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Finite-Time Consensus for Multi-Agent Networks with Second-Order Agent Dynamics ⋆

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Abstract: This paper considers the finite-time consensus problem for a multi-agent system with second-order individual dynamics. Local (non-smooth) time-invariant consensus protocols in different forms are constructed for each double-integrator agent dynamics in a quite unified way with help of Lyapunov function, graph theory, and homogeneity with dilation. Finite-time consensus can be obtained theoretically via the proposed non-smooth but continuous forms of distributed coordination controllers. Also, numerical analysis is given for illustration.

1. INTRODUCTION

Recent years have witnessed a large and growing literature concerned with the coordination of a group of mobile autonomous agents, partly due to a broad application of multi-agent systems including flocking and formation (e.g., Egerstedt et al. (2001); Reynolds (1987); Lin et al. (2005); Tanner et al. (2003); Olfati-Saber (2006); Hong et al. (2006a)). Distributed control analysis and design have been widely used and developed very fast in multi-agent systems. In some applications, a team of agents are required to agree upon certain quantities of interest, which is often called consensus or agreement problem (Cortés (2006); Hong et al. (2007); Ren et al. (2005)). To achieve the aim, suitable neighbor-based rules are usually adopted to interconnect the considered agents. Many results have been obtained with local rules applied to each agent in a considered multi-agent system. These neighbor rules for each agent are based on the average of its own information and that of its neighbors.

On the other hand, various finite-time stabilizing control laws have been proposed using continuous state feedback and output feedback controllers Bhat et al. (1998, 2000); Hong et al. (2001). Moreover, the finite-time control design has been extended to nth order systems with both parametric and dynamic uncertainties Hong et al. (2006), though the finite-time design is generally more difficult than that of asymptotically stabilizing control for the lack of effective analysis tools. Non-smooth finite-time control synthesis can improve the system behaviors in some aspects like high-speed, control accuracy, and disturbance-rejection. Thus, it is not surprising that finite-time control ideas have been applied to multi-agent systems with first-order agent dynamics using gradient flow and Lyapunov function Cortés (2006); Xiao et al. (2007).

In this paper, we will propose finite-time consensus protocols for multi-agent systems with each individual with second-order dynamics. In fact, the consensus analysis of second-order agent dynamics is more difficult than that of first-order dynamics, but it becomes a hot topic in multi-agent control (such as Olfati-Saber (2006); Hong et al. (2007); Tanner et al. (2003)) because many practical individual systems, especially mechanical systems, are of second-order dynamics. Using Lyapunov-based method and homogeneous properties, we design finite-time distributed control for multi-agent systems. To be specific, we assume each continuous-time agent with dynamics in the form of $\ddot{x}_i(t) = u_i(t)$, $t \geq 0$, or equivalently

$$\begin{cases} \dot{x}_i = v_i \\ \dot{v}_i = u_i, \quad i \in I_n = \{1, 2, ..., n\} \end{cases} \tag{1}$$

with initial conditions $x_i(0) = x_{i0}$, $v_i(0) = v_{i0}$, where $x_i(t) \in R^m$ denotes the position, $v_i(t) \in R^m$ the velocity, and $u_i(t) \in R^m$ the control input. Our objective of this paper is to seek a continuous distributed consensus protocol u_i involving information transmission of x_i and v_i between agents so that finite-time consensus is achieved.

The paper is organized as follows. First, problem formulation and preliminary results are given in Section 2. Then we focus on the finite-time design for a multi-agent network of second-order agent dynamics in Section 3 and Section 4, where finite-time consensus protocols are constructed to make the multi-agent system achieve consensus in finite time. Following that, an illustrative example is given. Finally, the paper is ended by concluding remarks.

2. PROBLEM FORMULATION

This paper is to deal with the finite-time consensus of multi-agent system (1) via time-invariant neighbor-based feedback laws (called consensus protocols) using the information from its neighbors.

To deal with multi-agent consensus, graph theory usually provides a big help. The information consensus problem appears frequently in coordination of multi-agent systems and involves finding a dynamic algorithm that enables a group of agents in a network to agree upon certain quantities of interest with undirected or directed information flow. In this paper, undirected graphs are employed

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to represent a multi-agent network in the study of its consensus problem based on its graph topologies (or information flow). Specifically, an undirected graph $\mathcal{G}$ consists of a (finite nonempty) set of nodes (representing agents), denoted by $\mathcal{V} = \{\pi_1, ..., \pi_n\}$ and the set of unordered pairs called edges, denoted by $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. If $(\pi_i, \pi_j) \in \mathcal{E}$, then π_i is said to be a neighbor of π_j and the set of all neighbor vertices of vertex π_j is denoted by $\mathcal{N}_j = \{i | (\pi_i, \pi_j) \in \mathcal{E}\}$. The weighted adjacency matrix $A = (a_{ij})_{n\times n} \in R^{n\times n}$ of a weighted undirected graph is defined on the following form: $a_{ij} = a_{ji}, \forall i \neq j$, since $(\pi_j, \pi_i) \in \mathcal{E}$ implies $(\pi_i, \pi_j) \in \mathcal{E}$. The weight between π_i and π_j, a_{ij}, is a positive constant, (which is set equal to 1 for simplicity in this paper) for all $(\pi_j, \pi_i) \in \mathcal{E}$ or $j \in \mathcal{N}_i$, and $a_{ij} = 0$ if there is no edge between π_i and π_j ($a_{ii} = 0$ since self loops are not allowed in a simple graph). The undirected graph is called connected if there is a path between any two vertices of the graph.

Finite-time stability of some equilibria of nonlinear systems has been studied in the literature (for example, Bhat et al. (2000); Haimo (1986); Hong et al. (2001, 2006)). Here we study a related problem for multi-agent systems, the global finite-time consensus.

Consider a multi-agent system in the form of (1) with denoting $X_i = (x_i^T, v_i^T)^T \in R^{2m}$. Without loss of generality in the consensus analysis, we take $m = 1$ in the sequel.

Suppose that, with a given consensus protocol u_i of the closed-loop system (1), for any initial condition $X_i(0) \in U$, where U is a neighborhood of the set $\{X_i = X_j,\ \forall i, j\}$, there is a settling time $T \in [0, \infty)$, such that its solution $X_i(t; 0, X_i(0))$ of system (1) is defined and $X_i(t; 0, X_i(0)) \in U/\{0\}$ for $t \in [0, T)$, and satisfies

$$\lim_{t\to T} |X_i(t; 0, X_i(0)) - X_j(t; 0, X_j(0))| = 0,$$

and

$$X_i(t; 0, X_i(0)) = X_j(t; 0, X_j(0)), \quad \forall t \geq T.$$

Then we say local finite-time consensus is achieved. Here T is called the settling time. When $U = R^{2n}$, then the (global) finite-time consensus is achieved.

Obviously, finite-time consensus is closely related to finite-time stability. The main difference between the two problems is that finite-time consensus is to make the considered multi-agent system converge to an invariant manifold (described by $\{X_i = X_j, \forall i, j\}$ or equivalently, $\{x_i = x_j,\ v_i = v_j, \forall i, j\}$. As we know, finite-time stability naturally results in non-smoothness, and so is finite-time consensus. In the study of non-smooth dynamics, some conventional results for stability analysis cannot be applied directly. For example, the celebrated LaSalle Invariance Principle was given first for smooth systems. In non-smooth cases, different extension versions of this principle have been given. Here we use a version of the non-smooth LaSalle Invariance Principle given in Rouche et al. (1977).

Lemma 1. Let $x(t)$ be a solution of $\dot{x} = f(x)$, $x(0) = x_0 \in R^k$, where $f : U \to R^k$ is continuous with U an open subset of R^k, and let $V : U \to R$ be a locally Lipschitz function such that $D^+V(x(t)) \leq 0$, where D^+ denotes the upper Dini derivative (referring to Rouche et al. (1977)). Then, with denoting the positive limit set as $\Lambda^+(x_0)$, $\Lambda^+(x_0) \cap U$ is contained in the union of all solutions that remain in $S = \{x \in U\ :\ D^+V(x) = 0\}$.

Note that Lemma 1 does not need the uniqueness of the solution to the considered (non-smooth) system.

Next, let us introduce the homogeneity with dilation (see Rosier (1992) for details) for finite-time convergence analysis.

A function $V(x)$ of $x \in R^k$ is homogeneous of degree $\sigma \geq 0$ with dilation coefficients $(r_1, ..., r_k)$, if

$$V(\epsilon^{r_1} x_1, ..., \epsilon^{r_k} x_k) = \epsilon^{\sigma} V(x), \quad \epsilon > 0.$$

If $r_1 = ... = r_k = 1$, then the dilation is called trivial.

Consider k-dimensional system

$$\dot{x} = f(x), \quad x = (x_1, ..., x_k)^T \in R^k \tag{2}$$

A (continuous) vector field $f(x) = (f_1(x), ..., f_k(x))^T$ is homogeneous of degree $\sigma \in R$ with dilation $(r_1, ..., r_k)$, if

$$f_i(\epsilon^{r_1} x_1, ..., \epsilon^{r_k} x_k) = \epsilon^{\sigma + r_i} f_i(x), \quad i = 1, ..., k, \quad \epsilon > 0.$$

Definition 1. System (2) is called homogeneous if its vector field is homogeneous. Moreover,

$$\dot{x} = f(x) + \hat{f}(x), \quad \hat{f}(0) = 0, \quad x \in R^k \tag{3}$$

is called locally homogeneous if f is homogeneous of degree $\sigma \in R$ with dilation $(r_1, ..., r_k)$ and $\hat{f}$ is a continuous vector field satisfying

$$\lim_{\epsilon\to 0} \frac{\hat{f}_i(\epsilon^{r_1} x_1, ..., \epsilon^{r_k} x_k)}{\epsilon^{\sigma + r_i}} = 0, \quad \forall x \neq 0, \quad i = 1, ..., k. \tag{4}$$

Sometimes, system (2) is called the leading homogeneous system of system (3).

For convenience, in the sequel, set

$$sig(y)^{\alpha} = |y|^{\alpha} sgn(y), \quad \alpha > 0$$

where $sgn(\cdot)$ denotes the sign function and $|y|$ denotes the absolute value of real number y as Haimo (1986) did. Clearly,

$$\frac{d}{dy}|y|^{\alpha+1} = (\alpha + 1) sig(y)^{\alpha}$$

and

$$\frac{d}{dy} sig(y)^{\alpha+1} = (\alpha + 1)|y|^{\alpha} \qquad \alpha > 0$$

Several lemmas are given for the finite-time analysis in the following.

The following lemma has been known (see Bhat et al. (1998, 2000); Hong (2002)).

Lemma 2. Suppose system (2) is homogeneous of degree σ with dilation $(r_1, ..., r_k)$, f is continuous and $x = 0$ is its asymptotically stable equilibrium. If homogeneity degree $\sigma < 0$, the equilibrium of system (2) is finite-time stable. Moreover, if (4) holds, then the equilibrium of system (3) is locally finite-time stable.

In multi-agent systems design, for agent i, u_i only depends on its own variables (*i.e.*, x_i and v_i) and the relative difference between its own variables and other of its neighbors (*i.e.*, $x_i - x_j$ and $v_i - v_j$ for some $j \in \mathcal{N}_i$)

Set $z_i = x_i - x_{i+1}$, $w_i = v_i - v_{i+1}$, $i = 1, ..., n-1$. Then the system (1) can be expressed in $2(n-1)$ equations with variables z_i, w_i ($i = 1, ..., n-1$), that is,

$$\begin{cases} \dot{z}_i = w_i \\ \dot{w}_i = u_i - u_{i+1}, \quad i = 1, ..., n-1. \end{cases} \tag{5}$$

In what follows, the considered consensus problem of system (1) can be viewed somehow as the stability problem of system (5).

The next lemma is quite obvious, and the proof is omitted here.

Lemma 3. Suppose that, for some given protocol $u_i(x,v),\ i = 1,...,n$, system (1) with $(x_1,...,x_n,v_1,...,v_n)$ is homogeneous of degree σ with dilation $(\underbrace{r_1,...,r_1}_{n},\underbrace{r_2,...,r_2}_{n})$. Then, with the same protocol, system (5) with variables $(z_1,...,z_{n-1},w_1,...,w_{n-1})$ is also homogeneous of homogeneity degree σ with dilation $(\underbrace{r_1,...,r_1}_{n-1},\underbrace{r_2,...,r_2}_{n-1})$.

3. FINITE-TIME CONSENSUS

In this section, we consider the finite-time consensus problem for multi-agent system (1).

Here is our main result.

Theorem 1. Suppose that the undirected graph $\mathcal{G}$ with A of system (1) is connected. Then the globally finite-time stable is achieved under the distributed consensus protocol given in the following form:

$$u_i = \sum_{j=1}^{n} a_{ij}[\psi_1(sig(x_j - x_i)^{\alpha_1}) + \psi_2(sig(v_j - v_i)^{\alpha_2})], \quad (6)$$

with

$$0 < \alpha_1 < 1, \quad \alpha_2 = \frac{2\alpha_1}{1+\alpha_1} \quad (7)$$

where ψ_1 and ψ_2 are continuous odd functions with $y\psi_i(y) > 0$ $(\forall y \neq 0 \in R)$ and $\psi_i(y) = c_i y + o(y)$ (around $y = 0$) for some positive numbers c_i $(i = 1,2)$.

Proof. Take a Lyapunov function

$$V = \sum_{i=1}^{n}\sum_{j=1}^{n}\int_0^{x_i - x_j} a_{ij}\psi_1(sig(s)^{\alpha_1})ds + \frac{1}{2}\sum_{i=1}^{n} v_i^2, \quad (8)$$

which is positive definite with respect to $x_i - x_j$ $(\forall i \neq j)$ and v_i $(\forall i \in I_n)$.

Note that $sig(x_i - x_j)^{\alpha}$ is an odd function. Consider the derivative of V along the trajectories of the closed-loop system, and then we have

$$\begin{aligned}
\dot{V} &= \sum_{i=1}^{n} v_i\dot{v}_i + \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\psi_1(sig(x_i - x_j)^{\alpha_1})v_i \\
&= \sum_{i=1}^{n} v_i \sum_{j=1}^{n} a_{ij}[\psi_1(sig(x_j - x_i)^{\alpha_1}) + \psi_2(sig(v_j - v_i)^{\alpha_2})] \\
&\quad + \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\psi_1(sig(x_i - x_j)^{\alpha_1})v_i \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} v_i a_{ij}\psi_2(sig(v_j - v_i)^{\alpha_2}) \\
&= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}[(a_{ij} + a_{ij})v_i\psi_2(sig(v_j - v_i)^{\alpha_2})] \\
&= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(v_i - v_j)a_{ij}\psi_2(sig(v_j - v_i)^{\alpha_2}) \le 0.
\end{aligned}$$

since $(v_i - v_j)sig(v_j - v_i)^{\alpha_2} = |v_i - v_j|^{1+\alpha_2}$.

Then we employ LaSalle invariance principle (Lemma 1). Since the constructed Lyapunov function is smooth, Dini derivative becomes the regular derivative. Denote the invariant set $S = \{(x_1, v_1, ..., x_n, v_n) | \dot{V} = 0\}$. Note that when the undirected graph is connected, $\dot{V} \equiv 0$ implies that $v_i \equiv v_j = \bar{v},\ \forall j \neq i$, which in turn implies that $u_i = u_j,\ \forall j \neq i$. Because $v_i \equiv v_j,\ \forall j \neq i$, it follows from (6) that

$$u_i = \sum_{j=1}^{n} a_{ij}\psi_1(sig(x_j - x_i)^{\alpha_1}), \quad i \in I_n.$$

Moreover, $\sum_{i=1}^{n} u_i = 0$ because $a_{ij} = a_{ji}$, which implies $u_i \equiv 0, \quad i \in I_n$, which in turn implies that

$$\sum_{j=1}^{n} a_{ij}\psi_1(sig(x_i - x_j)^{\alpha_1}) \equiv 0.$$

Then we obtain

$$\sum_{i=1}^{n} x_i \sum_{j=1}^{n} a_{ij}\psi_1(sig(x_i - x_j)^{\alpha_1}) = 0.$$

Then we have

$$\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x_i - x_j)\psi_1(sig(x_i - x_j)^{\alpha_1}) = 0.$$

Since undirected graph of A is connected, we have $x_j = x_i, \quad \forall j \neq i$. (The proof is similar to Theorem 1 of Olfati-Saber et al. (2004)). Thus, we obtain $v_i = v_j \equiv \bar{v},\ x_i = x_j = \bar{v}t + \bar{x},\ \forall j \neq i$, where $\bar{v}$ and $\bar{x}$ are some constants. Thus, according to Lemma 1, $x_i - x_j \to 0,\ v_i - v_j \to 0,\ \forall i,j \in I_n$ as $t \to \infty$.

According to the assumptions given to odd functions ψ_1 and ψ_2 (that is, $\psi_i(y) = c_i y + o(y)$ $(i = 1,2)$), we can rewrite the protocol (6) as $u_i = u_i^0 + \hat{u}_i$ with

$$u_i^0 = \sum_{j=1}^{n} a_{ij}[c_1 sig(x_j - x_i)^{\alpha_1} + c_2 sig(v_j - v_i)^{\alpha_2}]$$

$$\hat{u}_i = \sum_{j=1}^{n}[o(sig(x_j - x_i)^{\alpha_1}) + o(sig(v_j - v_i)^{\alpha_2})].$$

With u_i^0 (by setting $\hat{u}_i \equiv 0$), it is easy to find that system (1) with variables $(x_1,...,x_n,v_1,...,v_n)$ is homogeneous of degree $\sigma = \alpha_1 - 1 < 0$ with dilation $(\underbrace{2,...,2}_{n},\underbrace{1+\alpha_1,...,1+\alpha_1}_{n})$. Therefore, system (1) is locally homogeneous of degree σ with the same dilation under the protocol $u_i^0 + \hat{u}_i$ (that is, protocol (6). By Lemma 3, it is not hard to see that system (5) with variables $(z_1,...,z_{n-1},w_1,...,w_{n-1})$ is also locally homogeneous of the same homogeneity degree with dilation $(\underbrace{2,...,2}_{n-1},\underbrace{1+\alpha_1,...,1+\alpha_1}_{n-1})$.

Consider a modified Lyapunov function

$$V_0 = \sum_{i=1}^{n}\sum_{j=1}^{n}\int_0^{x_i - x_j} a_{ij}\psi_1(sig(s)^{\alpha_1})ds + \frac{1}{2}\sum_{i=1}^{n}(v_i - \bar{v})^2.$$

Considering the derivative of V_0 and following almost the same arguments as above, we can show the Lyapunov stability of the invariant manifold $\{x_i = x_j,\ v_i = v_j,\ \forall i, j\}$ for system (1), which implies the Lyapunov stability of the origin of system (5). Thus, the origin of system (5) is globally asymptotically stable.

To show finite-time convergence, let us focus on system (5). From the above discussion, system (5) is (globally) asymptotically stable and also locally homogeneous with degree $\sigma < 0$. From Lemma 2, system (5) is locally finite-time stable.

Note that, if the equilibrium of a nonlinear system is globally asymptotically stable and locally finite-time convergent, then it is globally finite-time stable. The reason is that globally asymptotical stability implies finite-time convergence to any given bounded neighborhood of the equilibrium. Based on this observation, system (5) is globally finite-time stable. In other words, for system (1), we have $x_i - x_j \to 0,\ v_i - v_j \to 0,\ \forall i, j = 1, ..., n$ in finite time. Thus, the conclusion follows. ∎

Remark 1. If $\alpha_1 = 1$, then the finite-time consensus becomes the asymptotical consensus (from the proof of Theorem 1) and (6) becomes a conventional consensus protocol. Moreover, with protocol (6), the steady-state velocity of all the agents will be $\bar{v} = \sum_{i=1}^n v_i(0)/n$ since $\sum_{i=1}^n \dot{v}_i(t) \equiv 0$, and the collective motion in the steady state will be $x_i(t) = \sum_{i=1}^n v_i(0)t/n + \sum_{i=1}^n x_i(0)/n$.

Note that ψ_i $(i = 1, 2)$ are very general functions. In fact, we can take some special forms of ψ_i $(i = 1, 2)$. For example, we will take a simple form of consensus protocol as follows:

$$u_i = \sum_{j=1}^n a_{ij}[sig(x_j - x_i)^{\alpha_1} + sig(v_j - v_i)^{\alpha_2}]. \quad (9)$$

Moreover, (6) is not necessary to be bounded. In other words, if we choose $\psi_i(\cdot)$ as bounded functions, then we have consensus protocols in saturation forms, for example,

$$u_i = \sum_{j=1}^n a_{ij}[\tanh(sig(x_j - x_i)^{\alpha_1}) + \tanh(sig(v_j - v_i)^{\alpha_2})],$$

or

$$u_i = \sum_{j=1}^n a_{ij}[sat(sig(x_j - x_i)^{\alpha_1}) + sat(sig(v_j - v_i)^{\alpha_2})], \quad (10)$$

where $sat(\cdot)$ denotes the well-known (non-smooth) saturation function. Thus, finite-time consensus can be achieved by bounded protocols.

Remark 2. In some cases (for example, flocking or rendezvous), we may also need to make $v_i = 0$ $(1, ..., n)$ in finite time. Still suppose that the undirected graph $\mathcal{G}$ with A of system (1) is connected. Then the globally finite-time stable is achieved under the distributed consensus protocol as follows:

$$u_i = \sum_{j=1}^n a_{ij}\psi_1(sig(x_j - x_i)^{\alpha_1}) - \psi_2(sig(v_i)^{\alpha_2}) \quad (11)$$

with (7), that is, $0 < \alpha_1 < 1, \alpha_2 = \frac{2\alpha_1}{1+\alpha_1}$, where ψ_1 and ψ_2 are continuous odd functions with $y\psi_i(y) > 0$ $(\forall y \neq 0 \in R)$ and $\psi_i(y) = c_i y + o(y)$ for some positive number c_i $(i = 1, 2)$. The proof is similar to Theorem 1. Still consider the Lyapunov function (8). Differentiating V along the trajectories of the closed-loop system is given by

$$\begin{aligned}\dot{V} &= \sum_{i=1}^n \sum_{j=1}^n a_{ij}\psi_1(sig(x_j - x_i)^{\alpha_1})v_i + \sum_{i=1}^n v_i \dot{v}_i \\ &= \sum_{i=1}^n \sum_{j=1}^n a_{ij}\psi_1(sig(x_j - x_i)^{\alpha_1})v_i \\ &+ \sum_{i=1}^n v_i \left[\sum_{j=1}^n a_{ij}\psi_1(sig(x_j - x_i)^{\alpha_1}) - \psi_2(sig(v_i)^{\alpha_2})\right] \\ &= -\sum_{i=1}^n v_i a_{ij}\psi_2(sig(v_i)^{\alpha_2}) \leq 0,\end{aligned}$$

which straightforwardly implies the Lyapunov stability of the origin of system

$$\dot{z}_i = v_i - v_{i+1} \quad \dot{v}_i = u_i, \quad i \in I_n = \{1, 2, ..., n\} \quad (12)$$

Note that $\dot{V} \equiv 0$ implies that $v_1 = ... = v_n = 0$, and then $u_i = \sum_{j=1}^n a_{ij}\psi_1(sig(x_j - x_i)^{\alpha_1}) \equiv 0$. Then, system (12) is globally finite-time stable. In other words, for system (1), $x_i - x_j \to 0,\ v_i \to 0,\ \forall i, j = 1, ..., n$ in a finite time.

The main difference between Theorem 1 and Remark 2 is whether or not $v_i = 0, \forall i$. With protocol (11), the velocities of all the agents will be zero in finite time and the steady-state position will be $x_i = \bar{x} = \sum_{i=1}^n x_i(0)/n$. It is not hard to see that once we can get a protocol to make finite-time multi-agent consensus like (6), we will easy get a protocol like (11) to make finite-time consensus along with with v_i $(i = 1, ..., n)$ vanishing in finite time. Therefore, in what follows, we only consider the finite-time consensus protocols without requiring vanishing velocities.

In fact, there are many other finite-time consensus protocols, which can be viewed as a generalized form of (6). For example, we take

$$\begin{aligned}u_i = \sum_{j=1}^n a_{ij}[&\psi_1(sig(x_j - x_i)^{\alpha_1}) \\ &+\psi_2(sig(v_j - v_i)^{\alpha_2}|x_j - x_i|^{\alpha_3})]\end{aligned} \quad (13)$$

with

$$0 < \alpha_i < 1,\ (i = 1, 2, 3) \quad 2(\alpha_1 - \alpha_3) = (1 + \alpha_1)\alpha_2$$

where ψ_1 and ψ_2 are continuous odd functions with $y\psi_i(y) > 0$ $(\forall y \neq 0 \in R)$ and $\psi_i(y) = c_i y + o(y)$ for some positive number c_i $(i = 1, 2)$. The following the same proof idea of Theorem 1, we have

Corollary 1. Suppose that the undirected graph $\mathcal{G}$ with A of system (1) is connected. Then the globally finite-time consensus can be achieved under the distributed consensus protocol (13).

The proof is almost the same as given in Theorem 1. Consider Lyapunov function (8) and find that

$$\dot{V} = \frac{1}{2}\sum_{i=1}^n (v_i - v_j)a_{ij}\psi_2(sig(v_j - v_i)^{\alpha_2}|x_j - x_i|^{\alpha_3}) \leq 0.$$

Still employ Lemma 1 and obtain that system (5) is globally asymptotically stable. Then it is easy to see the global consensus of system (1) can be achieved in finite time. The detailed proof is omitted here.

4. STEADY-STATE VELOCITY

Clearly, with the protocols given in the last section (including protocols (6) and (13)), the steady-state velocity of all the agents will still be $\bar{v} = \sum_{i=1}^{n} v_i(0)/n$ since $\sum_{i=1}^{n} \dot{v}_i(t) \equiv 0$, and the collective motion in the steady state will be $x_i(t) = \sum_{i=1}^{n} v_i(0)t/n + \sum_{i=1}^{n} x_i(0)/n$.

However, in some cases, we do not want $\bar{v} = \sum_{i=1}^{n} v_i(0)/n$. Thus, in this section, we present a new protocol with different steady-state velocities.

To make the steady-state velocity maybe different from $\bar{v} = \sum_{i=1}^{n} v_i(0)/n$, a protocol can be taken in the following form:

$$u_i = \sum_{j=1}^{n} a_{ij}[\psi_1(sig(x_j - x_i)^{\alpha_1}) + v_i^r \psi_2(sig(v_j - v_i)^{\alpha_2})], \tag{14}$$

where $0 < r$, $r = \frac{p}{q}$, $(p,q) = 1$, and p and q are odd integers such that

$$0 < \alpha_1 < 1, \quad \alpha_2 = \frac{2\alpha_1}{1+\alpha_1} - r > 0. \tag{15}$$

Here, ψ_1 and ψ_2 are continuous odd functions with $y\psi_i(y) > 0$ ($\forall y \neq 0 \in R$) and $\psi_i(y) = c_i y + o(y)$ (around $y = 0$) for some positive numbers c_i ($i = 1, 2$).

Theorem 2. Suppose that the undirected graph $\mathcal{G}$ with A of system (1) is connected. Then the globally finite-time stable is achieved under the distributed consensus protocol (14).

Proof. Also consider the Lyapunov function in the form of (8). Note that $sig(x_i - x_j)^{\alpha}$ is an odd function. Consider the derivative of V along the trajectories of the closed-loop system, and then we have

$$\begin{aligned}
\dot{V} &= \sum_{i=1}^{n} v_i \dot{v}_i + \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\psi_1(sig(x_i - x_j)^{\alpha_1})v_i \\
&= \sum_{i=1}^{n} v_i \sum_{j=1}^{n} a_{ij}[\psi_1(sig(x_j - x_i)^{\alpha_1}) + v_i^r \psi_2(sig(v_j - v_i)^{\alpha_2})] \\
&\quad + \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\psi_1(sig(x_i - x_j)^{\alpha_1})v_i \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} v_i^r a_{ij}\psi_2(sig(v_j - v_i)^{\alpha_2}) \\
&= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} (v_i^r - v_j^r) a_{ij}\psi_2(sig(v_j - v_i)^{\alpha_2}) \leq 0.
\end{aligned}$$

since $(v_i - v_j)(v_i^r - v_j^r) \geq 0$. Denote the invariant set $S = \{(x_1, v_1, ..., x_n, v_n) | \dot{V} = 0\}$. Note that we continue the same step in Theorem 1, then $\dot{V} \equiv 0$ implies that $v_i \equiv v_j = \bar{v}$, $x_i \equiv x_j = \bar{v}t + \bar{x}, \forall j \neq i$.

Similar to the proof of Theorem 1, system (5) is locally finite-time stable, and moreover, it is globally finite-time stable, which implies the conclusion. ∎

Remark 3. Note that, with the protocols given in the last sections, $\sum_{i=1}^{n} v_i$ is unchanged along the time since its derivative is zero. However, this is not true in this section. Therefore, the steady-state velocity with the proposed protocol (14) may not be $\bar{v} = \sum_{i=1}^{n} v_i(0)/n$, and its value will be different with different protocols.

5. EXAMPLE

In this section, we give two simple examples for illustration.

Example 1. A numerical simulation is given for illustration. Here we consider a 5-agent system described by an undirected graph $\mathcal{G}$ shown in Fig 1.

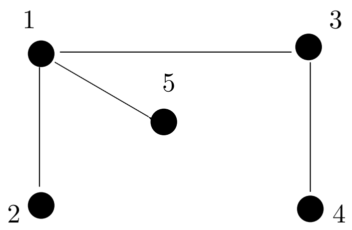

Fig. 1. $\mathcal{G}$ for a system with 5 agents.

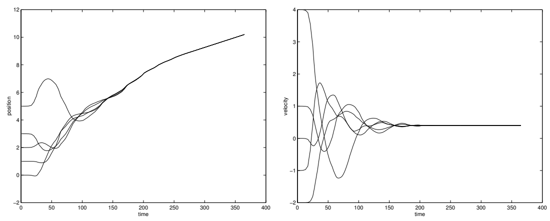

Fig. 2. Positions and velocities of the 5 agents with (9).

Take two special forms of protocol (6): (9) and (10), where nonzero $a_{ij} = 1$. In the simulations $\alpha_1 = 3/5$ (and therefore, $\alpha_2 = 3/4$). The initial conditions are randomly selected as follows:

$$x_1(0) = 0,\ x_2(0) = 1,\ x_3(0) = 3,\ x_4(0) = 2,\ x_5(0) = 5,$$
$$v_1(0) = -1,\ v_2(0) = 0,\ v_3(0) = -2,\ v_4(0) = 1,\ v_5(0) = 4.$$

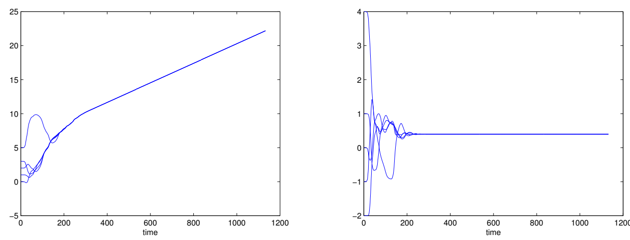

Fig. 3. Positions and velocities of the 5 agents with (10).

The numerical results given in Figs. 2 (protocol (9)) and 3 (protocol (10)) show the effectiveness of the given consensus protocol. Also, it is easy to see that the steady-state velocity of the multi-agent system is 0.4, which is consistent with the discussion given in Remark 1. Also, we van find that the convergence rates of the agents with protocol (9) are larger than those with protocol (10) since the latter one is in the saturation form.

Example 2. Still consider the system shown in Fig. 1. for protocol

$$u_i = \sum_{j=1}^{n} a_{ij}[(sig(x_j - x_i)^{\alpha_1}) + v_i^r sig(v_j - v_i)^{\alpha_2}] \quad (16)$$

with nonzero $a_{ij} = 1$ and $\alpha = 1/5$ and $r = 1/5$ (and therefore $\alpha_2 = 2/15$ according to (15)).

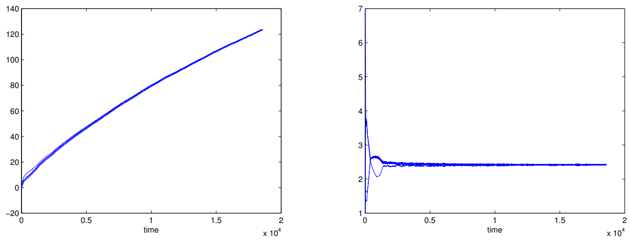

Fig. 4. Positions and velocities of the 5 agents with (16).

The initial conditions are randomly selected as follows:

$$x_1(0) = 0,\ x_2(0) = 1,\ x_3(0) = 3,\ x_4(0) = 2,\ x_5(0) = -1,$$
$$v_1(0) = 1,\ v_2(0) = 7,\ v_3(0) = 3,\ v_4(0) = 1,\ v_5(0) = 4$$

The numerical results given in Fig. 4 show the effectiveness of the given consensus protocol. It is not hard to see that the steady-state velocity is not $\sum_{i=1}^{5} v_i(0)/5 = 3.2$, unlike the results shown in Example 1.

6. CONCLUSIONS

In this paper, finite-time consensus algorithms were developed for multi-agent networks with second-order agent dynamics. With proposed consensus protocols, finite-time consensus as a generalization of conventional consensus was achieved in order to obtain the consensus in finite time. Lyapunov function, homogeneous properties, and graph theory were used in the theoretical analysis.

However, much remains to be done and the extension will be carried out for the cases with variable network topologies and communication limitations as well as various uncertainties. Moreover, many practical concerns such as avoiding collision and optimal location were not taken into consideration in this multi-agent consensus study, but they should certainly be taken care of when we deal with practical problems.

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