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Bounded Real Lemma for Linear Discrete-Time Descriptor Systems

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Abstract: Under some rank condition, a new version of bounded real lemma, which is expressed in terms of an admissible solution of a generalized discrete-time algebraic Riccati equation (GDARE) rather than inequality, is presented for linear discrete-time descriptor systems. When a linear discrete-time descriptor system is admissible, with the H^∞-norm of its transfer matrix less than a prescribed positive number γ, a constructive procedure is also given to obtain an admissible solution of the above-mentioned GDARE.

1. INTRODUCTION

For more than two decades, the analysis and design of descriptor systems has attracted considerable attention. It has been shown that the descriptor system model is a more natural representation of dynamic systems and can describe a larger class of systems than the conventional state-space model; see for example, Dai [1989], Lewis [1986], Luenberger [1977], Newcomb and Dziurla [1989], and Verghese et al. [1981].

On the other hand, bounded realness has played an important role in circuit and network synthesis Anderson [1973], stability analysis, and control systems analysis and design Lozano et al. [2000]. A well-known characterization of the bounded real property in terms of state-space realization is the so-called *bounded real lemma*; see Anderson [1973], Petersen et al. [1991], Stoorvogel [1992], Yung and Yang [1999], Zhou and Khargonekar [1988] and the references therein. Although bounded real lemma has been developed over the last two decades, most of the results were built upon state-space model.

Recently, among many other things, a version of bounded real lemma, expressed in terms of an admissible solution of a certain generalized continuous-time algebraic Riccati equation (GCARE), has been proposed for linear continuous-time descriptor systems with application to solving the H^∞ control problem Wang et al. [1998]. In Kawamoto et al. [1999], some properties of GCARE were also studied. Most recently, among other things, a version of bounded real lemma based on matrix inequality for linear discrete-time descriptor systems has been addressed in Hsiung and Lee [1999] and Xu and Yang [2000]; and a version of bounded real lemma based on the conjugation has been proposed in Katayama [1996]. Moreover, an LQG-type matrix equation for discrete-time descriptor systems was considered in Nikoukhah et al. [1992].

Motivated by the work of Wang et al. [1998], the main purpose of this paper is to derive a version of bounded real lemma for linear discrete-time descriptor systems, expressed in terms of an admissible solution of a certain generalized discrete-time algebraic Riccati equation (hereafter abbreviated GDARE). Motivation for using GDARE rather than matrix inequality stems from the fact that, in the H^∞ control problem for conventional state-space systems, the central controller obtained in Doyle et al. [1989], which is constructed from the stabilizing solutions of two celebrated algebraic Riccati equations (AREs) rather than matrix inequalities, has the minimum entropy property Glover and Mustafa [1989]. It is expected that this is also true for the discrete-time descriptor systems case. From this viewpoint, it is thus prefered to use GDARE to characterize the bounded realness of discrete-time descriptor systems.

This paper is organized as follows: In Section 2, we briefly review some basic definitions and preliminary results concerning descriptor systems. Section 3 is the main body of the paper. Finally, in Section 4 we give some concluding remarks.

2. ELEMENTS OF DESCRIPTOR SYSTEMS THEORY

In this section, we summarize some basic definitions and preliminary results concerning descriptor systems; see Dai [1989], for example, for more details. Let A and E be $n \times n$ constant real matrices. Assume that rank$E = r \leq n$. The (ordered) pair (E, A) is said to be *regular* if there exists a scalar λ (may be real or complex) such that $\det(\lambda E - A) \neq 0$. Clearly, if $\det E \neq 0$, (E, A) is regular. A scalar

λ is called a finite eigenvalue of (E, A) if $\det(\lambda E - A) = 0$. Let $q \triangleq \deg\det(\lambda E - A)$. Then it is quite well known that (E, A) has q finite dynamic modes, $r - q$ noncausal modes (called impulsive modes for continuous-time case) and $n - r$ nondynamic modes. Furthermore, if $r = q$, there exist no noncausal modes and in this case the system is said to be *causal* (impulse-free for continuous-time case). (E, A) is called *stable* if all the finite eigenvalues of (E, A) lie within the open unit disk. (E, A) is called *admissible* if (E, A) is regular, causal and stable.

The following important fact is taken from Gantmacher [1959].

Proposition 1. The pair (E, A) is regular if and only if there exist invertible matrices W and V such that

$$\bar{E} \triangleq WEV = \begin{bmatrix} I_q & 0 \\ 0 & N \end{bmatrix}, \quad \bar{A} \triangleq WAV = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n-q} \end{bmatrix}, \tag{1}$$

where I_k is the $k \times k$ identity matrix; N is a nilpotent matrix, that is, $N^p = 0$ for some positive integer p. The minimum positive integer p_0 such that $N^{P_0} = 0$ is called the *index* of N. $(\bar{E}, \bar{A})$ is called the *Weierstrass canonical form* of (E, A). □

In the Weierstrass canonical form, the eigenvalues of the $q \times q$ matrix A_1 coincide with the finite eigenvalues of the pair (E, A). Thus, in terms of its Weierstrass canonical form (1), (E, A) is stable if and only if A_1 is stable, i.e., all the eigenvalues of A_1 lie within the open unit disk. Furthermore, (E, A) is causal if and only if the nilpotent matrix N has index one, i.e., $N = 0$.

Now consider a discrete-time descriptor system described by the following equations:

$$\begin{aligned} Ex(k+1) &= Ax(k) + Bu(k), \\ y(k) &= Cx(k) + Du(k), \end{aligned} \tag{2}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ are the input and output signals, respectively. A and E are defined as above, B, C and D are constant real matrices of compatible dimensions. Equation (2) has a unique solution for any given initial condition $Ex(0)$ and any discrete forcing function u if and only if (E, A) is regular. In what follows, it is assumed that (E, A) is regular. Then, according to Proposition 1, a suitable coordinate transformation always exists so that (2) can be put in the following Weierstrass form:

$$\begin{aligned} \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} \begin{bmatrix} \hat{x}_1(k+1) \\ \hat{x}_2(k+1) \end{bmatrix} &= \begin{bmatrix} A_1 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \hat{x}_1(k) \\ \hat{x}_2(k) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(k), \\ y(k) &= [\, C_1 \; C_2 \,] \begin{bmatrix} \hat{x}_1(k) \\ \hat{x}_2(k) \end{bmatrix} + Du(k), \end{aligned} \tag{3}$$

where $WB \triangleq \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$, $CV \triangleq [\, C_1 \; C_2 \,]$. The concepts of controllability and observability for descriptor systems are needed in the subsequent development, and thus introduced in the following. System (2), or briefly the triple (E, A, B), is termed *finite dynamics stabilizable and noncausality controllable* if there exists a constant real matrix K such that the pair $(E, A + BK)$ is admissible. Dually, System (2) or the triple (E, A, C) is termed *finite dynamics detectable and noncausality observable* if there exists a constant real matrix L such that the pair $(E, A + LC)$ is admissible.

Remark 2. The term *noncausality controllable* (or observable) is not a standard terminology for the discrete-time descriptor systems. For continuous-time descriptor systems, it is called *impulse controllable* (observable). If the continuous-time descriptor system do possess impulsive modes, then its transfer function would be improper. For discrete-time descriptor systems, an improper transfer function implies that the system is noncausal. That is the reason why we adopt the term *noncausality.*

The next result is a variation of Lyapunov stability theorem for discrete-time descriptor systems given in Ishihara and Terra [2003]. See also Syrmos et al. [1995], Hsiung and Lee [1999], and Stykel [2002] for more relevant work.

Proposition 3. Consider (2) and the Lyapunov equation

$$A^T XA - E^T XE + C^T C = 0. \tag{4}$$

positive semidefinite. Suppose that (E, A) is regular. Suppose also that (E, A, C) is finite dynamics detectable and noncausality observable. Then, if there exists a symmetric solution $X \in \mathbb{R}^{n\times n}$ of (4) with $E^T XE \geq 0$, (E, A) is admissible. Conversely, suppose that (E, A) is admissible. Let W and V be $n \times n$ invertible matrices that transform (E, A) into the Weierstrass form (1). Let $V^T C^T CV$ be partitioned compatibly with (1) as $V^T C^T CV = \begin{bmatrix} Q_1 & Q_2^T \\ Q_2 & Q_3 \end{bmatrix}$. Suppose, in addition, that

$$\mathcal{K}er A_1 \subset \mathcal{K}er Q_2. \tag{5}$$

Then there exists a symmetric matrix $X \in \mathbb{R}^{n\times n}$, with $E^T XE \geq 0$, which satisfies (4). □

Proof. *Sufficiency.* See Ishihara and Terra [2003].

Necessity. Note that finding a symmetric real solution X of (4), with $E^T XE \geq 0$, amounts to finding a symmetric real solution $\bar{X} \triangleq W^{-T} XW^{-1}$, with $\bar{E}^T \bar{X}\bar{E} \geq 0$, of the equation

$$\bar{A}^T \bar{X}\bar{A} - \bar{E}^T \bar{X}\bar{E} + V^T C^T CV = 0. \tag{6}$$

Writing $\bar{X}$ compatibly as $\bar{X} = \begin{bmatrix} X_1 & X_2^T \\ X_2 & X_3 \end{bmatrix}$, with X_1 and X_3 symmetric, (6) is equivalent to

$$\begin{aligned} A_1^T X_1 A_1 - X_1 + Q_1 &= 0, \\ A_1^T X_2^T + Q_2^T &= 0, \\ X_3 + Q_3 &= 0. \end{aligned} \tag{7}$$

Thus, $X_1 = \sum_{i=0}^{\infty} (A_1^T)^i Q_1 A_1^i = X_1^T \geq 0$ since A_1 is stable and Q_1 is positive semidefinite. This in turn implies that $\bar{E}^T \bar{X}\bar{E} \geq 0$. In addition, $X_3 = -Q_3$. Furthermore, hypothesis (5) implies that (7) admits a solution X_2. This completes the proof. ■

The following result is taken from Stoorvogel [1992], which is a version of bounded real lemma for discrete-time state-space systems.

Lemma 4. The following statements are equivalent:

(1) The system
$$x(k+1) = Ax(k) + Bu(k),$$
$$y(k) = Cx(k) + Du(k), x(0) = 0, \quad (8)$$
is internally stable(i.e., A is stable) and $\|C(zI - A)^{-1}B + D\|_\infty < \gamma$.

(2) There exists a symmetric, positive semidefinite real matrix P satisfying the following:
(a) $N \triangleq \gamma^2 I - D^T D - B^T PB > 0$,
(b) $A^T PA - P + C^T C + (A^T PB + C^T D)N^{-1}(B^T PA + D^T C) = 0$,
(c) $A + BN^{-1}(B^T PA + D^T C)$ is stable.

□

3. MAIN RESULTS

First, we consider one kind of fractional matrix equation, namely the GDARE, which assumes the following form:

$$A^T XA - E^T XE + Q - (A^T XB + S)(B^T XB + R)^{-1}(B^T XA + S^T) = 0 \quad (9)$$

where A, E, and Q are given $n \times n$ real matrices, R is a given $m \times m$ real matrix, B and S are given $n \times m$ real matrices, and X is an $n \times n$ real matrix to be determined. It is assumed that Q and R are symmetric. The matrix E is, in general, noninvertible.

Definition 5. A real matrix X is said to be a solution of (9) if $B^T XB + R$ is invertible and X satisfies (9). □

Thus, it is implicitly implied that the matrix $B^T XB + R$ is invertible if X is a solution of (9).

Definition 6. A solution X of (9) is called *admissible* if the pair $(E, A - B(B^T XB + R)^{-1}(B^T XA + S^T))$ is admissible. □

The main result of this paper, namely bounded real lemma for discrete-time descriptor systems, is summarized in the following statements.

Theorem 7. (bounded real lemma) Consider System (2) with $Ex(0) = 0$. Suppose that (E, A) is regular. Let $T_{yu}(z) \triangleq C(zE - A)^{-1}B + D$. Then, (E, A) is admissible and $\|T_{yu}\|_\infty < \gamma$ if the GDARE

$$A^T XA - E^T XE + C^T C - (A^T XB + C^T D)(B^T XB + D^T D - \gamma^2 I)^{-1}(B^T XA + D^T C) = 0 \quad (10)$$

has a symmetric, admissible solution X^-, with $E^T X^- E \geq 0$ and $B^T X^- B + D^T D - \gamma^2 I < 0$. Conversely, suppose System (2) is admissible and $\| T_{yu} \|_\infty < \gamma$. Suppose, in addition, that the following assumption holds.

Assumption (A1): $\operatorname{rank}\left[\, A \;\; B \,\right] = \operatorname{rank}\begin{bmatrix} A & B \\ C & D \end{bmatrix}$.

Then the GDARE (10) has a symmetric, admissible solution X^-, with $E^T X^- E \geq 0$ and $B^T X^- B + D^T D - \gamma^2 I < 0$. □

Proof. *Sufficiency.* Set $M \triangleq \gamma^2 I - D^T D - B^T X^- B > 0$ and $K \triangleq M^{-1}(B^T X^- A + D^T C)$. Then the GDARE (10) (with $X = X^-$) can be written in the form of the Lyapunov equation (4):

$$A^T X^- A - E^T X^- E + \begin{bmatrix} C \\ M^{\frac{1}{2}} K \end{bmatrix}^T \begin{bmatrix} C \\ M^{\frac{1}{2}} K \end{bmatrix} = 0. \quad (11)$$

Let $L = \begin{bmatrix} 0 & BM^{\frac{-1}{2}} \end{bmatrix}$. Then the pair $(E, A + L\begin{bmatrix} C \\ M^{\frac{1}{2}} K \end{bmatrix}) = (E, A + BM^{-1}(B^T X^- A + D^T C))$ is admissible. Thus the triple $(E, A, \begin{bmatrix} C \\ M^{\frac{1}{2}} K \end{bmatrix})$ is finite dynamics detectable and noncausality observable. Thus the pair (E, A) is admissible by Proposition 3. Next, we show that $\|C(zE - A)^{-1}B + D\|_\infty < \gamma$. Define a function $F(x(k)) \triangleq x^T(k)E^T X^- Ex(k) \geq 0$. Let

$$\begin{aligned}
H(x(k), u(k)) &\triangleq F(x(k+1)) - F(x(k)) + \|y(k)\|^2 \\
&\quad -\gamma^2\|u(k)\|^2 \\
&= x^T(k+1)E^T X^- Ex(k+1) \\
&\quad -x^T(k)E^T X^- Ex(k) \\
&\quad +\|Cx(k) + Du(k)\|^2 - \gamma^2\|u(k)\|^2 \\
&= (Ax(k) + Bu(k))^T X^-(Ax(k) \\
&\quad +Bu(k)) - x^T(k)E^T X^- Ex(k) \\
&\quad +(Cx(k) + Du(k))^T \\
&\quad (Cx(k) + Du(k)) - \gamma^2 u^T(k)u(k) \\
&= x^T(k)[A^T X^- A - E^T X^- E \\
&\quad +C^T C + (A^T X^- B + C^T D) \\
&\quad M^{-1}(B^T X^- A + D^T C)]x(k) \\
&\quad +(A^T X^- B + C^T D)M^{-1} \\
&\quad (B^T X^- A + D^T C)]x(k) \\
&\quad -[(B^T X^- A + D^T C)x(k) \\
&\quad -Mu(k)]^T M^{-1}[(B^T X^- A \\
&\quad +D^T C)x(k) - Mu(k)] \\
&= -[(B^T X^- A + D^T C)x(k) \\
&\quad -Mu(k)]^T M^{-1}[(B^T X^- A + D^T C) \\
&\quad x(k) - Mu(k)].
\end{aligned}$$

Since $M > 0$, it follows that for each nonnegative integers k, we have

$$H(x(k), u(k)) \leq 0. \quad (12)$$

Hence summation of (12) from $k = 0$ to $k = \infty$ yields

$$F(x(\infty)) - F(x(0)) + \sum_{k=0}^{\infty}(\|y(k)\|^2 - \gamma^2\|u(k)\|^2) \leq 0.$$

As a result, $\sum_{k=0}^{\infty}(\|y(k)\|^2 - \gamma^2\|u(k)\|^2) \leq 0$. Thus we prove that $\|T_{yu}\|_\infty \leq \gamma$. To complete the proof, we need to show that the strict inequality holds. Since X^- is a solution of (10), the following equality holds for any complex number z with $|z| = 1$:

$$\begin{aligned}
&(z^{-1}E^T - A^T)X^-(zE - A) + (z^{-1}E^T - A^T)X^- A \\
&\quad + A^T X^-(zE - A) = E^T X^- E - A^T X^- A \\
&= C^T C + (A^T X^- B + C^T D)M^{-1}(B^T X^- A + D^T C).
\end{aligned}$$

Now pre-multiply the above equality by $B^T(z^{-1}E^T - A^T)^{-1}$ and post-multiply by $(zE-A)^{-1}B$ to get

$$\begin{aligned}&B^TX^-B + B^TX^-A(zE-A)^{-1}B + \\ &B^T(z^{-1}E^T-A^T)^{-1}A^TX^-B \\ =&B^T(z^{-1}E^T-A^T)^{-1}C^TC(zE-A)^{-1}B \\ &+B^T(z^{-1}E^T-A^T)^{-1}(A^TX^-B+C^TD)M^{-1} \\ &(B^TX^-A+D^TC)(zE-A)^{-1}B.\end{aligned}$$

Then we have

$$\begin{aligned}&\gamma^2I - T_{yu}^T(z^{-1})T_{yu}(z) \\ =&\gamma^2I-(C(z^{-1}E-A)^{-1}B+D)^T \\ &(C(zE-A)^{-1}B+D) \\ =&\gamma^2I-D^TD - B^T(z^{-1}E^T-A^T)^{-1} \\ &C^TC(zE-A)^{-1}B \\ &-B^T(z^{-1}E^T-A^T)^{-1}C^TD \\ &-D^TC(zE-A)^{-1}B \\ =&\gamma^2I-D^TD+B^T(z^{-1}E^T-A^T)^{-1}(A^TX^-B \\ &+C^TD)M^{-1}(B^TX^-A+D^TC)(zE-A)^{-1}B \\ &-B^TX^-B-B^TX^-A(zE-A)^{-1}B - \\ &B^T(z^{-1}E^T-A^T)^{-1}A^TX^-B \\ &-B^T(z^{-1}E^T-A^T)^{-1}C^TD-D^TC(zE-A)^{-1}B \\ =&M + B^T(z^{-1}E^T-A^T)^{-1}(A^TX^-B+ \\ &C^TD)M^{-1}(B^TX^-A+D^TC)(zE-A)^{-1}B \\ &-(B^TX^-A+D^TC)(zE-A)^{-1}B \\ &-B^T(z^{-1}E^T-A^T)^{-1}(A^TX^-B+C^TD) \\ =&G^T(z^{-1})G(z), \qquad (13)\end{aligned}$$

where

$$G(z) \triangleq M^{\frac{1}{2}} - M^{\frac{-1}{2}}(B^TX^-A+D^TC)(zE-A)^{-1}B. \qquad (14)$$

Suppose, by contradiction, that there exists a z_0 with $|z_0| = 1$ such that $\|C(z_0E-A)^{-1}B+D\|_\infty = \gamma$. Then (13) implies that there exists a nonzero vector v_0 such that $G(z_0)v_0 = 0$. Thus, we obtain $\det G(z_0) = 0$. Now, by a standard result on determinants, we have $\det(I - M^{-1}(B^TX^-A+D^TC)(z_0E-A)^{-1}B) = 0$. This leads to a contradiction, for the pair $(E, A+BM^{-1}(B^TX^-A+D^TC))$ being admissible, which implies that $\det(z_0E-A-BM^{-1}(B^TX^-A+D^TC)) = \det(z_0E-A)\det(I-(z_0E-A)^{-1}BM^{-1}(B^TX^-A+D^TC)) = \det(z_0E-A)\det(I-M^{-1}(B^TX^-A+D^TC)(z_0E-A)^{-1}B) \neq 0$. Hence it is concluded that $\|C(zE-A)^{-1}B+D\|_\infty < \gamma$.

Necessity. Since System (2) is admissible, there exist invertible matrices W and V that transform System (2) into the Weierstrass form (3), with A_1 stable, $N = 0$, $WB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$, and $CV = [\, C_1 \ C_2 \,]$. Then, the transfer matrix of System (2) is given by $T_{yu}(z) = C(zE-A)^{-1}B + D = C_1(zI-A_1)^{-1}B_1 + D_1$, where $D_1 \triangleq D - C_2B_2$. Since A_1 is stable and $\| T_{yu} \|_\infty < \gamma$, it follows from Lemma 4 that there exists a matrix $X_0 = X_0^T \geq 0$ satisfying the following:

(1) $M_0 \triangleq \gamma^2 I - D_1^TD_1 - B_1^TX_0B_1 > 0$,
(2) $A_1^TX_0A_1 - X_0 + C_1^TC_1 + (A_1^TX_0B_1 + C_1^TD_1)M_0^{-1}(B_1^TX_0A_1 + D_1^TC_1) = 0$,
(3) $A_1 + B_1M_0^{-1}(B_1^TX_0A_1 + D_1^TC_1)$ is stable.

Note that Assumption (A1) is equivalent to the following:

$$\text{rank}\,[\, WAV \ \ WB \,] = \text{rank}\begin{bmatrix} WAV & WB \\ CV & D \end{bmatrix},$$

that is, $\text{rank}\begin{bmatrix} A_1 & 0 & B_1 \\ 0 & I & B_2 \end{bmatrix} = \text{rank}\begin{bmatrix} A_1 & 0 & B_1 \\ 0 & I & B_2 \\ C_1 & C_2 & D \end{bmatrix}$.

But $\text{rank}\begin{bmatrix} A_1 & 0 & B_1 \\ 0 & I & B_2 \end{bmatrix}$
$= \text{rank}(\begin{bmatrix} A_1 & 0 & B_1 \\ 0 & I & 0 \end{bmatrix}\begin{bmatrix} I & 0 & 0 \\ 0 & I & B_2 \\ 0 & 0 & I \end{bmatrix})$
$= \text{rank}\begin{bmatrix} A_1 & 0 & B_1 \\ 0 & I & 0 \end{bmatrix}$, and $\text{rank}\begin{bmatrix} A_1 & 0 & B_1 \\ 0 & I & B_2 \\ C_1 & C_2 & D \end{bmatrix}$
$= \text{rank}(\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & C_2 & I \end{bmatrix}\begin{bmatrix} A_1 & 0 & B_1 \\ 0 & I & 0 \\ C_1 & 0 & D_1 \end{bmatrix}$
$\begin{bmatrix} I & 0 & 0 \\ 0 & I & B_2 \\ 0 & 0 & I \end{bmatrix}) = \text{rank}\begin{bmatrix} A_1 & 0 & B_1 \\ 0 & I & 0 \\ C_1 & 0 & D_1 \end{bmatrix}$, we have

$$\text{rank}\,[\, A_1 \ \ B_1 \,] = \text{rank}\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}. \qquad (15)$$

Thus, there exists a matrix X_1 satisfying

$$X_1^T[A_1 \quad B_1] = -C_2^T[C_1 \quad D_1]. \qquad (16)$$

Let $X^- \triangleq W^T \begin{bmatrix} X_0 & X_1 \\ X_1^T & -C_2^TC_2 \end{bmatrix} W = (X^-)^T$. With (16) in mind, it is straightforward to show that $B^TX^-B + D^TD - \gamma^2 I = -M_0 < 0$, that $E^TX^-E = V^{-T}\begin{bmatrix} X_0 & 0 \\ 0 & 0 \end{bmatrix}V^{-1} \geq 0$, and that X^- satisfies the following:

$$\begin{aligned}&A^TX^-A - E^TX^-E + C^TC - (A^TX^-B + C^TD) \\ &(B^TX^-B + D^TD - \gamma^2I)^{-1}(B^TX^-A + D^TC) \\ &= V^{-T}\begin{bmatrix} \Gamma & A_1^TX_1 + C_1^TC_2 \\ X_1^TA_1 + C_2^TC_1 & 0 \end{bmatrix}V^{-1} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},\end{aligned}$$

where $\Gamma \triangleq A_1^TX_0A_1 - X_0 + C_1^TC_1 + (A_1^TX_0B_1 + C_1^TD_1)M_0^{-1}(B_1^TX_0A_1 + D_1^TC_1)$. Furthermore, it is also easy to verify that the pair

$$\begin{aligned}&(WEV, WAV - WB(B^TX^-B + D^TD - \gamma^2I)^{-1} \\ &\qquad (B^TX^-A + D^TC)V) \\ &= (\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_1 + B_1M_0^{-1}(B_1^TX_0A_1 + D_1^TC_1) & 0 \\ B_2M_0^{-1}(B_1^TX_0A_1 + D_1^TC_1) & I \end{bmatrix})\end{aligned}$$

which is admissible since $A_1 + B_1M_0^{-1}(B_1^TX_0A_1 + D_1^TC_1)$ is stable. Accordingly, X^- is a symmetric, admissible solution of the GDARE (10). This completes the proof. ■

Remark 8. Note that the necessity proof of Theorem 7 provides a method to construct an admissible solution of the GDARE (10) when System (2) is admissible with $\|T_{yu}\|_\infty < \gamma$.

Remark 9. In the above proof, it has been seen that Assumption (A1) is equivalent to the condition (15), which is in turn equivalent to the following condition:

$$\mathcal{K}er[A_1 \quad B_1] \subset \mathcal{K}er[C_1 \quad D_1].$$

In fact, in view of (16), Assumption (A1) can be replaced by a weaker condition as follows:

$$\text{Assumption (A2):}\mathcal{K}er[A_1 \quad B_1] \subset \mathcal{K}er[C_2^T C_1 \quad C_2^T D_1].$$

It is easy to see that in the case of $E = I$, that is, System (2) reduces to a state-space system (8), Theorem 7, with Assumption (A1) replaced by Assumption (A2), coincides with Lemma 4. Accordingly, Theorem 7, with Assumption (A1) replaced by Assumption (A2), can be regarded as an extension of the bounded real lemma in Lemma 4 for discrete-time state-space systems to the case of discrete-time descriptor systems.

4. CONCLUSIONS

Under some rank condition, a new version of bounded real lemma for linear discrete-time descriptor systems has been proposed. Rather than matrix inequality, the condition obtained are expressed in terms of an admissible solution of a certain GDARE. A method has also been given to construct an admissible solution of the above-mentioned GDARE when a linear discrete-time descriptor system is admissible with the H^∞-norm of its transfer matrix less than a prescribed positive number γ. The main result given here can be regarded as an extension of bounded real lemma for discrete-time state-space systems case to discrete-time descriptor systems case. The application of the result of this paper to the H^∞ control problem for discrete-time descriptor systems is left as our future work.

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