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An LTP/LPV approach to orbit control of spacecraft on elliptical orbits [1]

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Abstract: The problem of orbit control for spacecraft on elliptical orbits is analysed and an approach to the design of optimal constant gain controllers for the periodic dynamics of relative motion is proposed. In particular, it is shown how the proposed approach can guarantee closed loop stability and optimal performance both in the case of circular and elliptical orbits.

Keywords: Satellite control, Time-varying systems, Optimal control, Output feedback.

1. INTRODUCTION

The problem of formation keeping in its simplest form, which involves two satellites flying along the same orbit, can be cast as a trajectory tracking one, assuming that the position of the leader spacecraft is known with sufficient accuracy. What the control system must achieve is then an accurate regulation of the error between the desired trajectory and the actual one. In the case of a circular reference orbit, the control problem can be formulated with reference to the well known Euler-Hill linearised model for relative motion. Since this model is time-invariant, it leads to a conventional problem formulation. When dealing with elliptical reference orbits, on the other hand, the equations of relative motion turn out to be linear time-periodic (LTP, see e.g., Inalhan et al. [2002]), because of the periodic variation of orbital velocity as a function of spacecraft anomaly. A number of approaches to this problem have been proposed in the literature, ranging from predictive control (see for example Rossi and Lovera [2001]) to non linear dynamic inversion (see, e.g., Marcos et al. [2007]) and optimal periodic control theory (as in Schubert [2001], Theron et al. [2007]). While resorting to optimal periodic control appears to be a very natural approach to the problem, there is a significant drawback to it: the obtained controller will be itself time-varying, which leads to a number of critical issues from the implementation point of view. Alternatively, considering the anomaly as a scheduling parameter, the equations of relative motion can be viewed as a linear parametrically-varying (LPV) system. Not surprisingly, therefore, among the numerous approaches which have been proposed in the literature to this control problem, LTP and LPV control appear to be the most promising ones. LPV control is very attractive, because of the wide body of theoretical results and numerical tools now available, but it might prove to be a conservative approach, as it is based on the embedding of the actual periodic dynamics of the system in a much more general parameter-dependent framework. The above mentioned difficulties with LTP control, on the other hand, have been partially solved in previous work (see, e.g., Vigano' and Lovera [2007], Vigano' et al. [2007]) by applying novel design tools for the design of constant-gain optimal controllers for periodic systems to the in-plane and out-of-plane dynamics of relative orbital motion. Therefore, the aim of this paper is to propose a novel approach to controller design for this problem, with specific emphasis on practical aspects associated with their on-board implementation. In particular, the proposed approach is the result of a combination of LPV and LTP techniques: the explicit parameter dependence is exploited in order to simplify the formulation of the actual optimal periodic control problem. Simulation results will be presented to demonstrate the feasibility of the proposed approach on the candidate reference orbit for the Proba-3 ESA mission.

2. SPACECRAFT ORBIT DYNAMICS

The dynamics of the relative motion of a satellite with respect to a reference point on a circular orbit (for instance, a *leader* spacecraft) can be expressed by means of the well-known Euler-Hill equations,which can be generalised to the case of elliptical orbits, as proposed in Inalhan et al. [2002]. In the following the considered dynamical models for relative orbital motion with respect to elliptical orbits will be described.

2.1 Coordinate frames

For the orbit control system of an Earth orbiting spacecraft the following reference systems are adopted:

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- Earth Centered Inertial (ECI) reference frame. The origin of these axes is in the Earth's centre. The X-axis is parallel to the line of nodes, and is positive in the Vernal equinox direction (Aries point). The Z-axis is defined as being parallel to the Earth's geographic north-south axis and pointing north. The Y-axis completes the right-handed orthogonal triad.
- Orbital Axes (X_0, Y_0, Z_0). The origin of these axes is in the satellite centre of mass. The X-axis is the unit vector of the radius $\mathbf{r}$ (i.e. the position vector of the spacecraft in the ECI coordinate system). The Z-axis is the unit vector of the momentum $\mathbf{h}$ (where $\mathbf{h} = \mathbf{r} \times \mathbf{v}$, with $\mathbf{v}$ the velocity vector of the satellite). The Y-axis completes the right-handed orthogonal triad.

2.2 Orbit mechanics: elliptical orbits

For elliptical orbits, the relative motion of the follower spacecraft with respect to the leader can be given the following form (see Inalhan et al. [2002])

$$\begin{aligned} x'_{in}(\theta) &= A_{in}(\theta)x_{in}(\theta) + B_{in}(\theta)u_{in}(\theta) \\ y_{in}(\theta) &= C_{in}(\theta)x_{in}(\theta) \end{aligned} \tag{1}$$

$$\begin{aligned} x'_{out}(\theta) &= A_{out}(\theta)x_{out}(\theta) + B_{out}(\theta)u_{out}(\theta) \\ y_{out}(\theta) &= C_{out}(\theta)x_{out}(\theta) \end{aligned} \tag{2}$$

where the state vectors x_{in} and x_{out} are, respectively, $[x', x, y', y]$ and $[z', z]$ and

$$A_{in}(\theta) = \begin{bmatrix} \frac{2e\sin\theta}{1+e\cos\theta} & \frac{3+e\cos\theta}{1+e\cos\theta} & 2 & -\frac{2e\sin\theta}{1+e\cos\theta} \\ 1 & 0 & 0 & 0 \\ -2 & \frac{2e\sin\theta}{1+e\cos\theta} & \frac{2e\sin\theta}{1+e\cos\theta} & \frac{e\cos\theta}{1+e\cos\theta} \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{3}$$

$$B_{in}(\theta) = \frac{\left(1-e^2\right)^3}{\left(1+e\cos\theta\right)^4 n^2} \begin{bmatrix} 1/m & 0 \\ 0 & 0 \\ 0 & 1/m \\ 0 & 0 \end{bmatrix} \tag{4}$$

$$A_{out}(\theta) = \begin{bmatrix} \frac{2e\sin\theta}{1+e\cos\theta} & -\frac{1}{1+e\cos\theta} \\ 1 & 0 \end{bmatrix} \tag{5}$$

$$B_{out}(\theta) = \frac{\left(1-e^2\right)^3}{\left(1+e\cos\theta\right)^4 n^2} \begin{bmatrix} 1/m \\ 0 \end{bmatrix} \tag{6}$$

where θ represents the true anomaly, n is the natural frequency of the reference orbit, e is the eccentricity, (x, y, z) denote the tracking errors in orbital axes and the operator $(.)'$ represents derivation with respect to θ. The mass of the spacecraft is indicated with m, while $u_{in} \in \mathbf{R}^2$ and $u_{out} \in \mathbf{R}$ represent the control forces acting on the center of mass of the satellite. The decoupling of the out-of-plane dynamics from the in-plane dynamics clearly simplifies the control design problem. In addition, observe that the equations are LTP (with period $T = 2\pi$) with respect to the true anomaly θ. For elliptical orbits, the time rate of change of the true anomaly $\dot{\theta}$ can be written as (Inalhan et al. [2002])

$$\dot{\theta} = \frac{n(1+e\cos(\theta))^2}{(1-e^2)^{3/2}}. \tag{7}$$

Remark 1. Systems (1)-(2) are expressed in the θ domain, but they can be given an equivalent form by using time as the independent variable, instead of the true anomaly, by simply exploiting the fact that derivation over time (represented by $\dot{(\cdot)}$) and derivation over θ (represented by $(\cdot)'$) are related as follows

$$\dot{(\cdot)} = (\cdot)'\dot{\theta} \qquad \ddot{(\cdot)} = (\cdot)''\dot{\theta}^2 + \dot{\theta}\dot{\theta}'(\cdot)'. \tag{8}$$

2.3 Open loop analysis

Defining the quantities

$$\alpha(\theta) = 1 + e\cos(\theta), \quad \beta(\theta) = 2e\sin(\theta), \tag{9}$$

it is easy to see that matrices (3)-(6) can be written in the form

$$A_{in}(\theta) = \frac{1}{\alpha(\theta)} \begin{bmatrix} \beta(\theta) & 2+\alpha(\theta) & 2\alpha(\theta) & -\beta(\theta) \\ \alpha(\theta) & 0 & 0 & 0 \\ -2\alpha(\theta) & \beta(\theta) & \beta(\theta) & \alpha(\theta)-1 \\ 0 & 0 & \alpha(\theta) & 0 \end{bmatrix} \tag{10}$$

$$B_{in}(\theta) = \frac{\left(1-e^2\right)^3}{\alpha^4(\theta)n^2} \begin{bmatrix} 1/m & 0 \\ 0 & 0 \\ 0 & 1/m \\ 0 & 0 \end{bmatrix} \tag{11}$$

$$A_{out}(\theta) = \frac{1}{\alpha(\theta)} \begin{bmatrix} \beta(\theta) & -1 \\ \alpha(\theta) & 0 \end{bmatrix} \tag{12}$$

$$B_{out}(\theta) = \frac{\left(1-e^2\right)^3}{\alpha^4(\theta)n^2} \begin{bmatrix} 1/m \\ 0 \end{bmatrix}, \tag{13}$$

which, as will be demonstrated in the following, is specially useful in the formulation of the control design problem. In this Section, on the other hand, the focus will be on the analysis of the open-loop properties of the dynamics of relative motion, with the following objectives: to investigate the role of periodicity in the dynamics and how orbit eccentricity affects periodicity.

Analysis of the A matrix The first issue to be dealt with is whether it is truly necessary to keep periodicity into account in the formulation of the control problem. To this purpose, averaged LTI approximations of the in-plane dynamics (3) have been computed, for increasing values of the eccentricity, and their eigenvalues have been analysed. The results are summarised in Figure 1, from which it is apparent that the averaged model exhibits a real, positive eigenvalue for $e > 0$, a fact that is in marked contrast with the well known fact that the characteristic multipliers of (3) are equal to 1 for all e (see Inalhan et al. [2002]). Therefore, the averaged LTI model is not even able to capture the open loop stability characteristics of the actual LTP system.

Analysis of the B matrix The effect of a non zero eccentricity shows up even more dramatically when one turns to the analysis of the $B_{in}(\theta)$ and $B_{out}(\theta)$ matrices. In particular, it is easy to see that with respect to the $e = 0$ case, the non zero eccentricity introduces a time-periodic scaling factor $\gamma(\theta)$ given by

$$\gamma(\theta) = \frac{(1-e^2)^3}{\alpha^4(\theta)}. \tag{14}$$

A plot of $\gamma(\theta)$ for increasing values of e is shown in the upper portion of Figure 3. As can be seen, increasing eccentricity implies that the control effectiveness is significantly

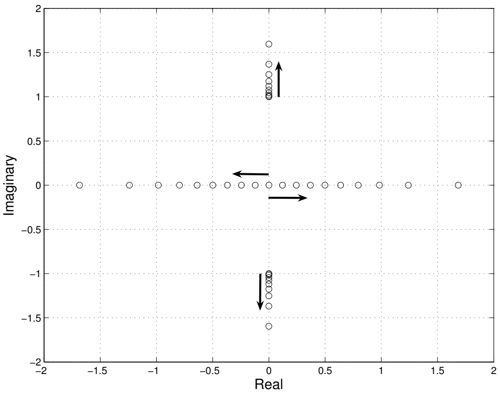

Fig. 1. Eigenvalues of the averaged $A_{in}(\theta)$ matrix (arrows indicate increasing eccentricity, from 0 to 0.9).

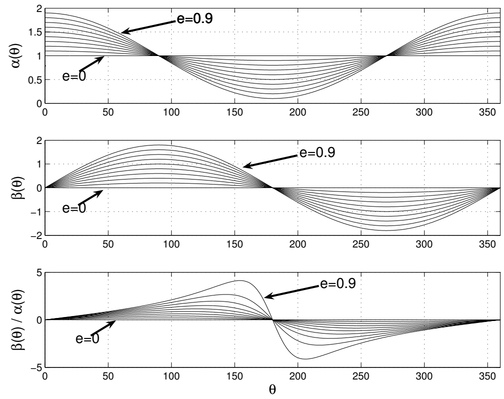

Fig. 2. Effect of increasing e on the elements of the periodic $A_{in}(\theta)$ matrix.

reduced near perigee ($\theta = 0^o$) and significantly increased near apogee ($\theta = 180^o$). The importance of this effect can

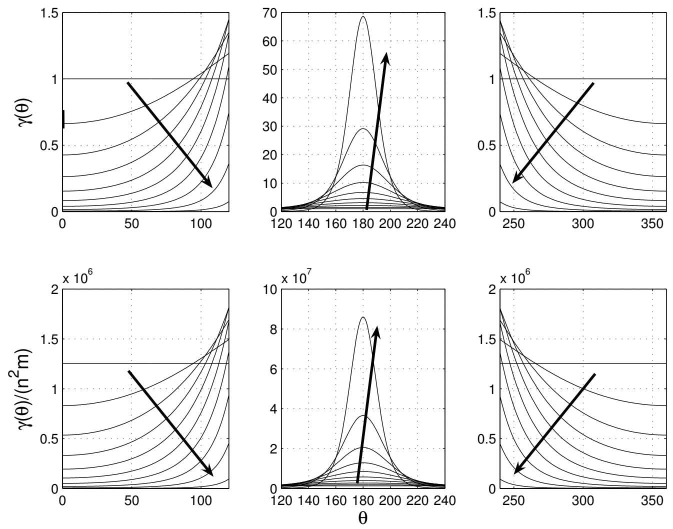

Fig. 3. Effect of increasing e on $\gamma(\theta)$ (top) and on the control effectiveness (bottom) - arrows indicate increasing eccentricity, from 0 to 0.9.

be fully appreciated by looking at the effect of increasing e on the overall *control effectiveness*, i.e., the quantity $\gamma(\theta)/(n^2 m)$, depicted in the lower portion of Figure 3 (for $n = 7.3 \times 10^{-5}$ rad/s and $m = 150$ kg).

3. DESIGN APPROACH

A number of approaches have been proposed in the literature in order to design suitable control laws for the dynamics of relative motion. In most of the existing literature the dynamics of relative motion is dealt with as it is, without making any attempt at compensating the variability of the gain or at decoupling the in-plane axes. In Marcos et al. [2007], on the other hand, a direct dynamic inversion approach is adopted, i.e., the entire dynamics is compensated and replaced with a desired acceleration vector computed from a suitable reference model. In this paper, a somewhat intermediate approach is proposed, in the sense that the availability of an accurate knowledge of the dynamics of relative motion is exploited in order to reduce the strong variability of the $B_{in}(\theta)$ and $B_{out}(\theta)$ matrices, and to transform the open-loop system into a set of three second order subsystems, associated with relative motion on each axis. Two different approaches are then proposed for the control of the decoupled axes, namely an eigenvalue assignment one (leading to the design of a periodic gain controller) and an optimal periodic control one (leading to the design of a constant gain controller) based on the method presented in Vigano' et al. [2007]. In particular, specific attention is dedicated to the problem of translating time-domain specifications on the desired closed-loop dynamics into suitable θ-domain specifications. The following Sections will discuss the main steps in the proposed design approach, namely gain compensation, decoupling and control of the decoupled x, y and z axes.

4. GAIN COMPENSATION AND DECOUPLING

Consider first the in-plane dynamics (1) and let

$$u_{in} = \frac{\alpha^3(\theta) n^2 m}{(1-e^2)^3}(K_1 x_{in} + v_{in}), \tag{15}$$

where v_{in} is an auxiliary control variable, so that (1) can be written as

$$x'_{in}(\theta) = A_{c1}(\theta) x_{in}(\theta) + B_{c1}(\theta) v_{in}(\theta) \tag{16}$$

where

$$A_{c1}(\theta) = A_{in}(\theta) + \frac{1}{\alpha(\theta)} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} K_1, \quad B_{c1}(\theta) = \frac{1}{\alpha(\theta)} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}. \tag{17}$$

It is now easy to see, by inspection of $A_{in}(\theta)$, that choosing

$$K_1(\theta) = \begin{bmatrix} 0 & 0 & -2\alpha(\theta) & \beta(\theta) \\ 2\alpha(\theta) & -\beta(\theta) & 0 & 0 \end{bmatrix} \tag{18}$$

the $A_{c1}(\theta)$ matrix reduces to

$$A_{c1}(\theta) = \frac{1}{\alpha(\theta)} \begin{bmatrix} \beta(\theta) & 2+\alpha(\theta) & 0 & 0 \\ \alpha(\theta) & 0 & 0 & 0 \\ 0 & 0 & \beta(\theta) & \alpha(\theta)-1 \\ 0 & 0 & \alpha(\theta) & 0 \end{bmatrix}, \tag{19}$$

so that the components of relative motion along the x and y axes have been effectively decoupled. The control problem therefore reduces to the one of stabilising the three, decoupled, second-order LTP systems associated to the x, y and z axes. In the following, two different approaches to the problem will be presented. The first one is based on simple eigenvalue assignment ideas and leads to the design of a (closed-form) time-periodic gain; the second one, on the other hand, relies on the LQ approach for LTP systems presented in Vigano' et al. [2007] and leads to the design of a constant gain.

5. SINGLE AXIS PERIODIC GAIN CONTROL

With respect to (19), consider the subsystem associated with the x axis, i.e., the matrices

$$A_{c1}^{x}(\theta) = \frac{1}{\alpha(\theta)} \begin{bmatrix} \beta(\theta) & 2+\alpha(\theta) \\ \alpha(\theta) & 0 \end{bmatrix}, \quad B_{c1}^{x}(\theta) = \frac{1}{\alpha(\theta)} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{20}$$

then the control problem at hand can be formulated as the one of determining a periodic control gain $K_2^x(\theta)$ such that the closed-loop dynamics $A_{c2}^x = A_{c1}^x(\theta) + B_{c1}^x(\theta)K_2^x(\theta)$ is given by

$$A_{c2}^{x} = \begin{bmatrix} -a_1^x & -a_0^x \\ 1 & 0 \end{bmatrix}, \tag{21}$$

where a_1^x, a_0^x are the coefficients of the desired closed-loop characteristic polynomial $p_x(\lambda) = \lambda^2 + a_1^x\lambda + a_0^x$. Clearly, the solution is given by

$$K_2^x(\theta) = \left[-a_1^x\alpha(\theta) - \beta(\theta) \;\; -a_0^x\alpha(\theta) - 2 - \alpha(\theta)\right], \tag{22}$$

and similarly, for the y and z axes, by

$$K_2^y(\theta) = \left[-a_1^y\alpha(\theta) - \beta(\theta) \;\; -a_0^y\alpha(\theta) + 1 - \alpha(\theta)\right] \tag{23}$$

$$K_2^z(\theta) = \left[-a_1^z\alpha(\theta) - \beta(\theta) \;\; -a_0^z\alpha(\theta) + 1\right]. \tag{24}$$

Care must be taken, however, in the choice of the desired characteristic polynomials $p_x(\lambda)$, $p_y(\lambda)$ and $p_z(\lambda)$, which reflect the specifications for the desired closed-loop behaviour of the relative motion along the three axes. Indeed, one must keep in mind that the design model given by (1)-(2) is expressed with respect to θ as independent variable, while the control specifications for each axis are likely to be formulated in the time domain. Therefore, the remarks of Section 2 on the relationship between t and θ must be taken into account in the design of the desired characteristic polynomials.

Assume that for a generic axis the desired closed-loop dynamics in the time domain is given by the time-invariant second order system

$$\dot{z} = A_d^t z = \begin{bmatrix} -a_1 & -a_0 \\ 1 & 0 \end{bmatrix}. \tag{25}$$

Then, according to Section 2 the corresponding representation of (25) in the θ domain is given by

$$z' = A_d^\theta z, \tag{26}$$

with

$$A_d^\theta(\theta) = \begin{bmatrix} (-a_1\dot{\theta} - \ddot{\theta})/\dot{\theta} & -a_0/\dot{\theta}^2 \\ 1 & 0 \end{bmatrix} \simeq \begin{bmatrix} -a_1/\dot{\theta} & -a_0/\dot{\theta}^2 \\ 1 & 0 \end{bmatrix}, \tag{27}$$

which is now an LTP system. The time variability of the right-hand side of (25) makes the problem of defining a direct correspondence between the time behaviour of the system and the coefficients of matrix A_d^θ a rather critical one. Furthermore, as can be seen from Figure 4, in the case of high eccentricity orbits the variability of $\dot{\theta}$ with respect to its mean value is such that replacing (26) with the corresponding *averaged* LTI system (see Khalil [1992]) is likely to lead to very crude approximations of the actual transients of the system. A different approach consists

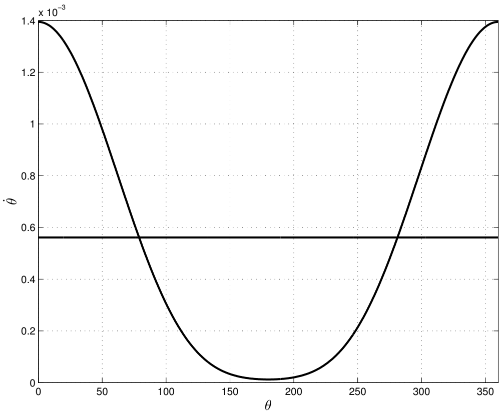

Fig. 4. Plot of $\dot{\theta}$ as a function of true anomaly for the nominal Proba-3 orbit ($e = 0.829$).

in using a *local* LTI approximation for the LTP system (26), valid for values of the anomaly corresponding to the portion of the orbit where the formation keeping controller will have to operate. For the case of apogee control (see, again, the Proba-3 mission), one can replace $\dot{\theta}$ in (26) with the value it takes for $\theta = 180^o$, i.e., in correspondence with its minimum over one period. Therefore, the design of the desired characteristic polynomial can be carried out on the basis of the LTI approximation

$$z' = \begin{bmatrix} -a_1/\dot{\theta}_{min} & -a_0/\dot{\theta}_{min}^2 \\ 1 & 0 \end{bmatrix} z, \tag{28}$$

where $\dot{\theta}_{min}$ is given by

$$\dot{\theta}_{min} = n\frac{(1-e)^2}{(1-e^2)^{3/2}} \tag{29}$$

and is plotted in Figure 5 as a function of eccentricity. As

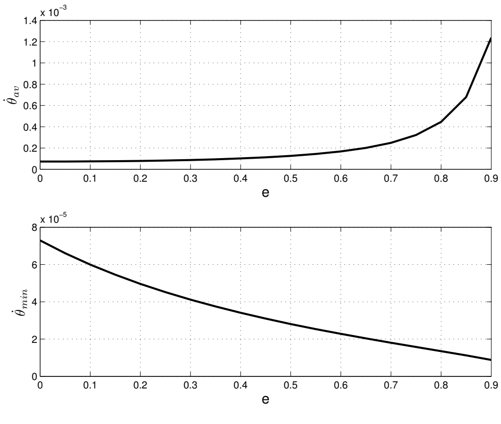

Fig. 5. $\dot{\theta}_{av}$ (top) and $\dot{\theta}_{min}$ (bottom) as functions of eccentricity.

an example, Figure 6 shows the time response of systems (26) and (28) for the case of $e = 0.829$ and

$$A_d^t = \begin{bmatrix} -2(2\pi/2000)0.7 & -(2\pi/2000)^2 \\ 1 & 0 \end{bmatrix}, \tag{30}$$

with initial state $z(0) = [0\ 10]^T$ and initial anomaly $\theta(0) = 170^o$. As can be seen from the Figure, the LTI system (28) provides a fairly reliable approximation of the actual LTP one.

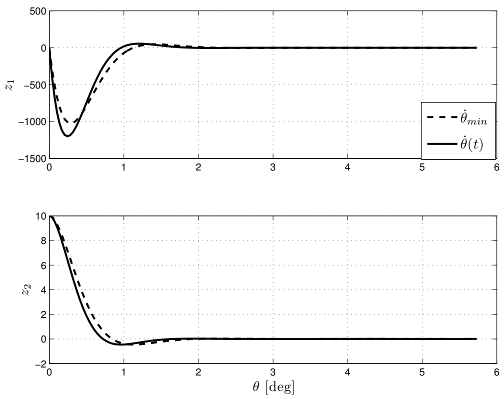

Fig. 6. Time response of systems (26) and (28) for the case of $e = 0.829$ and $\theta(0) = 170^o$.

6. SINGLE AXIS CONSTANT GAIN CONTROL

An alternative approach to the control of the decoupled x, y and z axis corresponds to the application of tools for LQ-optimal constant gain design for continuous-time LTP systems, first presented in Vigano' et al. [2007]. In this Section, a short overview of the above mentioned results will be provided, and their application to the problem at hand will be subsequently discussed.

Consider the LTP system

$$\begin{aligned}\dot{x}(t) &= A(t)x(t) + B(t)u(t)\\ y(t) &= C(t)x(t)\end{aligned} \quad (31)$$

where $A(t) \in \mathbf{R}^{n\times n}, B(t) \in \mathbf{R}^{n\times m}, C(t) \in \mathbf{R}^{p\times n}$ are T-periodic matrices, and the quadratic performance index

$$J = E\left\{\int_{t_0}^{\infty}\left[x^T(t)Q(t)x(t) + u^T(t)R(t)u(t)\right]dt\right\} \quad (32)$$

with $Q(t) = Q^T(t) \geq 0, R(t) = R^T(t) > 0$ T-periodic matrices and where the expectation is taken over the initial condition x_0, modelled as a random variable with zero mean and known covariance $X_0 = \left\{x_0 x_0^T\right\}$. The optimal output feedback control problem can be formulated as follows: find the constant feedback matrix F of optimal control action

$$u^*(t) = Fy(t) \quad (33)$$

which minimizes the performance index J of (32). Holding (33), the closed loop dynamics can be written as

$$\dot{x} = \left[A(t) + B(t)FC(t)\right]x = \bar{A}(t)x \quad (34)$$

where $\bar{A}(t) = A(t) + B(t)FC(t)$ represents the closed loop dynamic matrix, which is obviously periodic. Therefore matrix $\bar{A}(t)$ is associated with the transition matrix $\Phi_{\bar{A}}(t, t_0)$ satisfying

$$\dot{\Phi}_{\bar{A}}(t, t_0) = \bar{A}(t)\Phi_{\bar{A}}(t, t_0),\;\; \Phi_{\bar{A}}(t_0, t_0) = I. \quad (35)$$

The minimization of the performance index given by (32) can be carried out using either gradient-free or gradient-based methods, provided that an analytical expression for the gradient of the performance index with respect to the F matrix is available. In both cases, an initial stabilizing matrix F_0 must be employed. In the following Proposition (see Vigano' et al. [2007] for details), necessary conditions for optimality (and therefore the required gradient expression) will be presented.

Proposition 1. Let F be a constant stabilizing output feedback gain and assume that the matrices $\bar{A}(t)$, $\bar{Q}(t)$ and $X(t)$ are given respectively by $\bar{A} = A + BFC$, $\bar{Q} = Q + C^T F^T RFC$ and $X(t) = \Phi_{\bar{A}}(t, t_0)X_0\Phi_{\bar{A}}^T(t, t_0)$; hence, the expressions for the performance index (32) and its gradient are given by

$$J(F, X_0) = tr\left(P_0 X_0\right)$$

$$\nabla_F J(F, X_0) = 2\int_{t_0}^{t_0+T}\left[B^T(t)P(t) + \right.$$
$$\left. + R(t)FC(t)\right]\Phi_{\bar{A}}(t, t_0)V\Phi_{\bar{A}}^T(t, t_0)C^T(t)dt$$

where the symmetric periodic matrices $P(t)$ and V satisfy, respectively, the periodic Lyapunov differential equation (PLDE)

$$-\dot{P}(t) = \bar{A}^T(t)P(t) + P(t)\bar{A}(t) + \bar{Q}(t)$$

and the discrete Lyapunov equation (DLE)

$$V = \Psi V \Psi^T + X_0.$$

The optimization of (32) requires, as already said, that the LTP system (31) is output stabilizable and, at each iteration i, the matrix F_i belongs to the set $\mathcal{S}_F \subset \mathbf{R}^{m\times p}$ of the stabilizing feedback gain matrices. Formally, the optimization problem can be stated as

$$\min_{F\in\mathcal{S}_F} J(F, X_0). \quad (36)$$

The stopping criterion, indicating convergence to a global or, at least, a local solution of (36) will be simply $\|\nabla_F J\| < tol$.

Remark 2. The problem of output feedback control for LTP systems has been recently treated in Farges et al. [2006], in the discrete-time case. In particular, a characterization of these controllers is provided, which relies on the solution of bilinear matrix inequalities ($\mathcal{BMI}$s). The designed ellipsoids offer the interesting property of being *resilient*, which means that the resulting closed-loop system is robustly stable with respect to uncertainty of the control law parameters. While this approach lends itself to the formulation of very general control problems, it suffers from a significant drawback, i.e., it is currently limited to relatively small scale problems (both in terms of order and period) when compared to techniques relying on the solution of periodic Lyapunov and Riccati equations.

The application of the above described approach for the design of constant gain controllers for LTP systems to the stabilisation of the decoupled single axes of the relative motion dynamics does not pose any specific difficulty, but deserves a few comments. As discussed with reference to the eigenvalue assignment approach, it is worth noting that care must be taken if one aims at achieving a prescribed level of time-domain performance while carrying out the design in the θ domain. As an example, assume that the designer aims at minimising the time-domain quadratic performance index

$$J_t = E\left\{\int_{t_0}^{\infty}\left[x^T(t)Q(t)x(t) + u^T(t)R(t)u(t)\right]dt\right\} \quad (37)$$

for one the three axes (i.e., $x = [\dot{z}\; z]^T$), with $Q(t) = Q^T(t) \geq 0, R(t) = R^T(t) > 0$ given T-periodic matrices.

Typically the weighting matrices would be chosen constant as in the LTI case; in particular, we will make the simplifying, though not overly restrictive, assumption that Q has the block diagonal structure

$$Q = \begin{bmatrix} Q_{\dot{z}} & 0 \\ 0 & Q_z \end{bmatrix}. \quad (38)$$

The problem is then to choose the θ-domain 2π-periodic matrix weights $Q_\theta(\theta) = Q_\theta^T(\theta) \geq 0, R_\theta(\theta) = R_\theta^T(\theta) > 0$ such that the cost function

$$J_\theta = E\left\{\int_{\theta_0}^{\infty} \left[x^T(\theta)Q^\theta(\theta)x(\theta) + u^T(\theta)R^\theta(\theta)u(\theta)\right] d\theta\right\} \quad (39)$$

is equivalent to J_t. The problem is easily solved by recalling that according to (8) we have $dt = \dot{\theta}^{-1}d\theta$, so that in order to get $J_\theta = J_t$, $Q^\theta(\theta)$ and $R^\theta(\theta)$ should be chosen as

$$Q^\theta(\theta) = \begin{bmatrix} Q^\theta_{z'}(\theta) & 0 \\ 0 & Q^\theta_z(\theta) \end{bmatrix}, \quad (40)$$

$$Q^\theta_{z'}(\theta) = Q_{\dot{z}}\dot{\theta}, \quad Q^\theta_z(\theta) = Q_z\dot{\theta}^{-1}, \quad (41)$$

and

$$R^\theta(\theta) = R\dot{\theta}^{-1}. \quad (42)$$

The desired time-domain specification can therefore be achieved from the θ-domain formulation of the problem by making use of suitable time-periodic weights in the LQ problem.

Finally, as is apparent from equation (39), the obtained control gain will also be a function of the chosen initial anomaly θ_0, which therefore should be chosen in accordance with the expected operating conditions of the controller (e.g., apogee formation keeping).

7. SIMULATION RESULTS

The highly eccentric orbit which is currently being used as a reference for Proba-3 studies has a period of $24h$, a corresponding orbital frequency of $n = 2\pi/24h = 7.2910^{-5}$ rad/s and an eccentricity $e = 0.829$; a satellite mass $m = 200$ Kg has been assumed. The initial state vector given by $x^0_{in} = [0, 100, 0, 100]$ and $x^0_{out} = [0, 10]$ has been chosen. This initial condition is representative of the variation in relative position which the Proba-3 spacecraft should achieve while moving from one FF configuration to another, according to the preliminary study in Facility [2005]. The controller has been tuned using both the proposed approaches. More precisely, in the case of the eigenvalue assignment approach presented in Section 5 the desired closed-loop time domain dynamics for each axis has been chosen as in equation (30), while for the LQR approach presented in Section 6 the weighting matrices have been selected in order to obtain a similar closed-loop time response. Only the results for the x axis are presented, both for the sake of brevity and because the performance obtained along the y and z axes is essentially identical. As can be seen from Figure 7, the proposed control design methods lead to very similar closed-loop performance, which is very close to the one specified via the reference model (30) in the eigenvalue assignment approach. Current work is aiming at providing a more detailed assessment of the achievable performance in a more realistic simulation environment, taking also into account model uncertainty and measurement errors.

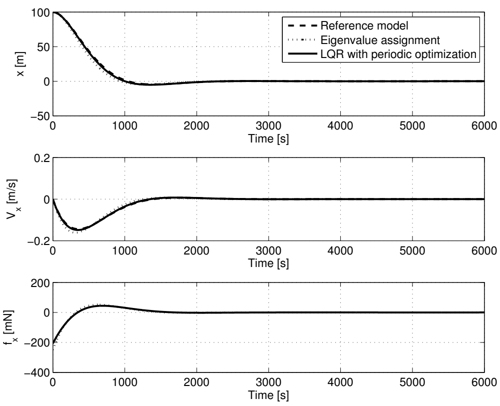

Fig. 7. Position error along x axis - comparison between eigenvalue assignment and periodic LRQ approaches.

8. CONCLUDING REMARKS

The problem of orbit control for a spacecraft in elliptical orbit has been considered and an approach based on decoupling and subsequent single axis control has been proposed. In particular, constant gain and periodic gain design methods have been developed. Simulation results demonstrate that excellent performance can be obtained.

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