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Structural Properties of Linear Discrete-Time Fractional-Order Systems

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Abstract: In this communication, some results on the analysis of the reachability and observability of linear discrete-time fractional order systems are given. Mathematical conditions for checking the controllability and the observability of such systems are developed. Furthermore, the concepts of the controllability realization index, the observability realization index and the structure realization index are introduced.

1. INTRODUCTION

The concept of non-integer derivative and integral is increasingly used to model the behavior of real systems in various £elds of science and engineering (Debnath [2003]), (Magin [2006]). These systems exhibit hereditary properties and long memory transients, which can be represented more accurately by fractional-order models. Some fundamental developments of the fractional calculus theory are given in (Oldham and Spanier [1974]), (Samko et al [1993]), (Oustaloup [1995]), (Kilbas et al [2006]).
In the particular domain of control theory, several authors have been interested by this aspect since the sixties. The £rst contributions, (Manabe [1961]), (Oustaloup [1983]), (Axtell and Bise [1990]), gave the generalization of classical analysis methods for fractional-order systems (transfer function de£nition, frequency response, pole and zero analysis,...).
The state-space representation of fractional-order systems has been introduced in (Matignon [1994]), (Hotzel [1998]), (Raynaud and A. Zergainoh [2000]), (Hotzel [1998]), (Dorčak [2000]), (Sabatier et al [2002]), (Vinagre et al [2002]). The state-space representation has been exploited in the analysis of system performances. In fact, the solution of the state-space equation has been derived by using the Mittag-Le¤ler function (Mittag-Lef¤er [1904]). The stability of the fractional-order system has been investigated (Matignon [1996a]). A condition based on the argument principle has been established to guarantee the asymptotic stability of the fractional-order system. Further, the controllability and the observability properties have been de£ned and some algebraic criteria of these two properties have been derived (Matignon and d'Andréa-Novel [1996b]), (Bettayeb and Djennoune [2006]).
Linear discrete-time fractional-order systems modeled by a state space representation have been introduced in (Dzieliński and Sierociuk [2006a]), (Dzieliński and Sierociuk [2006b]), (Sierociuk and Dzielinski [2006]), (Dzieliński and Sierociuk [2005]). These contributions are devoted respectively to a stability condition, to the design of an observer, Kalman £lter design and £nally to an adaptive feedback control for discrete fractional-order systems. In (Guermah et al [2008]), some new results concerning the controllability and observability properties of linear discrete-time fractional-order systems have been derived. The objective of this work is to give some extensions of our previous results (Guermah et al [2008]) on the analysis of structural properties of the linear discrete time fractional order systems. The concept of Controllability Realization Index (*CRI*) already introduced for linear discrete-time systems with time-delay in state (Pen et al [2003]) is extended here to fractional order systems. Furthermore, the dual concept of Observability Realization Index (*ORI*) and Structural Realization Index (*SRI*) are proposed here. These concepts are useful in the understanding of the fractional order systems.
The rest of this paper is organized as follows: In Section 2, we recall some fundamental de£nitions on fractional derivatives and fractional-order systems, modeled by continuous models. Then we expose the discrete-time model derived, as de£ned in (Dzieliński and Sierociuk [2005]) and we introduce extra notations that reveal a new form, making it possible to take into account the past behavior of the system and to analyze the structural properties. Section 3 addresses the controllability and observability properties. The previous results developed in (Guermah et al [2008]) are recalled here. In Section 4, the concept of the Controllability Realization Index introduced for discrete-time system with time delay in state is extended to the discrete-time fractional order systems. The same concept concerning the observability is proposed. In Section 5, we present some numerical results corresponding to different cases of checking the controllability and the observability conditions for such systems.

2. LINEAR DISCRETE-TIME FRACTIONAL-ORDER SYSTEMS

There are different de£nitions of the fractional derivative, (Oldham and Spanier [1974]), (Samko et al [1993]), (Kilbas et al [2006]). The Grünwald-Letnikov de£nition which is the discrete approximation of the fractional order derivative is used here. The Grünwald-Letnikov fractional order derivative of a given function $f(t)$ is given by:

$$ {}_a^G D_t^\alpha f(t) = \lim_{h\to 0} \frac{{}_a\Delta_h^\alpha f(t)}{h^\alpha} \tag{1}$$

where the real number α denotes the order of the derivative, a is the initial time and h is a sampling time. The difference operator Δ is given by:

$$ {}_a\Delta_h^\alpha f(t) = \sum_{j=0}^{[\frac{t-a}{h}]} (-1)^j \binom{\alpha}{j} f(t-jh) \tag{2}$$

The binomial term can be obtained by the relation:

$$\binom{\alpha}{j} = \begin{cases} 1 & \text{for } j=0 \\ \dfrac{\alpha(\alpha-1)...(\alpha-j+1)}{j!} & \text{for } j>0 \end{cases} \tag{3}$$

and [ ] takes the integer part.
Now, let us consider the traditional discrete-time state-space model of integer order, i. e. when α is equal to unity:

$$x(k+1) = Ax(k) + Bu(k); x(0) = x_0 \tag{4a}$$

$$y(k) = Cx(k) + Du(k) \tag{4b}$$

Where $u(k) \in \mathbb{R}^p$ and $y(k) \in \mathbb{R}^q$ are respectively the input and the output vectors, $x(k) = [x_1(k) \quad x_2(k) \quad \dots \quad x_n(k)] \in \mathbb{R}^n$ is the state vector. Its initial value is denoted $x_0 = x(0)$ and can be set equal to zero without loss of generality. (A, B, C, D) are the conventional state space matrices with appropriate dimensions.
The £rst-order difference for $x(k+1)$ can be de£ned as:

$$\Delta^1 x(k+1) = x(k+1) - x(k)$$

Therefore, using Equation $(4a)$ we deduce that:

$$\begin{aligned}\Delta^1 x(k+1) &= Ax(k) + Bu(k) - x(k) \\ &= A_d x(k) + Bu(k)\end{aligned}$$

where $A_d = A - I_n$ and I_n is the n-dimensional identity matrix. The generalization of this integer-order difference to a non integer-order (or fractional-order) difference has been addressed in (Dzieliński and Sierociuk [2005]) where the discrete fractional-order difference operator with the initial time taken equal to zero is de£ned as follows:

$$\Delta^\alpha x(k) = \frac{1}{h^\alpha} \sum_{j=0}^{k} (-1)^j \binom{\alpha}{j} x(k-j) \tag{5}$$

In the sequel, the sampling time h is taken equal to 1. These results conducted to conceive the linear discrete-time fractional-order state-space model, using the equations:

$$\Delta^\alpha x(k+1) = A_d x(k) + Bu(k); x(0) = x_0 \tag{6}$$

In this model, the differentiation order α is taken the same for all the state variables $x_i(k)$, $i = 1, \dots, n$. This is referred to as commensurate order. Besides, from Equations (5) and (6) we have:

$$x(k+1) = A_d x(k) - \sum_{j=1}^{k+1} (-1)^j \binom{\alpha}{j} x(k-j+1) + Bu(k) \tag{7}$$

The discrete-time fractional order system is represented by the following state space model:

$$x(k+1) = \sum_{j=0}^{k} A_j x(k-j) + Bu(k); x(0) = x_0 \tag{8a}$$

$$y(k) = Cx(k) + Du(k) \tag{8b}$$

where $A_0 = A_d - c_1 I_n$ and $A_j = -c_{j+1} I_n$ for $j \geq 2$ with $c_j = (-1)^j \binom{\alpha}{j}$, $j = 1, 2, 3, ...$ This description can be extended to the case of non-commensurate fractional-order systems modeled in (Dzieliński and Sierociuk [2005]), (Dzieliński and Sierociuk [2006a]) by introducing the following vector difference operator:

$$\Delta^\Upsilon x(k+1) = A_d x(k) + Bu(k)$$

$$x(k+1) = \Delta^\Upsilon x(k+1) + \sum_{j=1}^{k+1} A_j x(k-j+1)$$

where:

$$\Delta^\Upsilon x(k+1) = \begin{bmatrix} \Delta^{\alpha_1} x_1(k+1) \\ \vdots \\ \Delta^{\alpha_n} x_n(k+1) \end{bmatrix}$$

Then, in the case of non commensurate-order, the system is described by Equations $(8a)$ and $(8b)$ where the matrices A_j, $j = 0, 1, 2, ...$ take the following expressions:

$$A_0 = A_d - diag\{-\binom{\alpha_i}{1}, i = 1, \dots, n\}$$

and

$$A_j = diag\{-(-1)^{j+1} \binom{\alpha_i}{j+1}, i = 1, \dots, n\}$$

for $j = 1, 2, 3, \dots$
The model described by (8) can be classi£ed as a discrete-time system with time delay in state. Whereas, the models addressed in (Debeljkovic et al. [2002]), (Pen et al [2003]), consider a £nite constant number of steps of time-delays, System (8) has a varying number of steps of time-delays, equal to k, i.e., increasing along with time. Let us de£ne matrices G_k such that:

$$G_k = \begin{cases} I_n & \text{for} \quad k = 0; \\ \sum_{j=0}^{k-1} A_j G_{k-1-j} & \text{for} \quad k \geq 1 \end{cases} \tag{9}$$

Theorem 1. (Guermah et al [2008]) The solution of Equation $(8a)$ is given by:

$$x(k) = G_k x_0 + \sum_{j=0}^{k-1} G_{k-1-j} Bu(j) \tag{10}$$

We deduce that the corresponding transition matrix can be de£ned as:

$$\Phi(k, j) = G_{k-j}, \quad \Phi(0, 0) = G_0 = I_n$$

Obviously, this transition matrix does not enjoy the semi group property as for the integer order case. In fact:

$$\Phi(k_2, k_0) \neq \Phi(k_2, k_1)\Phi(k_1, k_0); \forall \quad k_2 > k_1 > k_0 \geq 0$$

3. REACHABILITY AND OBSERVABILITY

In (Guermah et al [2008]), fundamental results concerning the reachability and observability of fractional-order systems modeled by Equations $(8a)$ and $(8b)$ are derived. In this section, we recall some of them. We begin by the reachability property.

Definition 1. The linear discrete-time fractional-order system modeled by Equations $(8a)$ and $(8b)$ is reachable if it is possible to find a control sequence such that an arbitrary state can be reached from the origin in finite time.

For the linear discrete-time fractional-order system modeled by Equations $(8a)$ and $(8b)$ we define:

(1) The controllability matrix:
$$\mathcal{C}_k = \begin{bmatrix} G_0B & G_1B & G_2B & \cdots & G_{k-1}B \end{bmatrix} \tag{11}$$
(2) The reachability Gramian:
$$W_r(0,k) = \sum_{j=0}^{k-1} G_j B B^T G_j^T \tag{12}$$
It is easy to show that $W_r(0,k) = \mathcal{C}_k\mathcal{C}_k^T$.

Theorem 2. (Guermah et al [2008]) The linear discrete-time fractional-order system modeled by Equation (9) is reachable if and only if there exists a finite time K such that: $rank(\mathcal{C}_K) = n$ or, equivalently, $rank(W_r(0,K)) = n$. Furthermore, an input sequence $\mathcal{U}_K = [u^T(K-1), u^T(K-2)\dots, u^T(0)]^T$ that transfers $x_0 = 0$ at $k = 0$ to x_f at $k = K$ is given by:
$$\mathcal{U}_K = \mathcal{C}_K^T W_r^{-1}(0,K)x_f \tag{13}$$

The same analysis is extented to the observability property.

Definition 2. The linear discrete-time fractional-order system modeled by Equations $(8a)$ and $(8b)$ is observable at time $k = 0$ if and only if there exists some $K > 0$ such that the state x_0 at time $k = 0$ can be uniquely determined from the knowledge of $u_k, y_k, k \in [0, K]$.

For the linear discrete-time fractional-order system modeled by Equations $(8a)$ and $(8b)$ we define:

(1) The observability matrix:
$$\mathcal{O}_k = \begin{bmatrix} CG_0 \\ CG_1 \\ CG_2 \\ \vdots \\ CG_{k-1} \end{bmatrix} \tag{14}$$
(2) The observability Gramian:
$$W_o(0,k) = \sum_{j=0}^{k-1} G_j^T C^T C G_j \tag{15}$$
It is easy to show that $W_o(0,k) = \mathcal{O}_k^T\mathcal{O}_k$.

Theorem 3. (Guermah et al [2008]) The linear discrete-time fractional-order system modeled by Equations $(8a)$ and $(8b)$ is observable if and only if there exists a finite time K such that: $rank(\mathcal{O}_K) = n$ or, equivalently, $rank(W_o(0,K)) = n$. Furthermore, the initial state x_0 at $k = 0$ is given by:
$$x_0 = W_o^{-1}(0,K)\mathcal{O}_K^T\left[\mathcal{Y}_K - \mathcal{M}_K\tilde{\mathcal{U}}_K\right] \tag{16}$$
with
$$\tilde{\mathcal{U}}_K = [u^T(0), u^T(1), \dots, u^T(K-1)]^T$$
$$\mathcal{Y}_K = [y^T(0), y^T(1), \dots, y^T(K-1)]^T$$
and
$$\mathcal{M}_K = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ CG_0B & 0 & \dots & 0 & 0 \\ CG_1B & 0 & \dots & 0 & 0 \\ CG_2B & CG_0B & \dots & 0 & 0 \\ \vdots & \dots & \vdots & \vdots & \\ CG_{k-2}B & CG_{k-4}B & \dots & CG_0B & 0 \end{bmatrix} \tag{17}$$

Remark 1. In the case of an integer order, it is well known that the rank of the controllability matrix $\mathcal{C}_k$ and the rank of the observability matrix $\mathcal{O}_k$ cannot increase for $k \geq n$. This in virtue of the Cayley-Hamilton theorem. In contrast, in the case of the linear discrete-time non-commensurate fractional-order system (8), the rank of $\mathcal{C}_k$ and $\mathcal{O}_k$ can increase for values of $k \geq n$. In other words, a final state x_f which can not be reached in n steps can be reached in a number of steps greater than n. Furthermore, n samples of input/output data may be non sufficient to detect an initial state x_0. This initial state may be observable from a number of input/output samples greater than n. This is due to the nature of the elements G_k which build the controllability matrix $\mathcal{C}_k$ and the observability matrix $\mathcal{O}_k$ and which exhibit the particularity of being time-varying, in the sense that they are composed of a number of terms A_j that grows with k.

Remark 2. In the particular case of commensurate fractional-order systems, the matrices G_k defined by Equation (9) are polynomials in A_0, that is:
$$G_k = A_0^k + \beta_{1_k}A_0^{k-1} + \beta_{2_k}A_0^{k-2} + \ldots + \beta_{k_k}I_n$$
where the real coefficients β_{j_k} are calculated from the coefficients c_j. In particular, we have:
$$G_n = A_0^n + \beta_{1_n}A_0^{n-1} + \beta_{2_n}A_0^{n-2} + \ldots + \beta_{n_n}I_n$$
From the Cayley-Hamilton theorem, A_0^n is a linear combination of $A_0^{n-1}, A_0^{n-2}, \ldots, I_n$. We deduce that G_{k+n}, for all $k \geq 0$ are linearly dependent on G_{n-1}, G_{n-2}, …, I_n. This implies the linear discrete-time fractional-order system modeled by Equations $(8a)$ and $(8b)$ in the commensurate case is reachable if and only if $rank(\mathcal{C}_n) = n$ and is observable if and only if $rank(\mathcal{O}_n) = n$. The controllability and the observability criteria of commensurate fractional-order systems are then similar to those of the integer-order case.

4. REACHABILITY AND OBSERVABILITY REALIZATION INDICES

In (Pen et al [2003]), the concept of controllability (or reachability) index is introduced. This concept is used for determining the controllability of discrete-time linear systems with time-delay in state. We extend these concepts to fractional order systems.

Definition 3. (Reachability Realization Index, *RRI*) For System (8), if there exists a positive integer K_r such that for any initial state $x(0) = x_0$, and any final state x_f, there exists input sequence $[u(K_r-1), u(K_r-2), \dots, u(0)]$ such that $x(K_r) = x_f$, then we call K_r the Reachability Realization Index (*RRI*) for System (8).

If System (8) is reachable, K_r is not unique since if $rank(\mathcal{C}_{K_r}) = n$ then $rank(\mathcal{C}_k) = n$ for all $k \geq K_r$. The smallest K_r is called the Minimal Reachability Realization Index and is denoted by *MinRRI*. We extend these definitions to the observability.

Definition 4. (Observability Realization Index, *ORI*) For System (8), if there exists a positive integer K_o such that for any initial state $x(0) = x_0$, the knowledge of the input and output sequences $[u(K_o-1), u(K_o-2), \dots, u(0)]$ and $[y(K_o-1), y(K_o-2), \dots, y(0)]$ is sufficient to determine the initial state x_0, then we call K_o the Observability Realization Index (*ORI*) for System (8).

If System (8) is observable, K_o is not unique since if $rank(\mathcal{O}_{K_o}) = n$ then $rank(\mathcal{O}_k) = n$ for all $k \geq K_o$. The smallest K_o is called the Minimal Observability Realization Index and is denoted by *MinORI*.

Definition 5. (Structure Realization Index, *SRI*)
The Structure Realization Index (*SRI*) is the integer r defined as $r = max(K_r, K_o)$. The Minimal Structure Realization Index (*MinSRI*) is the integer r_{min} defined as $r_{min} = max(MinRRI, MinORI)$.

In the case of integer order systems with p inputs and q outputs described by its transfer function matrix $\mathcal{H}(z) \in \mathbb{C}^{q\times p}$, the n-dimensional realization (A, B, C, D) where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$, $C \in \mathbb{R}^{q\times n}$ and $D \in \mathbb{R}^{q\times p}$ is minimal if it is controllable and observable. Let the expansion of $\mathcal{H}(z)$ in Laurent series be

$$\mathcal{H}(z) = H_o + H_1 z^{-1} + H_2 z^{-2} + \ldots$$

where the terms H_i, $i = 0, 1, 2, \ldots$ are the Markov parameters determined by the formulas:

$$H_0 = \lim_{z\to\infty} \mathcal{H}(z)$$
$$H_1 = \lim_{z\to\infty} z(\mathcal{H}(z) - H_0)$$
$$H_2 = \lim_{z\to\infty} z^2(\mathcal{H}(z) - H_0 - H_1 z^{-1})$$

and so forth.
It is well known (Antsaklis and Michel [1997]) that (A, B, C, D) is a realization of $\mathcal{H}(z)$ if and only if

$$H_0 = D$$
$$H_i = CA^{i-1}B; \quad i \geq 1$$

This realization of dimension n is minimal if the rank of the Hankel matrix

$$M_{\mathcal{H}}(n, n) = \begin{bmatrix} H_1 & H_2 & \ldots & H_n \\ H_2 & H_3 & \ldots & H_{n+1} \\ H_3 & H_4 & \ldots & H_{n+2} \\ \vdots & \vdots & \ldots & \vdots \\ \vdots & \vdots & \ldots & \vdots \\ H_n & H_{n+1} & \ldots & H_{2n-1} \end{bmatrix}$$

is equal to n. Let us return now to the case of the fractional-order system (8) which exhibits an infinite structure since the state space representation is composed by infinite number of state matrices namely A_j, $j \geq 0$ reflecting the long memory characteristic. The corresponding transfer function matrix $\mathcal{H}(z)$ is called infinite dimensional transfer function matrix. It is easy to show that the pulse response matrix of (8) due to a pulse input applied at time $j = 0$ is given by

$$\mathbf{H}(0) = D \tag{18a}$$

and

$$\mathbf{H}(k) = CG_{k-1}B; \quad k \geq 1 \tag{18b}$$

Then the transfer function matrix $\mathcal{H}(z)$ of (8) possesses the following expansion:

$$\mathcal{H}(z) = H_0 + H_1 z^{-1} + \ldots + H_k z^{-k} + \ldots \tag{19}$$

with

$$H_0 = D \tag{20a}$$
$$H_k = CG_{k-1}B; \quad k \geq 1 \tag{20b}$$

We shall call a finite dimensional structure state space representation if the number of the state matrices is finite. This representation is given by

$$x(k+1) = \sum_{j=0}^{k} A_j x(k-j) + Bu(k); k \leq N \tag{21a}$$

$$x(k+1) = \sum_{j=0}^{N} A_j x(k-j) + Bu(k); k > N \tag{21b}$$

$$y(k) = Cx(k) + Du(k) \tag{21c}$$

N represents the structure dimension. This representation considers only a finite (short) memory.

Definition 6. A state space representation with finite dimensional structure $(A_0, A_1, A_2, \ldots, A_N, B, C, D)$ where $A_j \in \mathbb{R}^{n\times n}, j = 0, 1, 2, \ldots, N$, $B \in \mathbb{R}^{n\times p}$, $C \in \mathbb{R}^{q\times n}$ and $D \in \mathbb{R}^{q\times p}$ is a finite dimensional structure realization of a given infinite dimensional transfer function matrix $\mathcal{H}(z) \in \mathbb{C}^{q\times p}$ if the terms of the Laurent series of $\mathcal{H}(z)$ satisfies the following relations

$$H_0 = D$$
$$H_k = CG_{k-1}B; k = 1, 2, \ldots, N$$

and if for any given output vector with desired fixed value y_f, there exists an initial condition x_0 of $x(k)$ and an input sequence $[u(N-1), u(N-2), \ldots, u(0)]$ which produce this output in a finite time interval $[0 \quad N]$. This finite dimensional structure realization is minimal if it is reachable and observable.

From the above, we can state the following result.

Theorem 4. Consider the fractional system (8) with the infinite dimensional structure $[(A_j; j \geq 0), B, C, D]$. Assume that (8) is both reachable and observable and let r_{min} be its Minimal Structure Realization Index, then the finite dimensional state space representation $[(A_0, A_1, \ldots, A_N),\ B, C, D]$ given by Equations $(21a)$, $(21b)$ and $(21c)$ where $N = r_{min} - 2$ is a finite dimensional minimal structure realization of (8).

5. NUMERICAL EXAMPLE

Let us consider the following discrete-time non-commensurate fractional-order of dimension $n = 4$, with:
$\alpha_1 = 0.2 \quad ; \alpha_2 = 0.3 \quad \alpha_3 = 0.6 \quad \alpha_4 = 0.7$ and

$$A_d = \begin{bmatrix} -0.2 & 0 & 0 & 0 \\ 0 & -0.3 & 0 & 0 \\ 0 & 0 & -0.6 & 0 \\ 0 & 0 & 0 & -0.7 \end{bmatrix}; B = [\,1\ 1\ 1\ 1\,]$$

$$C = [\,1\ 1\ 1\ 1\,]; D = 0$$

For the controllability analysis, we have achieved the determination of $rank(\mathcal{C}_k)$ over a set of $N = 20$ samples. We have found that $rank(\mathcal{C}_k) = 4$ at $K = 5$. We chose the final state equal to:

$$x_f = [\,1 \quad -0.5 \quad 3 \quad 0.3\,]$$

The input sequence that permitted to transfer the state from the origin $x_0 = [\,0 \quad 0 \quad 0 \quad 0\,]$ to x_f according to Equation (13) is :

$$\mathcal{U}_k = [\,1 \quad -0.5 \quad 3 \quad 0.3\,]$$

The objective has been reached with a sequence of input data greater than the system order, which comes up to be a particularity of discrete non-commensurate fractional-order systems. This is not verifed in the case of discrete commensurate fractional-order systems for which the full rank, n, if it can be reached, cannot be reached beyond a number of steps $K = n$. For the observability analysis, we have achieved the determination of $rank(\mathcal{O}_k)$ over a set of $N = 20$ samples. We have found that $rank(\mathcal{O}_k) = 4$ at $K = 5$. We chose the following input and output sequences over 5 steps:

$$\tilde{\mathcal{U}}_K = [\,1 \quad -0.5 \quad 3 \quad 0.3\,]$$
$$\mathcal{Y}_K = [\,1 \quad -0.5 \quad 3 \quad 0.3\,];$$

According to Equation (16), the initial state x_0 is detected :

$$x_0 = [\,1 \quad -0.5 \quad 3 \quad 0.3\,]$$

From the above, it follows that the reachability realization index and the observability index are $RRI = K = 5$ and $ORI = K = 5$, respectively. Hence, the minimal structure realization index is $r_{min} = K = 5$. Then the £nite dimensional structure representation $(A_0, A_1, A_2, A_3, B, C, D)$ is a minimal structure realization of the fractional order system. In fact, we can reach any £nal state position from the origin and we can detect any initial state from a given output/input data by considering only this minimal structure realization instead of the in£nite dimensional structure $(A_j; j \geq 0, B, C, D)$.

6. CONCLUSION

In this paper some new results concerning the analysis of reachability and observability of discrete-time fractional order system are given. The concepts of reachability, observability and structure realization indices are introduced. The preliminary results developed here can be useful for further investigation concerning control and £ltering of fractional order discrete-time systems.

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