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Robust control for uncertain linear systems with State and Control Constraints ⋆

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Abstract: The paper is devoted to the stabilization of linear uncertain systems having restricted states and controls. Two classes of uncertainties, namely norm bounded and polytopic, are studied for continuous time systems. Sufficient LMI conditions are given for the derivation of robust state-feedback controllers driving the system asymptotically to the origin without violating the constraints. Further, the determination of a large region of attraction for these systems is addressed. Moreover, numerical algorithms are provided for the enlargement of the volume of attraction region of the uncertain system. The approach is illustrated by an example for each case of uncertainties.

1. INTRODUCTION

Most real systems cannot be represented by linear dynamics. But, under some assumptions it is often possible to model the dynamical behavior of practical systems with a linear model having some uncertainties. These uncertainties are generally induced from the difference, sometimes considerable, between the real behavior of the system and the model used to design the controller. In the last two decades, several approaches have been proposed for designing robust controllers and many progress have been accomplished to take into account the uncertainties. Hence, several techniques have been used and included in the robust control theory. See Dahleh and Diaz-Bobiollo (1995); Garcia et al. (1994), Geromel et al. (1991) and the references therein. These methods can be divided into two classes : the first one studies robust controllers design and the second one analysis given controllers robustness.

On the other hand, constraints are inherent to any kind of physical, chemical or real process, Berstein and Michel (1995). These constraints may arise from physical limitations on the process, as approximations to obtain linear models, or may depend on its nature. Henceforth, constrained systems are connected to a wild spread of applications and their study is of continuing interest in control community. Numerous approaches have been proposed for linear systems involving constraints, see for example (without been exhaustive) for the positive invariance concept; see the overview of Blanchini (1999), Mesquine et al. (2004a); Benzaouia et al. (2006), Pitet et al. (1997) and the small and high gain concept Lin and Saberi (1995) and the l_1 concept Dahleh and Diaz-Bobiollo (1995).

Other general methods have been derived by applying the absolute stability analysis tools, such as the Circle and Popov Criteria, where the saturation is treated as a locally sector bounded nonlinearity and the domain of attraction is estimated by using quadratic and Lur'e type Lyapunov functions Hindi and Boyd (1998), Pittet et al. (1997).

Many works have studied constrained systems based on the positive invariance concept since the early result of Gutman and Hagander, (1985). The positive invariance concept seems to be appropriate for solving theoretically and numerically synthesis problems for linear systems subject to linear constraints.

Since the work of Gilbert and Tan (1991), many works focused on the characterization of the maximal set of attraction for constrained linear systems. Recently, new trends have emerged based on writing the saturation function as linear convex combination of some constrained variables Hu and Lin (2000), Hu et al. (2002). This approach, leads to sufficient conditions in terms of a huge (exponential) number of LMIs which are somewhat difficult to solve numerically, since the current available

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LMI softwares cannot afford the treatment of very large size problems.

Most of the realistic control problems involve both some type of domain constraints and model uncertainty. The simultaneous presence of uncertainties and constraints in physical systems, pushed many authors to combine the techniques of the robust control theory and those of constrained control Benzaouia and Mesquine (1994), Milani and Carvalho (1995), Mesquine et al. (2004b). Having in mind the same goal, this work is an extension to the case of uncertain systems of constrained control and state technique presented in Ait rami et al. (2007). A method is presented for designing a state feedback linear control law that will ensure the robust stability of a given linear uncertain system. Both structured and non structured uncertainties are studied in this work to show the generality of the approach. Sufficient LMI conditions are given for the derivation of robust state-feedback controllers driving the system asymptotically to the origin without violating the constraints. As the challenge in constrained state and/or control is to obtain as a large as possible regions of attraction, the determination of a large ellipsoidal set of attraction is also studied. An algorithmic LMI-based procedure is proposed. The convergence of such algorithm is also proven. The remainder of this paper is organized as follows. Section 2 states the framework of the synthesis problem for continuous-time norm bounded or polytopic uncertain systems subject to state and control constraints. After some preliminaries and some key lemmas presented in section 3, the robust stabilizing control problem is solved in section 4. In section 5 a numerical algorithm is provided to enlarge the domain of attraction together with the proof of its convergence. Examples are studied in section 6 to illustrate the application of our method. Finally, Section 7 gives some concluding remarks.

2. PROBLEM FORMULATION

Consider the following uncertain system given by:

$$\dot{x}(t) = A_r\, x(t) + B_r\, u(t), \tag{1}$$

Where $x(t) \in I\!R^n$ represents the system state belonging to the following set:

$$\mathcal{L}(F, v) = \{x \in I\!R^n \;/\; Fx \leq v\}, \tag{2}$$

where matrix $F \in I\!R^{q\times n}$ and vector $v \in I\!R^q$ are given. The control $u(t) \in I\!R^m$ is constrained to evolve in the following set:

$$\Gamma = \{u \in I\!R^m \quad / \quad |u_i| \;\leq 1, i = 1, \ldots, m\}. \tag{3}$$

The aim of the paper is to show how to find stabilizing robust state-feedback controllers $u(t) = Kx(t)$ and a set of initial conditions for which $u(t) \in \Gamma$ and $x(t) \in \mathcal{L}(F, v)$ for all $t > 0$ for two classes of uncertainties:

-Structured Norm Bounded Uncertainty

$$A_r = A + D\triangle(t)E_1 \;\text{ and }\; B_r = B + D\triangle(t)E_2 \tag{4}$$

where $\Delta(t) \in \Im = \left\{\triangle(t) \in I\!R^{d\times e} : \triangle(t)^T\triangle(t) \leq I\!I_e\right\}$ (5)

the matrices $D \in I\!R^{n\times d}$, $E_1 \in I\!R^{e\times n}$ and $E_2 \in I\!R^{e\times m}$ are given matrices that specify how the nominal state matrix A and input matrix B are affected by the uncertainty. Matrices $\triangle$(t) and $I\!I_e (\in I\!R^{e\times e})$ represent respectively the uncertain parameter and the identity matrix.

-Polytopic Uncertainty

Matrices A_r and B_r belong to the convex-bounded domains defined as

$$\{A_r \in I\!R^{n\times n}/A_r = \sum_{i=1}^{N}\lambda_i A_i, \sum_{i=1}^{N}\lambda_i = 1, \lambda_i \geq 0\}, \tag{6}$$

$$\{B_r \in I\!R^{n\times m}/B_r = \sum_{j=1}^{M}\theta_j B_j, \sum_{j=1}^{M}\theta_j = 1, \theta_j \geq 0\}. \tag{7}$$

Moreover, the robust state-feedback control must guarantee the asymptotic stability of the closed-loop system and respects the constraints for as large as possible domain of initial conditions. This problem of enlargement is addressed using an LMI based algorithm.
For notational convenience > 0 (respectively < 0) means positive definite (respectively negative definite) for matrices and component wise inequality for vectors.

3. PRELIMINARIES

This section recalls first asymptotic stability conditions for norm bounded and polytopic uncertain systems together with the definition of positive invariance. Second, some useful key lemmas which provide necessary and sufficient conditions for the inclusion of an ellipsoidal set into polyhedral sets are presented.

Theorem 1. Garcia, et al. (1994) The uncertain system given by (1),(4),(5) without control and state constraints, with state feedback $u(t) = Kx(t)$ is quadraticaly stable if there exists a matrix $P \in I\!R^{n\times n}$, $P = P^T > 0$ and a scalar $\epsilon > 0$ such that:

$$P\,A \;+\; A^T\,P \;+ PBK + K^TB^TP + \epsilon PDD^TP + \\ +\epsilon^{-1}(E_1^T + K^TE_2^T)(E_1 + E_2K) \leq 0. \tag{8}$$

Theorem 2. Bernussou, et al. (1989) The uncertain system (1),(6),(7) without control and state constraints, with state feedback $u(t) = Kx(t)$ is asymptotically stable if there exists a matrix $P \in I\!R^{n\times n}$, $P = P^T > 0$ such that

$$(\,A_k \;+\; B_l\,K\,)^TP + P\,(\,A_k \;+\; B_l\,K\,) \;<\; 0 \tag{9}$$
$$\forall \quad k = 1, \ldots, N, \;\forall \quad l \;=\; 1, \ldots, M.$$

For both cases of uncertain systems satisfying conditions (8) or (9), the function $v(x) = x^TPx$ is a Lyapunov function for the closed-loop system. Further, the Lyapunov level set

$$\Omega(P) = \left\{x \in I\!R^n \quad / \quad x^TPx \leq 1\right\}, \tag{10}$$

is positively invariant with respect to (w.r.t.) motion of the closed loop system in the sense of the following definition:

Definition 3. A subset $S \subset I\!R^n$ is said to be positively invariant w.r.t. motion of system (1) if for every initial state $x(0)$ inside S the trajectory $x(t)$ remains in S for all $t > 0$.

Now, let K_i denotes the i^{th} row of K and define:

$$\mathcal{L}(K) = \{x \in I\!R^n \ / \ |K_i x| \leq 1, i = 1, \ldots, m\}. \quad (11)$$

With regard to the preceding considerations, if $\Omega(P) \subset \mathcal{L}(K) \bigcap \mathcal{L}(F, v)$ and P satisfies the Lyapunov conditions (8) or (9), then for any initial condition inside the ellipsoid $\Omega(P)$ we have $u(t) \in \Gamma$ and $x(t) \in \mathcal{L}(F, v)$ for for all $t > 0$. Now, we are in position to establish the following.

Lemma 4. Ait rami, et al. (2007) The inclusion $\Omega(P) \subset \mathcal{L}(K)$ is equivalent to:

$$\begin{bmatrix} 1 & Y_i \\ Y_i^T & Q \end{bmatrix} \geq 0, \quad i = 1, \ldots, m, \quad (12)$$

where $Q = P^{-1}$ and $Y_i = K_i Q$ are the rows of the matrix Y.

Proof. Rewrite $x \in \Omega(Q^{-1})$ as $p(x) = x^T Q^{-1} x - 1 \leq 0$ and $x \in \mathcal{L}(K)$ as $q_i(x) = x^T K_i^T K_i x - 1 \leq 0$ for $i = 1, \ldots, m$. Since the condition $\Omega(P) \subset \mathcal{L}(K)$ is nothing than the implication $p(x) \leq 0 \Rightarrow q_i(x) \leq 0$, then by using the S-procedure lemma, this condition is equivalent to the existence of $\alpha_i > 0$ such that $x^T K_i^T K_i x - 1 \leq \alpha_i (x^T \ Q^{-1} \ x - 1)$ for $i = 1, \ldots, m$. Now, taking any arbitrary scalar β we have $\beta x^T K_i^T K_i x \beta - \beta^2 \leq \alpha_i \beta x^T \ P \ x \beta - \alpha_i \beta^2$. By making the change of variable $\tilde{x} = \beta x$, we obtain $\tilde{x}^T K_i^T K_i \tilde{x} - \beta^2 \leq \alpha_i \tilde{x}^T \ P \ \tilde{x} - \beta^2 \alpha_i$, or equivalently, we have for $i = 1, \ldots, m$:

$$\begin{bmatrix} \tilde{x} \\ \beta \end{bmatrix}^T \begin{bmatrix} K_i^T K_i - \alpha_i Q^{-1} & 0 \\ 0 & -1 + \alpha_i \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \beta \end{bmatrix} \leq 0.$$

The above inequality reduces to:

$$K_i^T K_i \ \leq \alpha_i Q^{-1},$$
$$\alpha_i \leq 1,$$

which, equivalently, leads to $K_i^T K_i \ \leq Q^{-1}$.
Let $Y_i = K_i Q$ and multiply by Q the left and right seeks of $K_i^T K_i \ \leq Q^{-1}$, we obtain $Y_i^T Y_i \ \leq Q$. Finally, by using Schur lemma, we have that

$$\begin{bmatrix} 1 & Y_i \\ Y_i^T & Q \end{bmatrix} \geq 0,$$

is equivalent to $\Omega(P) \subset \mathcal{L}(K)$ ■

It is worth to note that lemma above is well known as a sufficient condition given by Boyd et al. (1994). Here we proove that this condition is also necessary.

Lemma 5. Ait rami, et al. (2007) The inclusion $\Omega(P) \subset \mathcal{L}(F, v)$ is equivalent to the existence of positive scalars $\alpha_1 > 0, \ldots, \alpha_q > 0$ such that:

$$\begin{bmatrix} 4\alpha_i v_i - F_i Q F_i^T & 2\alpha_i \\ 2\alpha_i & 1 \end{bmatrix} \geq 0, \text{for } i = 1, \ldots, q, \quad (13)$$

where $Q = P^{-1}$ and F_i is the i^{th} row of F.

Proof. Using Schur lemma the LMI (13) is equivalent to:$-4 \ \alpha_i^2 + 4 \ \alpha_i \ v_i \ - \ F_i \ Q \ F_i^T \geq 0$, which by using again Schur lemma is equivalent to

$$\begin{bmatrix} -\alpha_i + \ v_i & -\frac{1}{2} F_i \\ -\frac{1}{2} F_i^T & \alpha_i Q^{-1} \end{bmatrix} \geq 0,$$

so that for any vector $[z \ \tilde{x}]^T \in I\!R \times I\!R^n$, we have

$$\begin{bmatrix} z \\ \tilde{x} \end{bmatrix}^T \begin{bmatrix} -\alpha_i \ + \ v_i & -\frac{1}{2} F_i \\ -\frac{1}{2} F_i^T & \alpha_i Q^{-1} \end{bmatrix} \begin{bmatrix} z \\ \tilde{x} \end{bmatrix} \geq 0,$$

by developing the above inequality, we obtain

$$\frac{z}{2} F_i \tilde{x} + \frac{z}{2} \tilde{x}^T F_i^T - v_i z^2 \leq \alpha_i \tilde{x}^T P \tilde{x} - \alpha_i z^2,$$

now, by making the change of variable $\tilde{x} \ = z \ x$,$Q = P^{-1}$ we have

$$\frac{1}{2} F_i x + \frac{1}{2} x^T F_i^T - v_i \leq \alpha_i x^T P x - \alpha_i,$$

or equivalently, we get

$$F_i x - \ v_i \leq \alpha_i (x^T \ P \ x \ - \ 1) \ \text{ for all } x \in I\!R^n,$$

then by using the S-procedure lemma the proof is complete ■

4. MAIN RESULTS

This section shows and demonstrates how the robust stabilizing state-feedback control laws, that respect state and control constraints, can be synthesized. Sufficient conditions are given for both norm bounded uncertainties and polytopic case. It is shown that our conditions can be formulated in terms of LMI's. Moreover, the proposed LMI's formulation enables us to compute adequately and efficiently the robust stabilizing state-feedback control law. In addition, an LMI-based algorithm is given to enlarge the volume of the invariant ellipsoid.

Theorem 6. The control law $u(t) = Kx(t)$ is robustly stabilizing for the constrained uncertain system (1),(4),(5) for any initial state in $\Omega(P)$ and respects constraints (2) - (3) if there exist matrices $Q = Q^T > 0$, $Y \in I\!R^{m \times n}$, and scalars $\epsilon > 0$, $\alpha_1 > 0, \ldots, \alpha_q > 0$ such that the following LMI's hold true:

$$\begin{bmatrix} AQ + QA^T + BY + Y^T B^T + \epsilon D D^T & QE_1^T + Y^T E_2^T \\ E_1 Q + E_2 Y & \epsilon I\!I_e \end{bmatrix} \leq 0, \quad (14)$$

$$\begin{bmatrix} 4\alpha_i v_i - F_i Q F_i^T & 2\alpha_i \\ 2\alpha_i & 1 \end{bmatrix} \geq 0, \quad \text{for } 1 \leq i \leq q, \quad (15)$$

$$\begin{bmatrix} 1 & Y_j \\ Y_j^T & Q \end{bmatrix} \geq 0, \quad \text{for } 1 \leq j \leq m, \quad (16)$$

where $P = Q^{-1}$, and $K = YQ^{-1}$.

Proof.
The LMI's (15) and (16) guarantee the inclusion of the ellipsoidal set $\Omega(P)$ in the intersection of $\mathcal{L}(F,v)\bigcap\mathcal{L}(K)$. According to Theorem (1), LMI (14) guarantees the quadratic stability of the closed loop uncertain system. Further, the ellipsoid $\Omega(P)$ is positively invariant w.r.t. the motion of the closed loop uncertain system. Hence, the linear behavior of the closed loop is always valid. It is then clear that once initialized in the set $\Omega(P)$ the motion does not leave it and consequently all constraints are respected for all $t > 0$
■

Theorem 7. The control law $u(t) = Kx(t)$ is robustly stabilizing for the constrained uncertain system (1),(6),(7) for any initial state in $\Omega(P)$ and respects constraints (2),(3) if there exist matrices $Y \in I\!R^{m\times n}$, $Q = Q^T > 0$ and scalars $\alpha_1 > 0, \ldots, \alpha_q > 0$, such that the following LMI's hold true:

$$QA_k^T + A_kQ + B_lY + Y^TB_l^T < 0, \quad \text{for } 1 \le k \le N, \text{ and for } 1 \le l \le M, \quad (17)$$

$$\begin{bmatrix} 4\alpha_i v_i - F_iQF_i^T & 2\alpha_i \\ 2\alpha_i & 1 \end{bmatrix} \ge 0, \quad \text{for } 1 \le i \le q, \quad (18)$$

$$\begin{bmatrix} 1 & Y_j \\ Y_j^T & Q \end{bmatrix} \ge 0, \quad \text{for } 1 \le j \le m, \quad (19)$$

where $P = Q^{-1}$ and $K = YQ^{-1}$.

Proof. The proof follows the same lines of argument of theorem 6 ■
With regard to the above result a relevant remark.

Remark 8. Let $u = Kx$ be any stabilizing state-feedback control, then the condition:
$\mathcal{L}(K)\bigcap\mathcal{L}(F,v)$ has a nonempty interior and contains the origin, is necessary and sufficient for the existence of an invariant ellipsoid inside $\mathcal{L}(K)\bigcap\mathcal{L}(F,v)$. If the LMI's:(14)-(15)-(16) or (17)-(18)-(19) are non feasible, this simply means that such condition does not hold for all stabilizing state-feedback controls.

5. ENLARGEMENT OF THE DOMAIN OF ATTRACTION

In the sequel, our goal is to find a robust state-feedback control law associated to a large invariant ellipsoid (w.r.t.) the state and control constraints (2)-(3). For this purpose, let us introduce the following theorem:

Theorem 9. The largest invariant ellipsod of the system (1),(4),(5) with state and control constraints (2),(3) is the ellipsoid $\Omega(P)$ for which P is solution to the following optimization problem:

$$\textbf{Maximize} \quad \textbf{Log}(\textbf{Det}(P^{-1}))$$
$$P > 0, Q > 0, \epsilon > 0, Y, \alpha_i$$
subject to:
relation (15), relation (16),

$$\begin{bmatrix} AQ + QA^T + BY + Y^TB^T + \epsilon DD^T & QE_1^T + Y^TE_2^T \\ E_1Q + E_2Y & \epsilon I\!I_e \end{bmatrix} \le 0, \quad (20)$$

$$\begin{bmatrix} P & I\!I \\ I\!I & Q \end{bmatrix} \ge 0. \quad (21)$$

Proof. The proof is based on the algorithm and lemmas given below. ■
A numerical procedure will be provided for the enlargement of the volume of the invariant ellipsoid. Let us introduce the following LMI-based algorithm:

Algorithm 5.1. Fix an accuracy δ (for example $\delta = 10^{-6}$).

- Step 0: set $P_i = I\!I$.
- Step 1: Find $P_{i+1}, Q_{i+1}, Y_{i+1}$ solution to:
$$\min \textbf{Tr}\,(P_i^{-1}\,P)$$
$$\textit{subject to:} \begin{cases} \text{relation (15), relation (16),} \\ \text{relation (20), relation (21).} \end{cases}$$
- Step 2: If $Det(P_{i+1}^{-1}) - Det(P_i^{-1}) < \delta$ then stop and compute $K = Y_{i+1}P_{i+1}$. Otherwise set $P_i \leftarrow P_{i+1}$ and go to step 1.

It will be shown that Algorithm 5.1 generates a sequence of invariant ellipsoids with increasing volume. To prove our claim, we will need the following well-known lemmas Horn and Johnson (1985), Boyd et al. (1994).

Lemma 10. Let $f(\cdot)$ be a concave real function, then for any variables x, y we have
$$f(x) - f(y) \le f'(y)(x - y).$$

Lemma 11. For positive definite matrix $P > 0$ the volume of the ellipsoid $\Omega(P)$ is proportional to $Det(P^{-1})$.

Lemma 12. Let be given any symmetric matrices R, S such that $R \ge S$; then if $\textbf{Tr}(R) = \textbf{Tr}(S)$ we have $R = S$.

Now we are in position to state the following result.

Theorem 13. Algorithm 5.1 generates a sequence of positive definite matrices P_i such that the associated ellipsoids $\Omega(P_i)$ have increasing volumes. Moreover, we have $P_i = Q_i^{-1}$ for every i and $\Omega(P_i)$ is an invariant ellipsoid for the state-feedback $u = Y_iP_ix$ with respect to the state and control constraints (2)-(3).

Proof. The function $F(P) = -log(Det(P^{-1}))$ is a concave function and its derivative at any point P_i is given by $F'(P_i)(P) = \textbf{Tr}(P_i^{-1}\,P)$. Then by using Lemma 10 we obtain:

$$-log(Det(P_{i+1}^{-1})) + log(Det(P_i^{-1})) \le \textbf{Tr}(P_i^{-1}(P_{i+1} - P_i)). \quad (22)$$

Since at Step 1 of Algorithm 5.1 we minimize the objective function $\textbf{Tr}(P_i^{-1}\,P)$ and P_{i+1} is its optimal solution, we necessarily have $\mathbf{Tr}(P_i^{-1}(P_{i+1} - P_i)) \leq 0$. So that inequality (22) implies

$$log(Det(P_i^{-1})) \leq log(Det(P_{i+1}^{-1})),$$

or equivalently $Det(P_i^{-1}) \leq Det(P_{i+1}^{-1})$. Note that by using Lemma 11 the volume of $\Omega(P)$ is proportional to $Det(P^{-1})$ then the above inequality shows that the volume of $\Omega(P_{i+1})$ is bigger than the volume of $\Omega(P_i)$.
Now let us prove $P_i = Q_i^{-1}$: By Schur lemma the LMI (21) in Algorithm 5.1 is equivalent to $P_i \geq Q_i^{-1}$. Then taking into account that $P_i \geq Q_i^{-1}$ and since we minimize the objective function $\mathbf{Tr}(P_{i-1}^{-1}P)$ (which is increasing in function of P) we must have at the optimum:

$$\mathbf{Tr}(P_{i-1}^{-1}P_i) = \mathbf{Tr}(P_{i-1}^{-1}Q_i^{-1}),$$

using the fact that $P_{i-1}^{1/2}P_iP_{i-1}^{1/2} \geq P_{i-1}^{1/2}Q_i^{-1}P_{i-1}^{1/2}$ and Lemma 12 we get $P_i = Q_i^{-1}$ and the rest of the proof is straightforward. ■

Corollary 14. Theorem 9 and algorithm 5.1 hold true by replacing the LMI (20) by the LMI (17) for polytopic uncertain systems

6. EXAMPLES

6.1 Structured Norm Bounded Uncertainty

Consider the following uncertain continuous-time system given in Garcia et al. (1994):

$$\dot{x}(t) = (A + D\Delta(t)E_1)x(t) + (B + D\Delta(t)E_2)u(t),$$

$$\text{where } A = \begin{bmatrix} -0.82 & 17.76 & 90.24 \\ 0.17 & -0.75 & -11.10 \\ 0.00 & 0.00 & -250.00 \end{bmatrix}, \quad B = \begin{bmatrix} -91.24 \\ 0.00 \\ 250.00 \end{bmatrix},$$

The structure of uncertainty is given by:

$$D = \begin{bmatrix} 0.80 \\ -0.25 \\ 0.00 \end{bmatrix}, E_1 = [\, 0.10 \ 0.10 \ 1.00 \,],$$

$$E_2 = 20.00, -1 \leq \Delta(t) \leq 1.$$

where the control law must fulfill: $|u| \leq 1$ and the state constraint set given by

$$F = \begin{bmatrix} 8.00 & 20.00 & 15.00 \\ -4.00 & 10.00 & 5.00 \\ 0.00 & 0.00 & 1.00 \\ -2.00 & -3.00 & 1.00 \end{bmatrix}, \quad v = \begin{bmatrix} 120.00 \\ 60.00 \\ -15.00 \\ 60.00 \end{bmatrix}.$$

By applying Theorem 6 and Algorithm 5.1, we have obtained the following robust state-feedback gain and its corresponding invariant ellipsoid $\Omega(P)$:

$$K = \begin{bmatrix} 0.0942 \\ 0.2371 \\ 0.0259 \end{bmatrix}, P = \begin{bmatrix} 0.0122 & 0.0089 & -0.0001 \\ 0.0089 & 0.1213 & -0.0017 \\ -0.0001 & -0.0017 & 0.0315 \end{bmatrix},$$

We obtain $\epsilon = 353.58$.

Figure 1 shows the system motion from initial condition $xo = [1 \ \ 1.8 \ \ 4]^T$, together with control evolution. It can be seen that robust feedback is achieved with respected state and control constraints.

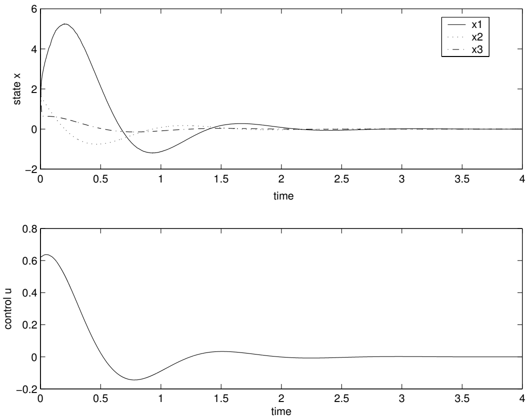

Figure 1 : Time-evolution of state and control action

6.2 Polytopic Uncertainty

Consider the following uncertain system Milani and Carvalho (1994):

$$\dot{x} = (\lambda A_1 + (1-\lambda)A_2)x(t) + (\theta B_1 + (1-\theta)B_2)u(t)$$

$$A_1 = \begin{bmatrix} 9.10 & 0.47 & -6.33 \\ 7.62 & 0.00 & 7.56 \\ 2.37 & -3.53 & 9.66 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 9.10 & 0.47 & -6.53 \\ 7.62 & 0.00 & 7.56 \\ 2.87 & -3.03 & 10.16 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 25.48 & 72.40 \\ 17.36 & -75.4 \\ 71.54 & 0.00 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 25.48 & 72.40 \\ 17.36 & -75.4 \\ 65.94 & 0.00 \end{bmatrix},$$

where the control law must fulfill: $|u| \leq 1$ and the state constraint set given by

$$F = \begin{bmatrix} 5.69 & 1.97 & -1.68 \\ 2.24 & -1.68 & 0.59 \\ 0.00 & 0.00 & 1.00 \end{bmatrix}, \quad v = \begin{bmatrix} 8.75 \\ 10.50 \\ 1.00 \end{bmatrix}.$$

By applying Theorem 7 and Algorithm 5.1, we have obtained the following robust state-feedback gain:

$$K = \begin{bmatrix} -0.3108 & -0.0713 & 0.3546 \\ -0.0337 & 0.1294 & -1.1322 \end{bmatrix},$$

and its corresponding invariant ellipsoid $\Omega(P)$, with

$$P = \begin{bmatrix} 0.4679 & 0.1124 & -0.0124 \\ 0.1124 & 0.0764 & -0.1284 \\ -0.0124 & -0.1284 & 1.3185 \end{bmatrix},$$

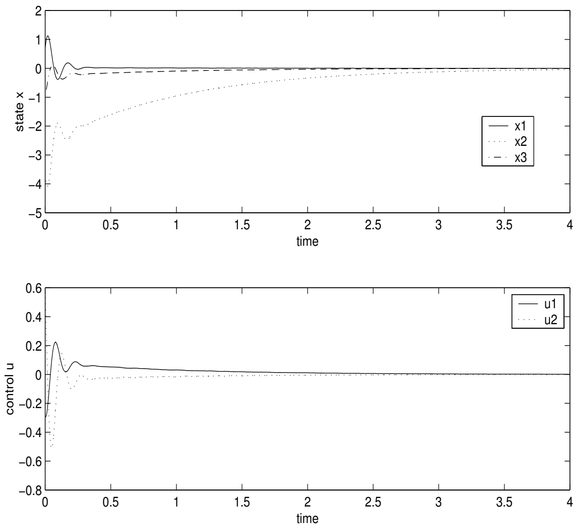

Figure 2 : Time-evolution of state and control action

Figure 2 shows the state trajectory from initial condition $xo = [0.7 \quad -3.68 \quad -0.95]^T$. The trajectory state are all driven to the origin by the provided robust state-feedback control, with respected constraints.

7. CONCLUSION

Stabilization of uncertain linear systems having restricted states and/or controls is considered. Norm bounded and polytopic uncertainties are studied. Computation of a large region of attraction for these systems is shown. Sufficient LMI conditions are provided to derive constrained stabilizing robust state-feedback controllers. In addition, a numerical algorithm is proposed to enlarge the volume of the invariant ellipsoid. Some examples are worked out to demonstrate the effectiveness of the proposed approach. Further, this approach can be seen as the LMI's formulation of the robust constrained regulator problem using the positive invariance concept.

REFERENCES

M. Ait Rami, H. Ayad & F. Mesquine (2007). "Enlarging ellipsoidal invariant sets for constrained linear systems", *IJICIC* , Vol. 3, no 5, pp. 1097 - 1108.

A. Benzaouia & F. Mesquine (1994). "Regulator problem for uncertain linear discrete time systems with constrained control", *Int. J. of Robust and Nonlinear Control*, vol. 4, pp. 387-395.

A. Benzaouia, F. Tadeo & F. Mesquine (2006) " The regulator problem for linear systems with saturations on the control and its increments or rate an LMI approach" *IEEE Trans. Circuits and Systems*, vol. 53, no 12; pp. 2681-2691.

D.S. Berstein & A.N. Michel (1995). "A chronological bibliography on saturating actuators, *Int. J. of Robust and Nonlinear Control*, vol. 5, pp. 375-380.

J. Bernussou, P.L.D. Perez & J.C. Geromel (1989). "A linear programming oriented procedure for quadratic stabilization of uncertain system", *Sys. Control let.*, vol 13, pp. 65-72.

F. Blanchini, F. Mesquine, & S. Miani (1995). "Constrained stabilization with an assigned initial condition set", *Int.J. Control*, vol.62, no. 3, pp. 601-617.

F. Blanchini (1999). "Set Invariance in control", *Automatica* , vol. 35, pp. 1747-1767.

S. Boyd, L. ELGhaoui, E. Ferron & V. Balakrishnan (1994). "Linear Matrix inequality in systems and control theory", *SIAM.*

M.A. Dahleh & I.J. Diaz-Bobiollo (1995). "Control of uncertain systems : a linear programming approach", *Engelwoods Cliffs, NJ : Prentice Hall.*

E. Gilbert & K.T. Tan (1991). "Linear systems with state and control constraints: the theory and application of maximal output admissible sets",*IEEE Trans. Aut. Control*, vol. 36, pp. 1008-1020.

G. Garcia, J. Bernussou & P. Camozi (1994). "H_2 guaranteed cost control for uncertain systems with norm bounded uncertainties", *Proc. of IFAC Symposium on Robust Control Design.*

C. Geromel , P.L.D. Peres & J. Bernussou (1991) "On a convex parameter space method for linear control design of uncertain system", *SIAM J. on Control and Optimization*, vol 29, no. 2, pp. 381-402.

P.O. Gutman, & P. Hagander (1985). "A new design of constrained controllers for linear systems", *IEEE Trans. Aut. Control*,vol. 30, pp. 22-33, 1985.

H. Hindi & S. Boyd (1998). "Analysis of Linear Systems with Saturating Using Convex Optimization", *CDC*, pp. 903-908.

R. Horn and C. Johnson (1985). "Matrix analysis" *Cambridge University Press.*

T. Hu, & Z. Lin (2000). "On Enlarging the basin of Attraction for Linear Systems under Saturated Linear Feedback", ACC *, Chicago,Illinois, pp. 1766-1770.*

T. Hu, Z. Lin & B.M. Chen (2002). "An analysis and design method for linear systems subject to actuator saturation and disturbance", Automatica*, vol. 38, pp. 351-359.*

Z. Lin & A. Saberi (1995). "Semi global exponential stabilization of linear discrete time systems subject to input saturation via linear feedbacks", Systems and Control letters*, vol. 21, pp. 225-239.*

F. Mesquine F. Tadeo & A. Benzaouia (2004a). "Regulator problem for linear systems with constrained control on its increment or rate", Automatica*, vol. 40,pp. 1387-1395.*

F. Mesquine, A. Benlemkadem & A. Benzaouia (2004b). "Robust constrained regulator problem for linear uncertain systems", J. of Dynamical & Control systems*, vol. 10, no. 4, pp. 527-544.*

B.E.A. Milani and A.N. Carvalho (1994), "Robust optimal linear regulator for discrete-time linear systems under state and control constraints ", Proc. of the IFAC Symposium on Robust Control Design.

C. Pittet, S. Tarbouriech, & C. Burgat (1997). "Stability regions for Linear Systems with saturating controls via Circle and Popov Criteria", 36th CDC*, pp. 4518-4523.*