the Theorem 4. Then there exists a domain Ω such as in Theorem 4 such that the equation

$$\begin{gathered}\frac{\partial\mathbf{x}(v)}{\partial v}Sv=\\ A\mathbf{x}(v)+B(v)+f(\tilde{\mathbf{x}}(v))+g(\tilde{\mathbf{x}}(v))c(v).\end{gathered}\tag{16}$$

has a solution x for every $c,\tilde{\mathbf{x}}\in L^2(\Omega)$ on Ω.

Lemma 10. Let $\tilde{\mathbf{x}}_1(v),\tilde{\mathbf{x}}_2(v)$ and $c(v)$ be given. Denote by $\mathbf{x}_i(v),\ i\in\{1,2\}$ the solution of the equation

$$\frac{\partial\mathbf{x}_\mathrm{i}(v)}{\partial v}Sv=A\mathbf{x}_\mathrm{i}(v)+Bc(v)+f(\tilde{\mathbf{x}}_\mathbf{i}(v))+g(\tilde{\mathbf{x}}_i(v))c(v).$$

Moreover, assume the functions f, g are Lipschitz continuous with constants $K_f>0$, $K_g>0$. Then, there exists a constant $K>0$ such that

$$\|\mathbf{x}_1-\mathbf{x}_2\|\leq K\|\tilde{\mathbf{x}}_1-\tilde{\mathbf{x}}_2\|.\tag{17}$$

Proof: Denote $e(v)=\mathbf{x}_1(v)-\mathbf{x}_2(v)$. Then the function e satisfies the equation

$$\begin{gathered}\frac{\partial e(v)}{\partial v}Sv=Ae(v)+\\ f(\tilde{\mathbf{x}}_1(v))+g(\tilde{\mathbf{x}}_1(v))c(v)-f(\tilde{\mathbf{x}}_2(v))-g(\tilde{\mathbf{x}}_2(v))c(v).\end{gathered}$$

According to the estimate given in Theorem 13 there exists a constant $C>0$ such that

$$\begin{gathered}\|e\|\leq C(\|f(\tilde{\mathbf{x}}_1(v))-f(\tilde{\mathbf{x}}_2(v))\|+\\ \|g(\tilde{\mathbf{x}}_1(v))c(v)-g(\tilde{\mathbf{x}}_2(v))c(v)\|\leq\\ CK_f\|\tilde{\mathbf{x}}_1-\tilde{\mathbf{x}}_2\|+CK_g\|\tilde{\mathbf{x}}_1-\tilde{\mathbf{x}}_2\|\|c\|.\end{gathered}$$

Thus setting $K=CK_f+CK_g\|c\|$ proves the result. □

Now one can formulate the final result that guarantees existence of a solution of the equation (4).

Theorem 11. Let the assumptions **L1-L3** and **N1** be valid and, moreover, let $K<1$. Under the introduced notation, the mapping $\Pi:L^2(\Omega)\to L^2(\Omega)$ defined by $\Pi(\tilde{\mathbf{x}})=\mathbf{x}$ (where $\mathbf{x}$ solves the equation (16) with right-hand side as defined there) has a fixed point. Consequently, this fixed point is the solution of the equation (4).

Remark 12.
- Note that the constant K depends on the function c. For a function c with greater norm $\|c\|$ the constant K is greater.
- The functions f and g are $O(\|x\|^2)$ functions in a neighborhood of the origin. Thus, for a sufficiently small domain Ω the constant K is less that 1.

Lemma 13. Let the assumptions of Theorem 11 are satisfied for a certain function c_0. There exists a constant $C>0$ such that for every $c_1,c_2\in L^2(\Omega),\ \|c_1\|,\|c_2\|\leq\|c_0\|$ holds. $\|\mathbf{x}_1-\mathbf{x}_2\|\leq C\|c_1-c_2\|$. Moreover, there exists a constant $\tilde{C}>0$ such that $\|\mathrm{x}_1\|\leq C\|c_1\|$

Let x_i be a solution of (15) with right-hand side c_i. Then define auxiliary equations

$$\frac{\partial\zeta_i}{\partial v}Sv=A\zeta_i(v)+Bc_i(v)+f(\mathbf{x}_i(v))+g(\mathbf{x}_i(v))c_i(v).$$

Its solution are the functions ζ_i. According to Lemma 5 the estimate holds

$$\begin{gathered}\|\zeta_1-\zeta_2\|\leq C_1\|c_1-c_2\|+C_2\|\mathrm{x}_1-\mathrm{x}_2\|+\\ C_3\|\zeta_1-\zeta_2\|\|c_0\|+C_4\|\mathrm{x}_1\|\|c_1-c_2\|.\end{gathered}$$

The Banach fixed point theorem implies that if $\Pi:X\to X$ is a contraction with a contractivity constant $K\in(0,1)$ and having a fixed point $x\in X$ then there exists a continuous nondecreasing function $\Phi:(0,1)\to(0,+\infty)$ such that $\|x\|\leq\Phi(K)$. Thus, if $C_2+C_3\|c_0\|<1$ then

$$\|\zeta_1-\zeta_2\|\leq\frac{C_1+C_4\Phi(K)}{1-C_2-C_3\|c_0\|}\|c_1-c_2\|.$$

The last statement is obtained by setting $c_2 \equiv 0, \mathbf{x}_2 \equiv 0$ which implies also $\zeta_2 \equiv 0$. $\square$

3. ALGEBRAIC CONDITION

The algebraic condition (the zero-output condition) is undoubtedly a fundamental part of the Regulator Equation. There are basically two ways to satisfy it. The first approach, adopted by many authors (see e.g. the chapter 3.4 in Huang [2004]), is based on a partial computation of some variables (up to the relative degree of the plant) and on their substitution into the differential part. This allows to reduce the size of the differential equation to be solved. (In fact, one gets the system of equations (3.67) in Huang [2004].) However, the crucial property - nonzero eigenvalues of the Jacobi matrix of the right-hand of the solved differential equation - may not hold for the resulting equation. There are two ways of remedy: This property can be assumed for the system of differential equations after this substitution. Then, the method described in this paper can be applied to this system of equations without any change. However, this assumption is difficult to verify a-priori. Therefore, the approach described in Čelikovský and Rehák [2004], Rehák and Čelikovský [2008] seems to be more suitable. The original system is stabilized using a state feedback. Then, an iterative process converging to the solution of the Regulator Equation is started: each iteration consists of three steps. First, given a value of the control $c(v)$, the differential part is solved. Then a value of a penalty functional measuring the error made in the algebraic condition is computed. Using this, a new value of the function $c(v)$ is defined to decrease the penalty in the next step. Hence, the proof of existence of the solution of the differential part actually guarantees existence of each iteration in this iterative process.

4. CONCLUSION

An alternative proof of solvability of the differential part of the Regulator Equation arising from the Output Regulation Problem was presented. This is especially suitable to be combined with the optimization-based iterative method that constructs a convergent sequence approaching the solution of the whole Regulator Equation as this proof guarantees existence of the iterations.

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