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Input-to-State Stabilization of Nonlinear Systems with Quantized Feedback ⋆

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Abstract: This paper addresses the stabilization problem of nonlinear feedback systems with quantized measurements in the presence of bounded disturbances. This paper is an extension of [Liberzon, Nešić (2007)] to nonlinear systems. Using the scheme proposed in [Liberzon, Nešić (2007)], we show that input-to-state stability with respect to bounded disturbances is achievable for nonlinear systems with quantized feedback.

Keywords: quantization, controller design, disturbance, input-to-state stabilization

1. INTRODUCTION

The present work generalizes the contributions of Liberzon, Nešić (2007) to nonlinear systems. In Liberzon, Nešić (2007) it was shown that using the trajectory-based approach, it is possible to achieve global asymptotic stability for linear systems without disturbances and input-to-state stability (ISS) when the disturbance is introduced into the system. All results developed in Liberzon, Nešić (2007) are limited to linear systems. In this paper we consider the stabilization problem of nonlinear systems with quantized feedback in the presence of bounded disturbances. Our approach fits into the framework of control with limited information in the sense that the state of the system is not completely known.

In many recent applications of the networked control systems it may happen that each component of the system is allocated only a small portion of the bandwidth for communication between the plant and the controller. Since the quantization errors in this case can be large, the communication limitations have to be taken into account for a successful control design. In recent years, a number of researchers have analyzed various versions of this problem, including Brockett, Liberzon (2000), Delchamps (1990), Elia, Mitter (2001), Liberzon (2003), Nair *et. al.* (2007).

In this paper, we consider nonlinear time-invariant feedback systems with quantized measurements, when the system is perturbed by bounded disturbances. Under appropriate assumptions, our objective is to find the conditions under which the closed-loop system is ISS with respect to bounded disturbances. Building on the earlier work from Liberzon, Nešić (2007), our main result, Theorem 1 in Section 4 shows that if the parameters of the switching scheme and the parameters of the quantizer are adjusted appropriately, then the nonlinear plant is input-to-state stable with respect to bounded disturbances.

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The outline of the paper is as follows. In Section 2 we give definitions that are used in the sequel. In Section 3 we present the closed loop system and describe the switching rules and the protocol. We present the main results in Section 4. We make some concluding remarks and propose some future directions for the research in Section 5.

2. NOTATION AND PRELIMINARIES

In this section we introduce some notation and give the definitions that will make the discussed concepts precise. In what follows, $|\cdot|$ denotes the Euclidean norm, $\|\cdot\|$ denotes the corresponding matrix induced norm. The infinity-norm of a sequence of vectors on a time-interval $[k_1, k_2]$ is denoted $\|z\|_{[k_1,k_2]} := \sup_{k\in[k_1,k_2]} |z_k|$. A quantizer is a piecewise constant function $q : \mathbb{R}^n \to Q$, where Q is a finite subset of $\mathbb{R}^n$. We use the following assumption:

Assumption 1. There exist strictly positive numbers $M > \Delta > 0$ and Δ_0, such that the following holds: 1. If $|z| \leq M$ then $|z - q(z)| \leq \Delta$; 2. If $|z| > M$ then $|q(z)| > M - \Delta$; 3. For all $|z| \leq \Delta_0$ we have that $q(z) = 0$.

M is called the range of the quantizer; Δ is called the quantization error; Δ_0 is the dead-zone. The first condition gives a bound on the quantization error when the state is in the range of the quantizer, the second gives the possibility to detect saturation. The third condition is needed to preserve the origin as an equilibrium. We use the following definitions:

Definition 1. A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}_\infty$ if it is continuous, zero at zero, strictly increasing and unbounded.

Definition 2. A continuous function $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ is said to be class $\mathcal{KL}$ if, for each fixed s, the mapping $\beta(r, s)$ is strictly increasing and $\beta(0, s) = 0$, and, for each fixed r, the mapping $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to \infty$.

For $\chi \in \mathcal{K}_\infty$ we define $\underbrace{\chi \circ \chi \circ \cdots \circ \chi(s)}_{\text{n times}} \equiv \chi^n(s)$.

3. CLOSED-LOOP SYSTEM

Consider the continuous-time nonlinear system with a control input:

$$\dot{x}(t) = f(x(t), u(t), w(t)), \quad x(0) \in \mathbb{R}^n \tag{1}$$

where f is well defined for all x, u, w and is locally Lipschitz in x and w for each u; $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is a control input and $w \in \mathbb{R}^l$ is an unknown bounded disturbance. Define $t_k = kT$ for $k = 0, 1, 2, \ldots$, where $T > 0$ is a given sampling period. Let $k_0 = 0$. We shortly denote $x(t_k) = x_k$ and similarly for all other variables. Let $u_k =$ const. for all $t \in [kT, (k+1)T)$, $k \geq 0$. Assume that the exact discrete time plant model of the sampled-data plant (1) is the following:

$$x_{k+1} = x_k + \int_{kT}^{(k+1)T} f(x(\tau), u_k, w(\tau))d\tau = F(x_k, u_k, w_k), \tag{2}$$

where $x_0 \in \mathbb{R}^n$ and $F(x_k, u_k, w_k)$ is the solution of (1) at a time T starting at x_k and with the constant input u_k. This model is well-defined when the continuous model (1) does not exhibit finite-escape time. For simplicity we assume that (2) is known. For results on the stabilization of the nonlinear sampled-data systems based on their approximate discrete-time models we refer the reader to Nešić, Teel (2004-1), where it was shown that the controller that stabilizes a family of approximate discrete-time plant models also stabilizes the exact discrete-time plant model under some conditions.

To control the system (2) we use the nonlinear version of the quantized hybrid feedback that was introduced by Liberzon and Nešić in Liberzon, Nešić (2007). The quantized hybrid feedback is defined by:

$$u_k = U(\Omega_k, \mu_k, x_k), \tag{3}$$

$$U(\Omega_k, \mu_k, x_k) = \begin{cases} 0 & \text{if } \Omega_k = \Omega_{out} \\ \kappa(q_k) & \text{if } \Omega_k = \Omega_{in}, \end{cases} \tag{4}$$

where

$$q_k := \mu_k q\left(\frac{x_k}{\mu_k}\right), \quad \mu_0 > 0 \tag{5}$$

is a family of dynamic quantizers, μ_k is the adjustable parameter, called "zoom" variable, that is updated at discrete instants of time. The variable μ_k depends only on the quantized measurements of the state q_k. Geometrically, at each time instant $\mathbb{R}^n$ is divided into a finite number of quantization regions. Each region corresponds to a fixed value of the quantizer q_k. The variable Ω_k can take only two strictly positive values Ω_{out} and Ω_{in}, that will be defined next. If $\Omega_k = \Omega_{out}$ we say that the zoom-out condition is triggered at a time k. If $\Omega_k = \Omega_{in}$ we say that the zoom-in condition is triggered at a time k. During the zoom-out stage the system is running in an open loop: $u_k = 0$. During the zoom-in stage the certainty equivalence feedback $u_k = \kappa(q_k)$ is applied.

The protocol dynamics is described by the following:

$$\mu_{k+1} = G(\Omega_k, \mu_k, x_k), \quad \mu_0 \in \mathbb{R}_{>0} \tag{6}$$

$$G(\Omega_k, \mu_k, x_k) = \begin{cases} \chi(\mu_k) + c & \text{if } \Omega_k = \Omega_{out} \\ \psi(\mu_k) & \text{if } \Omega_k = \Omega_{in}, \end{cases} \tag{7}$$

where $c > 0$, $\chi, \psi \in \mathcal{K}_\infty$, $\psi(s) < s \ \forall s > 0$ that will be defined later. The adjustment policy for μ_k can be thought of as implemented synchronously on both ends of the communication channel from some known initial value μ_0. For each fixed μ the range of the quantizer is $M\mu$ and the quantization error is $\Delta\mu$. The adjustment policy for μ_k is composed of two stages: a zoom-out stage and a zoom-in stage. During the zoom-out stage the value of an adjustable parameter μ_k is increased at the rate faster than the growth of $|x_k|$ until the state can be adequately measured. During the zoom-in stage the value of an adjustable parameter μ_k is decreased in such way as to drive the state to the origin.

The hysteresis switching is used to switch between the zoom-in and the zoom-out stages. It is described by the following:

$$\Omega_k = H(\Omega_{k-1}, \mu_k, x_k), \quad \Omega_{-1} = \Omega_{out} \tag{8}$$

$$H(\Omega_{k-1}, \mu_k, x_k) = \begin{cases} \Omega_{out} & \text{if } |q_k| > l_{out}\mu_k \\ \Omega_{in} & \text{if } |q_k| < l_{in}\mu_k \\ \Omega_{k-1} & \text{if } |q_k| \in [l_{in}\mu_k, l_{out}\mu_k], \end{cases} \tag{9}$$

where l_{out} and l_{in} are strictly positive numbers such that $l_{out} := M - \Delta$, $l_{in} := \Delta_M - \Delta$, Δ_M will be defined later. Cancelling μ_k in (9) we can conclude, that the switching depends only on the value of $q\left(\frac{x_k}{\mu_k}\right)$, which can be interpreted as the fact that the switching is governed by the variable $\xi_k := \frac{x_k}{\mu_k}$ (see Remark below).

Remark 1. Consider the zoom-in switching condition in (9). Note, that whenever $\left|\frac{x_k}{\mu_k}\right| < l_{in} - \Delta$ holds, $\left|\mu_k q\left(\frac{x_k}{\mu_k}\right)\right| < l_{in}\mu_k$ holds. Also, the zoom-out switching condition in (9) $\left|\mu_k q\left(\frac{x_k}{\mu_k}\right)\right| > l_{out}\mu_k$ implies that $\left|\frac{x_k}{\mu_k}\right| > l_{out} + \Delta$.

We introduce some notation. For each $k \geq 0$ there are two possible cases: $\Omega_k = \Omega_{out}$ or $\Omega_k = \Omega_{in}$. Given an initial condition and a disturbance there is a sequence of the zoom-out and zoom-in intervals. We introduce $k_j \in \mathbb{N}$ such that

$$\Omega_k = \Omega_{out} \text{ if } k \in [k_{2i}, k_{2i+1} - 1], \quad i = 0, 1, \ldots, N$$
$$\Omega_k = \Omega_{in} \text{ if } k \in [k_{2i+1}, k_{2i+2} - 1],$$

That is: k_{2i+1} is the time instant at which the plant switches from the zoom-out stage to the zoom-in stage; k_{2i+2} is the time instant at which the plant switches from the zoom-in stage to the zoom-out stage. For simplicity we assume that the first interval is always the zoom-out: $\Omega_{-1} = \Omega_{out}$.

Next we state the assumptions that we use during the zoom-out and the zoom-in stages and consider the dynamics of the adjustable parameter μ and the switching variable ξ during the zoom-out and the zoom-in stages.

ZOOM-OUT STAGE:

By (4), during the zoom-out stage the control (3) is set to zero and the system is running in the open loop. We assume that during the zoom-out stage the system (2) with $u_k = 0$ is forward complete Sontag (1989):

Assumption 2. Assume that there exist a class $\mathcal{K}_\infty$ functions φ_1, φ_2, φ_3 and a constant $\bar{c}$ such that the following holds for the trajectories of the system (2) with $u_k = 0$ $\forall k \in [k_{2i}, k_{2i+1}]$:

$$|x_k| \leq \varphi_1(|x_{k_{2i}}|) + \varphi_2(\|w\|_{[k_{2i}, k-1]}) + \varphi_3(k - k_{2i}) + \bar{c}.$$

Since the disturbance w is bounded, there exists a time $k^* \in [k_{2i}, k_{2i+1}]$ such that the following holds $\forall k \in [k^*, k_{2i+1}]$:

$$|x_k| \leq \varphi_1(k-k_{2i}) + \varphi_2(k-k_{2i}) + \varphi_3(k-k_{2i}) + \bar{c}. \quad (10)$$

Now consider the dynamics of the adjustable parameter μ during the zoom-out stage. During the zoom-out stage, while the state is in the saturation region ($|x_k| > M\mu_k$), the range of the quantizer is increased (by increasing the value of the adjustable parameter μ_k at the rate faster than the growth of $|x_k|$) until the state can be adequately measured. We increase μ_k in a piecewise constant fashion, fast enough to dominate the rate of growth of $|x_k|$, refer to Proposition 1 below.

Proposition 1. In (7) let χ be such that

$$\chi(s) > as \quad \forall s \geq 0, \quad a > 1. \quad (11)$$

Then let $a > 1$ be such that the following holds:

$$as > \varphi_1(s) + \varphi_2(s) + \varphi_3(s) + \bar{c} \quad \forall s \geq 0. \quad (12)$$

where $\varphi_1, \varphi_2, \varphi_3, \bar{c}$ come from Assumption 2. Note, that we can overbound the sum of $\mathcal{K}_\infty$ functions $\varphi_1, \varphi_2, \varphi_3$ by letting a in (11) big enough. Then there exists a time instant $\bar{k}$ such that the following holds:

$$\mu_{\bar{k}+1} = \chi(\mu_{\bar{k}}) + c > \varphi_1(\bar{k}) + \varphi_2(\bar{k}) + \varphi_3(\bar{k}) + \bar{c} + c > |x_{\bar{k}+1}|.$$

To understand the operation of the plant, we have to consider dynamics of the switching variable ξ_k. During the zoom-out stage the dynamics of ξ_k evolves according to the following for all $k \in [k_{2i}, k_{2i+1}-1]$:

$$\xi_{k+1} = \frac{x_{k+1}}{\mu_{k+1}} = \frac{F(x_k, 0, w_k)}{\chi(\mu_k) + c} =: \mathcal{F}_{out}(\xi_k, \mu_k, w_k).$$

ZOOM-IN STAGE:

By (4), during the zoom-in stage the certainty equivalence feedback $u_k = \kappa(q_k)$ is applied. During the zoom-in stage we assume that the following holds for the solutions of the system (2) - (9):

Assumption 3. Assume that for any $\bar{T} \in [k_{2i+1}, k_{2i+2}]$ there exist $\chi_1, \chi_2 \in \mathcal{K}_\infty$, such that the following holds for the trajectories of the system (2) - (9) for all $k \in [k_{2i+1}, \bar{T}]$:

$$|x_k| \leq \chi_1(|x_{k_{2i+1}}|) + \chi_2(\|w\|_{[k_{2i+1}, k-1]}).$$

We also assume that during the zoom-in stage the closed-loop system (2) - (9) with with $u_k = \kappa(x_k + e_k)$ is input-to-state stable with respect to the measurement error e and the disturbance w:

Assumption 4. Assume that for all $\mu_0 > 0$ there exist functions $\beta \in \mathcal{KL}$ and $\gamma, \bar{\gamma} \in \mathcal{K}_\infty$ such that for every initial condition x_0 and every e_k, w_k the corresponding solution of the system (2) - (9) with $u_k = \kappa(x_k + e_k)$ satisfies the following for all $k \in [k_{2i+1}, k_{2i+2}]$:

$$|x_k| \leq \beta(|x_{k_{2i+1}}|, k) + \gamma(\|e\|_{[k_{2i+1}, k-1]}) + \bar{\gamma}(\|w\|_{[k_{2i+1}, k-1]}).$$

Remark 2. Note, that in analogy to Assumption 4 for nonlinear systems, in Liberzon, Nešić (2007) for linear system

$$x_{k+1} = (\Phi + \Gamma K)x_k + \Gamma K \left(q\left(\frac{x_k}{\mu_k}\right) - \frac{x_k}{\mu_k} \right) + w_k, \quad (13)$$

the authors let the matrix K be such that $\Phi + \Gamma K$ is Schur, which implies that the system (13) is input-to-state stable with respect to bounded disturbance w (the definition is given in the next section).

We remark that Assumption 4 can be restrictive (for the continuous-time systems refer to Freeman (1995)). Assumption 4 is equivalent to saying that there exists a smooth function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that for some class $\mathcal{K}_\infty$ functions $\alpha_1, \alpha_2, \alpha_3, \bar{\rho}_1, \bar{\rho}_2$, for all $x, e \in \mathbb{R}^n$ we have:

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \quad (14)$$

and

$$|x| \geq \bar{\rho}_1(|e|) + \bar{\rho}_2(|w|) \Longrightarrow$$
$$V(F(x, \kappa(x+e), w)) - V(x) \leq -\alpha_3(|x|).$$

Now consider the dynamics of the adjustable parameter μ during the zoom-in stage. During the zoom-in stage, while the state is in the range of the quantizer ($|x_k| \leq M\mu_k$), the quantization error is decreased (by decreasing the value of adjustable parameter μ_k) in such way as to drive the state to the origin. In (7) let ψ be such that the following holds:

$$\psi(\mu) < \mu \quad \forall \mu > 0. \quad (15)$$

Then μ_k is decreasing during the zoom-in stage in a piecewise constant fashion as $k \to \infty$.

During the zoom-in stage the dynamics of the switching variable ξ_k evolves according to the following for all $k \in [k_{2i+1}, k_{2i+2}-1]$:

$$\xi_{k+1} = \frac{x_{k+1}}{\mu_{k+1}} = \frac{F(x_k, u_k, w_k)}{\psi(\mu_k)} = \frac{F(x_k, \kappa(\mu_k(q(\xi_k))), w_k)}{\psi(\mu_k)}$$
$$= \frac{F(\mu_k \xi_k, \kappa(\mu_k \xi_k + \mu_k(q(\xi_k) - \xi_k)), w_k)}{\psi(\mu_k)}$$
$$=: \mathcal{F}_{in}(\xi_k, \mu_k, \nu_k, w_k), \quad (16)$$

where $\nu_k := q(\xi_k) - \xi_k$ with $|\nu_k| \leq \Delta$. We assume that there exists $\psi \in \mathcal{K}_\infty$ such that during the zoom-in stage ξ_k dynamics is ISS with respect to the bounded error ν and the bounded disturbance w, uniformly in μ.

Assumption 5. Assume that for all $\mu_0 > 0$ there exist functions $\tilde{\beta} \in \mathcal{KL}$ and $\tilde{\gamma}_1, \tilde{\gamma}_2 \in \mathcal{K}_\infty$ such that for every initial condition ξ_0 and every ν_k the following holds for the trajectories of the system (16) $\forall k \in [k_{2i+1}, k_{2i+2}]$:

$$|\xi_k| \leq \tilde{\beta}(|\xi_{k_{2i+1}}|, k) + \tilde{\gamma}_1(\|\nu\|_{[k_{2i+1}, k-1]}) + \tilde{\gamma}_2(\|w\|_{[k_{2i+1}, k-1]}).$$

Remark 3. In analogy to Assumption 5 for the nonlinear systems, in Liberzon, Nešić (2007) for the linear systems the authors show in Lemma III.2 that ξ dynamics is ISS with respect to the error, ν, and the disturbance, w.

We remark, that Assumption 5 requires uniformity of ISS property in μ, which can be restrictive. In other words, Assumption 5 is equivalent to the saying that there exists a smooth function $\tilde{V}(\xi, \mu) : \mathbb{R}^n : \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ such that for some class $\mathcal{K}_\infty$ functions $\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3, \tilde{\rho}_1, \tilde{\rho}_2$, for all $\xi, \nu \in \mathbb{R}^n$ we have:

$$\tilde{\alpha}_1(|\xi|) \leq \tilde{V}(\xi, \mu) \leq \tilde{\alpha}_2(|\xi|) \quad (17)$$

and

$$|\xi| \geq \tilde{\rho}_1(|\nu|) + \tilde{\rho}_2(|w|) \Longrightarrow$$
$$\tilde{V}(F_{in}(\xi, \mu, \nu, w), \psi(\mu)) - \tilde{V}(\xi, \mu) \leq -\tilde{\alpha}_3(|\xi|).$$

We remark that in general it is hard to find such $\tilde{V}$.

The following corollary basically says that the range of the quantizer M has to be large enough compared to the quantization error Δ (i.e. the quantizer takes sufficiently many levels).

Corollary 1. Let $\tilde{\beta}, \tilde{\gamma}_1, \tilde{\gamma}_2$ come from Assumption 5 and let strictly positive M and Δ be such that the following holds:

$$M > \tilde{\beta}(\Delta, 0) + \tilde{\gamma}_1(\Delta) + \tilde{\gamma}_2(\Delta) + 2\Delta. \tag{18}$$

Then there exist $\Delta_M > 0$ with $\Delta_M - \Delta > 0$ and $\Delta_w > 0$, such that whenever $|\xi_0| \leq \Delta_M$, $\|\nu\| \leq \Delta$ and $\|w\| < \Delta_w$, we have:

$$|q(\xi_k)| \leq M - \Delta \quad \text{and} \quad |\xi_k| \leq M \quad \forall k \geq 0. \tag{19}$$

Remark 4. Note that if we consider linear system the condition on the data rate

$$M > (2 + \hat{K} + \hat{\gamma}_1 + \hat{\gamma}_2)\Delta \tag{20}$$

(which is the condition used in Liberzon, Nešić (2007)) can be recovered from (18) with

$\tilde{\beta}(|\xi_0|, k) = \hat{K}\exp(-\lambda k)|\xi_0|$, $\tilde{\gamma}_1(\|\nu\|) = \hat{\gamma}_1\|\nu\|$ and $\tilde{\gamma}_2(\|w\|) = \hat{\gamma}_2\|w\|$.

Remark 5. Note, it is not hard to show that the stability bound that is valid at sampling instants t_k can be extended for all $t \geq 0$. We will analyze only the stability properties of the discrete-time system (2) - (9) induced by the sampled-data system (1). It was shown in Nešić *et. al.* (1999) how to use the underlying discrete-time model to conclude appropriate stability properties of the sampled-data system.

4. MAIN RESULTS

The main contributions of our work are presented in this section. We show that the closed loop system (2) - (9) is input-to-state stable in the following sense:

Definition 3. The system (2) is said to be input-to-state stable (ISS) if for all $\mu_0 > 0$ there exist $\gamma_1, \gamma_2, \gamma_3 \in \mathcal{K}_\infty$ such that for any initial conditions x_0 and every bounded disturbance w we have that μ_k is bounded for all $k \geq 0$ and:

$$|x_k| \leq \gamma_1(|x_0|) + \gamma_2(\|w\|) \quad \forall k \geq 0, \tag{21}$$

$$\lim_{k\to\infty} \sup |x_k| \leq \gamma_3(\lim_{k\to\infty} \sup |w_k|). \tag{22}$$

Note, that the gain functions $\gamma_1, \gamma_2, \gamma_3$ may depend on the choice of the initial value μ_0 of the zoom variable μ (but not on x_0 and w). It was shown in Sontag, Wang (1996) that for continuous systems the property expressed by inequalities (21), (22) is equivalent to the input-to-state stability with respect to w. In the present case, the closed-loop system contains an additional state μ and we talk about a partial stability property (in x) of the closed loop system. With some abuse of terminology, we will refer to the previous property as ISS of the closed-loop system.

The main contribution of our work is the following theorem, which presents conditions under which the system (2) - (9) is ISS.

Theorem 1. Consider the system (2) - (9) and let q be a quantizer fulfilling Assumption 1. Let Assumptions 2 - 5 hold. Let

- $\chi(s)$ be such that (11) and (12) hold,
- $\psi(s)$ be such that (15) hold,
- M and Δ be such that (18) holds,
- $l_{out} := M - \Delta$,
- $l_{in} := \Delta_M - \Delta$, where Δ_M comes from Corollary 1.

Then, the system (2) - (9) is ISS from w to x.

Remark 6. The first item of Theorem 1 is a condition on how fast the μ-subsystem has to be during the zoom-out stage; the second is a condition on how slow the μ-subsystem has to be during the zoom-in stage; the third is a condition on the data-rate of the channel; the forth and the fifth are the conditions on the switching parameters.

The proof of Theorem 1 relies on several lemmas that are given bellow. The following Lemma 1 implies that the zoom-out condition can be only triggered for finitely many time steps. Hence, if N is finite, then $k_{2N+2} = \infty$. In other words, there exists a $k_{2N+1} \in \mathbf{N}$ such that the zoom-in condition is triggered on the interval $[k_{2n+1}, \infty)$. Moreover, Lemma 1 establishes a bound on the state x during the zoom-out interval.

Lemma 1. Consider the system (2) - (9). Let q be a quantizer fulfilling Assumption 1. Let all conditions of Theorem 1 hold. Then there exist $\tilde{\varphi}_1, \tilde{\varphi}_2, \rho_1, \rho_2 \in \mathcal{K}_\infty$ such that for all $i = 0, 1, \ldots, N, x_{k_{2i}} \in \mathbb{R}^n, \mu_{k_{2i}} > 0, w \in \mathbb{R}^l$ the following holds:

$$k_{2i+1} - k_{2i} \leq \tilde{\varphi}_1(|x_{k_{2i}}|) + \tilde{\varphi}_2(\|w\|_{[k_{2i}, k_{2i+1}-1]}) \tag{23}$$

$$|x_k| \leq \rho_1(|x_{k_{2i}}|) + \rho_2(\|w\|_{[k_{2i}, k_{2i+1}-1]}), \quad k \in [k_{2i}, k_{2i+1}]. \tag{24}$$

Note, that functions $\tilde{\varphi}_i$ and ρ_i, $i = 1, 2$, are independent of μ.

Proof of Lemma 1. The proof will be carried out by contradiction. Suppose the zoom-out interval is unbounded, that is $k_{2i+1} = \infty$. For all $k \in [k_{2i}, k_{2i+1}]$ by Assumption 2 we have

$$|x_k| \leq \varphi_1(|x_{k_{2i}}|) + \varphi_2(\|w\|_{[k_{2i}, k-1]}) + \varphi_3(k - k_{2i}) + \bar{c}.$$

Dividing both sides of the inequality above by μ_k we have for all $k \in [k_{2i}, k_{2i+1}]$:

$$|\xi_k| = \left|\frac{x_k}{\mu_k}\right| \leq$$

$$\leq \frac{\varphi_1(|x_{k_{2i}}|) + \varphi_2(\|w\|_{[k_{2i}, k-1]}) + \varphi_3(k - k_{2i}) + \bar{c}}{\chi(\mu_{k-1}) + c}$$

$$\leq \frac{\varphi_1(|x_{k_{2i}}|) + \varphi_2(\|w\|_{[k_{2i}, k-1]}) + \varphi_3(k - k_{2i}) + \bar{c}}{\chi^{k-k_{2i}-1}(\mu_{k_{2i}})}.$$

Since the disturbance w is bounded, there exists a time instant $k^* \in [k_{2i}, k_{2i+1}]$ such that the following holds for all $k \in [k^*, k_{2i+1}]$:

$$|\xi_k| \leq \frac{\varphi_1(k - k_{2i}) + \varphi_2(k - k_{2i}) + \varphi_3(k - k_{2i}) + \bar{c}}{\chi^{k-k_{2i}-1}(\mu_{k_{2i}})}$$

$$< \frac{\chi(k - k_{2i})}{\chi(a^{k-k_{2i}-1}\mu_{k_{2i}})},$$

where the last inequality above is due to (11) and (12). Since $a > 1$ for sufficiently large k the following holds: $a^{k-k_{2i}-1}\mu_{k_{2i}} > k - k_{2i}$. We have for all $k \in [k^*, k_{2i+1}]$:

$$\lim_{k\to\infty} \frac{\chi(k - k_{2i})}{\chi(a^{k-k_{2i}-1}\mu_{k_{2i}})} = 0.$$

We can conclude, that the variable ξ_k is decreasing and eventually we must have $|\xi_k| < l_{in} - \Delta$, which implies that $|\mu_k q(\xi_k)| < l_{in}\mu_k$ and the zoom-in stage is triggered in a finite time. Hence, we came to the contradiction of the claim that $k_{2i+1} = \infty$ and we can conclude that $k_{2i+1} - k_{2i} - 1$ is bounded. Moreover, we can write for some φ continuous nondecreasing and bounded function that

$$k_{2i+1} - k_{2i} - 1 \le \varphi(|x_{k_{2i}}|, \|w\|_{[k_{2i},k_{2i+1}-1]}). \quad (25)$$

Note, that we can let $\varphi(0,0) = 0$ since if $x_{k_{2i}} = 0$ then $k_{2i+1} - k_{2i} = 1$. Hence, we can find $\tilde{\varphi}_1, \tilde{\varphi}_2 \in \mathcal{K}_\infty$ so that (23) holds. Note also that for all $k \in [k_{2i}, k_{2i+1}]$ we have

$$\begin{aligned}|x_k| &\le \varphi_1(|x_{k_{2i}}|) + \varphi_2(\|w\|_{[k_{2i},k_{2i+1}-1]}) + \varphi_3(k) + c\\ &\le \varphi_1(|x_{k_{2i}}|) + \varphi_2(\|w\|_{[k_{2i},k_{2i+1}-1]}) + \bar{\varphi}_3(k_{2i+1} - k_{2i})\\ &\le \varphi_1(|x_{k_{2i}}|) + \varphi_2(\|w\|_{[k_{2i},k_{2i+1}-1]})\\ &\quad + \bar{\varphi}_3(\tilde{\varphi}_1(|x_{k_{2i}}|) + \tilde{\varphi}_2(\|w\|_{[k_{2i},k_{2i+1}-1]}))\\ &\le \rho_1(|x_{k_{2i}}|) + \rho_2(\|w\|_{[k_{2i},k_{2i+1}-1]}),\end{aligned}$$

where $\rho_1, \rho_2 \in \mathcal{K}_\infty$ are such that $\rho_1(s) := \varphi_1(s) + \bar{\varphi}_3(2\tilde{\varphi}_1(s))$ and $\rho_2(s) := \varphi_2(s) + \bar{\varphi}_3(2\tilde{\varphi}_2(s))$. ■

The following Lemma 2 establishes a bound on μ at the end of each zoom-out interval in terms of the values of μ and x at the beginning of that interval and the infinity norm of the disturbance during the zoom-in interval.

Lemma 2. Consider the system (2) - (9) and let q be a quantizer fulfilling Assumption 1. Let all conditions of Theorem 1 hold. Then there exists a continuous bounded function ρ_μ^{out} such that for any $\mu > 0$ we have $\rho_\mu^{out}(\mu, 0, 0) > 0$ and the following is true for all $i = 0, 1, \ldots, N$ and all $\mu_{k_{2i}} > 0$, $x_{k_{2i}} \in \mathbb{R}^n$, $w \in \mathbb{R}^l$:

$$|\mu_{k_{2i+1}}| \le \rho_\mu^{out}(\mu_{k_{2i}}, |x_{k_{2i}}|, \|w\|_{[k_{2i},k_{2i+1}-1]})$$

Proof of Lemma 2. Note that we can find a continuous bounded strictly increasing function $\tilde{\chi}$ such that $\tilde{\chi}(s) > \chi(s) + c$. Then we have

$$|\mu_{k_{2i+1}}| < \tilde{\chi}^{k_{2i+1}-k_{2i}}(\mu_{k_{2i}}).$$

Since $k_{2i+1} - k_{2i} - 1$ is bounded by (23), we can find continuous bounded function ρ_μ^{out} such that the following holds:

$$\begin{aligned}\mu_{k_{2i+1}} &\le \tilde{\chi}^{\lceil \tilde{\varphi}_1(|x_{k_{2i}}|) + \tilde{\varphi}_2(\|w\|_{[k_{2i},k_{2i+1}-1]})\rceil + 1}(\mu_{k_{2i}})\\ &\le \rho_\mu^{out}(\mu_{k_{2i}}, |x_{k_{2i}}|, \|w\|_{[k_{2i},k_{2i+1}-1]}). \quad (26)\end{aligned}$$

■

The following Lemma 3 establishes an appropriate bound on the state x during the zoom-in intervals. This bound is a direct consequence of the fact that during the zoom-in interval the system behaves as a cascade of $x-$ and $\mu-$subsystems. The $x-$subsystem is ISS when μ is regarded as an input, and the $\mu-$subsystem is globally asymptotically stable (GAS).

Lemma 3. Consider the system (2) - (9) and let q be a quantizer fulfilling Assumption 1. Let all conditions of Theorem 1 hold. Then there exist functions $\bar{\beta} \in \mathcal{KL}$ and $\gamma_1 \in \mathcal{K}_\infty$ such that for all $k \in [k_{2i+1}, k_{2i+2}]$ the following holds:

$$|x_k| \le \bar{\beta}(|x_{k_{2i+1}}| + \mu_{k_{2i+1}}, k) + \gamma_1(\|w\|_{[k_{2i+1},k-1]}). \quad (27)$$

Proof of Lemma 3. Consider the closed-loop system

$$x_{k+1} = F(x_k, \kappa(x_k + e_k), w_k),$$

where the measurement error e_k is defined as $e_k = |\mu_k q(x_k/\mu_k) - x_k|$. During the zoom-in interval $[k_{2i+1}, k_{2i+2}]$ due to Assumption 4 the $x-$subsystem satisfies

$$|x_k| \le \beta(|x_{k_{2i+1}}|, k) + \gamma(\|e\|_{[k_{2i+1},k-1]}) + \bar{\gamma}(\|w\|_{[k_{2i+1},k-1]}),$$

and the $\mu-$subsystem evolves according to

$$\mu_{k+1} = \psi(\mu_k).$$

The $x-$subsystem is ISS when μ is regarded as an input, and the $\mu-$subsystem is globally asymptotically stable. This is a cascade of an ISS and GAS systems and, hence, the overall system during the zoom-in interval is ISS and (27) follows immediately. ■

The following Lemma 4 establishes different bound on the state x during the zoom-in intervals.

Lemma 4. Consider the system (2) - (9) and let q be a quantizer fulfilling Assumption 1. Let all conditions of Theorem 1 hold. Then there exists a continuous function $\rho_x^{in} : \mathbb{R}_{>0} \times \mathbb{R}_{\ge 0} \times \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$, with $\rho_x^{in}(\mu, 0, 0) = 0$ for all $\mu > 0$, and such that for any $s \ge 0$, $\rho_x^{in}(\cdot, \cdot, s)$ is nondecreasing in its first two arguments and for any $i = 0, 1, \ldots, N$, the following holds for all $\mu_{k_{2i+1}}, x_{k_{2i+1}}, w$ $\forall k \in [k_{2i+1}, k_{2i+2}]$:

$$|x_k| \le \rho_x^{in}(\mu_{k_{2i+1}}, |x_{k_{2i+1}}|, \|w\|_{[k_{2i+1},k_{2i+2}-1]}). \quad (28)$$

Proof of Lemma 4. In order to obtain the desired bound, we consider two cases:

- Case 1: $|x_{k_{2i+1}}| \ge \|w\|_{[k_{2i+1},k_{2i+2}-1]}$,
- Case 2: $|x_{k_{2i+1}}| \le \|w\|_{[k_{2i+1},k_{2i+2}-1]}$.

Case 1: Introduce T_x^* such that

$$\psi^{T_x^*}(\mu_{k_{2i+1}}) \le \mu_{k_{2i+1}}|x_{k_{2i+1}}|.$$

Hence, for all $k \ge k_{2i+1} + T^*$ we have

$$\begin{aligned}|x_k| &\le M\mu_k \le M\psi^{k-k_{2i+1}}(\mu_{k_{2i+1}}) \le M\psi^{T_x^*}(\mu_{k_{2i+1}})\\ &\le M\ \mu_{k_{2i+1}}\ |x_{k_{2i+1}}| =: \chi_1^x(\mu_{k_{2i+1}}, |x_{k_{2i+1}}|),\end{aligned}$$

where $\chi_1^x(\mu, \cdot) \in \mathcal{K}_\infty$ for each fixed $\mu > 0$.

Note that for such T_x^* Assumption 3 holds and for all $k \in [k_{2i+1}, k_{2i+1} + T_x^*]$ we have:

$$\begin{aligned}|x_k| &\le \chi_1(|x_{k_{2i+1}}|) + \chi_2(\|w\|_{[k_{2i+1},k-1]})\\ &\le \chi_1(|x_{k_{2i+1}}|) + \chi_2(|x_{k_{2i+1}}|) := \chi_2^x(|x_{k_{2i+1}}|).\end{aligned}$$

Note that $\chi_2^x(\cdot) \in \mathcal{K}_\infty$. Let $\chi^x(\mu, s) := \chi_1^x(\mu, s) + \chi_2^x(s)$, then for all $k \in [k_{2i+1}, k_{2i+2}]$ we have:

$$|x_k| \le \chi^x(\mu_{k_{2i+1}}, |x_{k_{2i+1}}|),$$

where $\chi^x(\mu, \cdot) \in \mathcal{K}_\infty$ for each fixed $\mu > 0$.

Case 2: The proof of this case follows exactly the same steps as the proof of Case 1 with the following modification. Introduce T_w^* such that

$$\psi^{T_w^*}(\mu_{k_{2i+1}}) \le \mu_{k_{2i+1}}\|w\|_{[k_{2i+1},k_{2i+2}-1]}.$$

Hence, for all $k \ge k_{2i+1} + T_w^*$ we have

$$\begin{aligned}|x_k| &\le M\mu_k \le M\psi^{k-k_{2i+1}}(\mu_{k_{2i+1}}) \le M\psi^{T_x^*}(\mu_{k_{2i+1}})\\ &\le M\ \mu_{k_{2i+1}}\ \|w\|_{[k_{2i+1},k_{2i+2}-1]}\\ &=: \chi_1^w(\mu_{k_{2i+1}}, \|w\|_{[k_{2i+1},k_{2i+2}-1]}),\end{aligned}$$

where $\chi_1^w(\mu, \cdot) \in \mathcal{K}_\infty$ for each fixed $\mu > 0$.

Note that for such T_x^* Assumption 3 holds and for all $k \in [k_{2i+1}, k_{2i+1} + T_w^*]$ we have:

$$\begin{aligned}|x_k| &\le \chi_1(|x_{k_{2i+1}}|) + \chi_2(\|w\|_{[k_{2i+1},k-1]})\\ &\le \chi_1(\|w\|_{[k_{2i+1},k_{2i+2}-1]}) + \chi_2(|x_{k_{2i+1}}|)\\ &:= \chi_2^w(\|w\|_{[k_{2i+1},k_{2i+2}-1]}).\end{aligned}$$

Note that $\chi_2^w(\cdot) \in \mathcal{K}_\infty$. Let $\chi^w(\mu, s) := \chi_1^w(\mu, s) + \chi_2^w(s)$, then for all $k \in [k_{2i+1}, k_{2i+2}]$ we have:

$$|x_k| \le \chi^w(\mu_{k_{2i+1}}, \|w\|_{[k_{2i+1},k_{2i+2}-1]}),$$

where $\chi^w(\mu, \cdot) \in \mathcal{K}_\infty$ for each fixed $\mu > 0$. The conclusion of the lemma follows by defining $\rho_x^{in}(\mu, s, p) := \chi^x(\mu, s) + \chi^w(\mu, p)$ and noting that χ^x and χ^w are nondecreasing in μ. ■

The following Lemma 5 establishes that if the zoom-in interval is bounded (i.e., is followed by the zoom-out interval) then at the end of the zoom-in interval we have that x and μ are bounded by the function of the disturbance only, i. e., the initial conditions are "forgotten".

Lemma 5. Consider the system (2) - (9) and let q be a quantizer fulfilling Assumption 1. Let all conditions of Theorem 1 hold. Consider arbitrary $i \in \{0, 1, \ldots, N\}$. If $k_{2i+2} < +\infty$, then $i < N-1$ and there exists $\tilde{\gamma} \in \mathcal{K}_\infty$ such that the following holds:

$$\max\left\{|x_{k_{2i+2}}|, \mu_{k_{2i+2}}\right\} \le \tilde{\gamma}(\|w\|_{[k_{2i+1},k_{2i+2}-1]}). \qquad (29)$$

Proof of Lemma 5. The inequality $i < N-1$ follows by the definition of N. Note, that by Corollary 1 a zoom-out can occur after a zoom-in only if there exists $k^* \in [k_{2i+1}, k_{2i+2}-1]$ such that

$$\Delta_w^{-1}|w_{k^*}| \ge \mu_{k^*}.$$

Indeed, if $\Delta_w^{-1}|w_k| \le \mu_k$ for all k during the zoom-in interval, then we have from Corollary 1 $|\zeta_k| = |w_k/\mu_k| \le \Delta_w$ and hence $|x_k| \le M\mu_k$ for all k. Moreover, during the zoom-in interval we must have $|x_{k^*}| \le M\mu_{k^*}$ and also

$$\Delta_w^{-1}M|w_{k^*}| \ge |x_{k^*}|.$$

Using (27) with $k = k^*$ we have:

$$|x_{k_{2i+2}}| \le \beta(|x_{k^*}| + \mu_{k^*}, k^*) + \bar{\gamma}(\|w\|_{[k_{2i+1},k_{2i+2}-1]})$$
$$\le \beta(\Delta_w^{-1}M|w_{k^*}| + \Delta_w^{-1}|w_{k^*}|, k^*) + \bar{\gamma}(\|w\|_{[k_{2i+1},k_{2i+2}-1]}).$$

From here we can find a function $\tilde{\gamma} \in \mathcal{K}_\infty$ such that (29) holds. ■

PROOF OF THEOREM 1. The proof is almost identical to the proof of Theorem 2 in Liberzon, Nešić (2007) with the only difference that $\tilde{\gamma}$ in (29) is $\mathcal{K}_\infty$, not the positive constant as in Liberzon, Nešić (2007).

5. CONCLUSIONS

This paper is an extension of Liberzon, Nešić (2007) to nonlinear systems. This paper addresses the stabilization problem for nonlinear feedback systems with quantized feedback in a presence of bounded disturbances. Using the trajectory-based scheme proposed in Liberzon, Nešić (2007) we derived the conditions under which the nonlinear plant is input-to-state stable with respect to bounded disturbances. Possible future directions for research include nonlinear gain l_p stabilization of nonlinear systems and considering the mismatch in the initialization of the adjustable parameter μ at the coder and at the decoder for nonlinear systems.

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