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Probabilistic Assurance of Constraint Fulfillment against Model Uncertainties and Disturbances

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Abstract: This paper addresses computations of a robustly safe region on the state space for uncertain constrained systems subject to disturbances based on a probabilistic approach. We first define a probabilistic output admissible (POA) set. This set is a subset of the state space which excludes with high probability initial states violating the constraint. Then, an algorithm for computing the POA set is developed based on a randomized technique. The utility of the POA set is demonstrated through a numerical simulation.

1. INTRODUCTION

Most of practical control systems inherently have state and control constraints due to nonlinear characteristics of actuators or for safety of hardware. It is thus required on design stage not only to achieve a good control performance but also to avoid constraint violations for the following reasons. Firstly, these constraints can lead to performance deterioration or even instability if not properly accounted for. Secondly, it is usually true that higher levels of performance are associated with operating on, or near, the constraint boundaries. A designer thus cannot ignore the constraints without incurring a performance penalty.

A constrained system operates safely if and only if its initial state lies within a certain subset of the state space (Blanchini [1999]). Constructing such subsets is crucial both in analysis and control of constrained systems. This paper concentrates on the constraint fulfillment of a closed-loop system. It is well known that a necessary and sufficient condition for the constraint fulfillment of a closed-loop system is given by a so-called maximal output admissible (MOA) set (Gilbert and Tan [1991]), which is the set of all initial states such that state trajectories starting from them never violate the infinite-time constraint. The MOA set not only gives an insight into analysis of constrained systems, but also has been extensively used in control system design schemes such as controller switching strategies (Hirata and Fujita [1998, 2000]), reference governors (Gilbert et al. [1995], Gilbert and Kolmanovsky [1999]), model predictive controls (Goodwin et al. [2004]). In addition, it also relates to the minimal l^{∞}-induced norm (Shamma [1996], Blanchini et al. [1997]).

Gilbert and Tan [1991] first defined the MOA set of linear time-invariant autonomous systems. Then, a lot of research works have been devoted to the computations of the MOA sets for various constrained systems (Blanchini [1994], Kolmanovsky and Gilbert [1995], Gilbert and Kolmanovsky [1998], Hirata and Ohta [2004, 2005], Raković et al. [2004]). Topics on invariant sets with close relationships to the MOA set are well summarized in the survey paper due to Blanchini [1999].

The authors addressed computations of the set of initial states of uncertain constrained systems which robustly guarantee the infinite-time constraint fulfillment on the basis of a probabilistic approach (Hatanaka and Takaba [2008a,b]). Hatanaka and Takaba [2008a] deals with time-invariant uncertainties and Hatanaka and Takaba [2008b] time-varying ones. Previous relevant works in the deterministic framework include Blanchini [1994] and Hirata and Ohta [2004], which consider several types of time-varying uncertainties and aim at the constraint fulfillment against *all* possible uncertainty outcomes. Meanwhile, we introduced a new concept of robust constraint fulfillment, where the guarantee of constraint fulfillment is intended in the probabilistic sense (satisfaction *in probability*). To be concrete, we constructed a subset of the state space such that if the system is initialized in any element of the set, the constraints can be violated by at most a fraction ϵ of the uncertainty family. This paper refers to the set as an *ϵ-level probabilistic output admissible (POA) set*. Our approach enables us to handle some kinds of uncertain systems such that the deterministic methods cannot deal with without introducing conservatism: e.g. the structured uncertainty, the time-invariant uncertainty and so on. Additionally, our approach allows us to incorporate information on probabilistic properties of uncertainties into the computations of the safe region.

This paper extends the result of Hatanaka and Takaba [2008a] to time-invariant uncertain constrained systems subject to disturbances. We redefine the POA set and present an algorithm for computing it based on the sequential randomized algorithm similarly to Hatanaka and Takaba [2008a]. It should be noted that the update rule of our algorithm is quite different from standard sequential randomized algorithms in robust control theory (Oishi

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[2007], Tempo et al. [2004]), since our objective is to compute not a *point* (a design parameter) but a *set* (an ϵ-level POA set).

The following notations will be used throughout this paper. The set $\mathbb{Z}_+$ is the set of nonnegative integers, namely, $\mathbb{Z}_+ = \{0, 1, 2, \cdots\}$. $\mathbf{0}$ is the matrix with appropriate dimension whose elements are 0. Let $v \in \mathbf{R}^n$, $M \in \mathbf{R}^{m\times n}$ and $\mathcal{Z} \subset \mathbf{R}^n$. Then, $|v|_\infty$ represents the ∞-norm of the vector v. $\|M\|$ the maximal singular value of M. $\mathrm{int}(\mathcal{Z})$ represents the interior of $\mathcal{Z}$. For a positive scalar α, $\alpha\mathcal{Z}$ represents the set $\{\alpha z |\ z \in \mathcal{Z}\}$. For a matrix $P = P^\top > 0$ and a scalar $\rho > 0$, $\Omega(P, \rho)$ denotes the ellipsoid $\Omega(P, \rho) := \{x \in \mathbf{R}^n |\ x^\top Px \le \rho^2\}$. $\lambda(A)$ represents an arbitrary eigenvalue of a square matrix $A \in \mathbf{R}^{n\times n}$.

2. PROBLEM STATEMENT

Consider the discrete-time uncertain system

$$\Sigma \begin{cases} x(t+1) = A(\Delta)x(t) + B(\Delta)w(t), & (1)\\ c(t) = C(\Delta)x(t) + D(\Delta)w(t), & (2)\end{cases}$$

where $t \in \mathbb{Z}_+$, $x(t) \in \mathbf{R}^n$ is the state of Σ, $w(t) \in \mathbf{R}^{n_w}$ is the disturbance and $c(t) \in \mathbf{R}^{n_c}$ is the constrained output which must be constrained within a prescribed set $\mathcal{C}$ as

$$c(t) \in \mathcal{C}\ \forall t \in \mathbb{Z}_+, \quad (3)$$
$$\mathcal{C} := \{c \in \mathbf{R}^{n_c} |\ |c|_\infty \le 1\}.$$

The matrix $\Delta \in \mathbf{R}^{s_1\times s_2}$ is the time-invariant uncertainty confined in a set $\mathbb{D}$. We assume that the set $\mathbb{D}$ is endowed with a σ-algebra $\mathcal{D}$, and that the probability measure $\mathrm{P}_{\boldsymbol{\Delta}}$ is defined over $\mathcal{D}$. This paper denotes the system with a fixed Δ by $\Sigma(\Delta)$. Without loss of generality, we assume that $\mathbf{0} \in \mathrm{int}(\mathbb{D})$, and refer to $\Sigma(\mathbf{0})$ as the nominal system. Assume that the disturbance function $w(\cdot) : \mathbb{Z}_+ \to \mathbf{R}^{n_w}$ satisfies

$$w(\cdot) \in \mathcal{W}_{\mathrm{f}} := \{w(\cdot) |\ w(t) \in \mathbb{W}\ \ \forall t \in \mathbb{Z}_+\},$$
$$\mathbb{W} := \{w \in \mathbf{R}^{n_w} |\ |w|_\infty \le 1\}.$$

In addition, we make the following assumptions.

Assumption 1.

- **(a)** $A(\Delta)$, $B(\Delta)$, $C(\Delta)$ and $D(\Delta)$ are Lebesgue measurable functions of Δ.
- **(b)** $\sup_{\Delta\in\mathbb{D}} \|C(\Delta)\|$ and $\sup_{\Delta\in\mathbb{D}} \|D(\Delta)\|$ are finite.
- **(c)** Samples can be efficiently drawn according to $\mathrm{P}_{\boldsymbol{\Delta}}$.
- **(d)** $(A(\mathbf{0}), C(\mathbf{0}))$ is an observable pair.

3. PROBABILISTIC OUTPUT ADMISSIBLE SET

3.1 Definitions

In this subsection, we define several important sets and functions.

We first define the MOA set of $\Sigma(\Delta)$.

Definition 2. (MOA Set of $\Sigma(\Delta)$). Let $c(t; x, \Delta, w(\cdot))$ denote the response of $\Sigma(\Delta)$ for the initial state x and the disturbance $w(\cdot) \in \mathcal{W}_{\mathrm{f}}$. Then, the MOA set $\mathcal{S}(\Delta)$ and the i-step output admissible sets $\mathcal{K}_i(\Delta)$, $i \in \mathbb{Z}_+$ of $\Sigma(\Delta)$ are defined by

$$\mathcal{S}(\Delta) := \{x \in \mathbf{R}^n |\ c(t; x, \Delta, w(\cdot)) \in \mathcal{C}\ \ \forall w(\cdot) \in \mathcal{W}_{\mathrm{f}} \text{ and } t \in \mathbb{Z}_+\},$$
$$\mathcal{K}_i(\Delta) := \{x \in \mathbf{R}^n |\ c(t; x, \Delta, w(\cdot)) \in \mathcal{C}\ \ \forall w(\cdot) \in \mathcal{W}_{\mathrm{f}} \text{ and } t \in \{0, 1, \cdots, i\}\}.$$

By the definition, we have

$$\mathcal{S}(\Delta) = \mathcal{K}_\infty(\Delta) := \bigcap_{i\in\mathbb{Z}_+} \mathcal{K}_i(\Delta). \quad (4)$$

Now, we review several important properties of the sets $\mathcal{S}(\Delta)$ and $\mathcal{K}_i(\Delta)$.

Proposition 3. (Gilbert and Kolmanovsky [1999]).

- **(i)** Suppose that $A(\Delta)$ is stable and $\mathbf{0} \in \mathrm{int}(\mathcal{S}(\Delta))$ holds. Then, the MOA set $\mathcal{S}(\Delta)$ is finitely determined, i.e. there exists a finite $i \in \mathbb{Z}_+$ satisfying $\mathcal{S}(\Delta) = \mathcal{K}_i(\Delta)$.
- **(ii)** If $\mathcal{K}_i(\Delta) = \mathcal{K}_{i+1}(\Delta)$ holds for some $i \in \mathbb{Z}_+$, then we obtain $\mathcal{S}(\Delta) = \mathcal{K}_i(\Delta)$.
- **(iii)** If $\mathbf{0} \in \mathrm{int}(\mathcal{S}(\Delta))$, then the set $\mathcal{S}(\Delta)$ is a convex polyhedron. Additionally, $\mathcal{S}(\Delta)$ is bounded if $(C(\Delta), A(\Delta))$ is an observable pair.

If $A(\Delta)$ is stable and $\mathbf{0} \in \mathrm{int}(\mathcal{S}(\Delta))$, then a finite characterization of $\mathcal{S}(\Delta)$ ($\mathcal{S}(\Delta) = \mathcal{K}_i(\Delta)$) is possible from Proposition 3. However, $\mathcal{S}(\Delta)$ does not always satisfy $\mathbf{0} \in \mathrm{int}(\mathcal{S}(\Delta))$, and can be empty even if $A(\Delta)$ is stable. Gilbert and Kolmanovsky [1999] show that $\mathcal{S}(\Delta) \ne \emptyset$ if and only if the origin $\mathbf{0}$ is admissible for any possible disturbance $w(\cdot) \in \mathcal{W}_{\mathrm{f}}$, that is,

$$c(t; \mathbf{0}, \Delta, w(\cdot)) \in \mathcal{C}\ \forall t \in \mathbb{Z}_+ \text{ and } w(\cdot) \in \mathcal{W}_{\mathrm{f}}. \quad (5)$$

However, this paper introduces another condition since (5) cannot always be checked efficiently. Given $j \in \{1, \cdots, n_c\}$ and $\Delta \in \mathbb{D}$, we define $\beta_j^*(\Delta)$ as

$$\beta_j^*(\Delta) := \begin{cases} \sup\limits_{t\in\mathbb{Z}_+, w(\cdot)\in\mathcal{W}_{\mathrm{f}}} |c_j(t; \mathbf{0}, \Delta, w(\cdot))|, \\ \qquad\qquad \text{if} \quad \max|\lambda(A(\Delta))| < 1, \\ \infty, \quad \text{otherwise,}\end{cases}$$

where $c_j(t; \mathbf{0}, \Delta, w(\cdot))$ is the j-th element of $c(t; \mathbf{0}, \Delta, w(\cdot))$. If $A(\Delta)$ is asymptotically stable, then $\beta_j^*(\Delta)$ is a finite value. Though it is difficult to obtain the exact value of $\beta_j^*(\Delta)$, we can compute its approximate value to an arbitrary accuracy $\sigma > 0$ (Dahleh and Diaz-Bobillo [1995]). Namely, it is possible to obtain $\beta_j^{\mathrm{l}}(\Delta)$ and $\beta_j^{\mathrm{u}}(\Delta)$ satisfying $\beta_j^{\mathrm{l}}(\Delta) < \beta_j^*(\Delta) < \beta_j^{\mathrm{u}}(\Delta)$ and $\beta_j^{\mathrm{u}}(\Delta) - \beta_j^{\mathrm{l}}(\Delta) \le \sigma$. In this paper, we thus fix the accuracy $\sigma > 0$ and define

$$\beta_j(\Delta) := \begin{cases} \beta_j^{\mathrm{u}}(\Delta) \text{ if } \ \max|\lambda(A(\Delta))| < 1, \\ \infty, \quad \text{otherwise.}\end{cases}$$

Then, $\beta_j(\Delta)$ satisfies $\beta_j(\Delta) \in (\beta_j^*(\Delta), \beta_j^*(\Delta) + \sigma)$ for any $\Delta \in \mathbb{D}$ such that $\max|\lambda(A(\Delta))| < 1$. By using the functions $\beta_j^*(\Delta)$ and $\beta_j(\Delta)$, we get the following proposition.

Proposition 4. If

$$\beta_j(\Delta) \le 1\ \ \forall j \in \{1, \cdots, n_c\} \quad (6)$$

holds, then we have $\mathbf{0} \in \mathrm{int}(\mathcal{S}(\Delta))$.

Proof . The inequality (6) implies the stability of $\Sigma(\Delta)$ and the existence of $\gamma > 0$ satisfying $\max_j \beta_j^*(\Delta) = 1 - \gamma$. Now, we have

$$\begin{aligned}
&\sup_{t\in\mathbb{Z}_+, w(\cdot)\in\mathcal{W}_\mathrm{f}} |c(t;x,\Delta,w(\cdot))|_\infty \\
&\qquad\le \sup_{t\in\mathbb{Z}_+} |c(t;x,\Delta,\mathbf{0}_\mathrm{f}(\cdot))|_\infty + \max_j \beta_j^*(\Delta) \\
&\qquad= \sup_{t\in\mathbb{Z}_+} |c(t;x,\Delta,\mathbf{0}_\mathrm{f}(\cdot))|_\infty + 1-\gamma, \qquad (7)
\end{aligned}$$

where $\mathbf{0}_\mathrm{f}(\cdot) : \mathbb{Z}_+ \to \{\mathbf{0}\}$. From the stability of $\Sigma(\Delta)$, there exists an open ball $\mathcal{B}$ centered at the origin satisfying $\sup_{t\in\mathbb{Z}_+} |c(t;x,\Delta,\mathbf{0}_\mathrm{f}(\cdot))|_\infty \le \gamma \ \ \forall x \in \mathcal{B}$, and such a ball is a subset of $\mathcal{S}(\Delta)$ because of (3) and (7). This completes the proof. □

We next define the MOA set of the uncertain system Σ.

Definition 5. (MOA set of Σ). The MOA set $\mathcal{S}$ and the i-step output admissible sets $\mathcal{K}_i$ of Σ are defined by

$$\begin{aligned}
\mathcal{S} &:= \{x \in \mathbf{R}^n |\ c(t;x,\Delta,w(\cdot)) \in \mathcal{C} \ \ \forall \Delta \in \mathbb{D}, \\
&\qquad w(\cdot) \in \mathcal{W}_\mathrm{f} \text{ and } t \in \mathbb{Z}_+\}, \\
\mathcal{K}_i &:= \{x \in \mathbf{R}^n |\ c(t;x,\Delta,w(\cdot)) \in \mathcal{C} \ \ \forall \Delta \in \mathbb{D}, \\
&\qquad w(\cdot) \in \mathcal{W}_\mathrm{f} \text{ and } t \in \{0,\cdots,i\}\}.
\end{aligned}$$

By the definition, $x(0) \in \mathcal{S}$ is necessary and sufficient for the constraint fulfillment in the face of all possible uncertainties $\Delta \in \mathbb{D}$ and disturbances $w(\cdot) \in \mathcal{W}_\mathrm{f}$. However, it is practically impossible to construct the MOA set $\mathcal{S}$ even if $\mathbf{0} \in \mathrm{int}(\mathcal{S})$ (Hatanaka and Takaba [2008a]), and the determination of $\mathcal{S} = \emptyset$ is not always possible.

We relax the problem by accepting a certain risk level of constraint violations. To be concrete, we aim at computing a probabilistic output admissible (POA) set defined below.

Definition 6. (ϵ-level POA set). Define

$$\begin{aligned}
\mathbb{B}(\mathcal{X}) := \{\Delta \in \mathbb{D}|\ \exists t \in \mathbb{Z}_+,\ x \in \mathcal{X} \text{ and } w(\cdot) \in \mathcal{W}_\mathrm{f} \\
\text{s.t. } c(t;x,\Delta,w(\cdot)) \notin \mathcal{C}\}
\end{aligned}$$

for any $\mathcal{X} \subseteq \mathbf{R}^n$. Then, for a real number $\epsilon \in (0,1)$, if a nonempty set $\mathcal{X} \subseteq \mathbf{R}^n$ satisfies

$$\mathcal{X} \subseteq \mathcal{S}(\mathbf{0}), \qquad (8)$$

$$\mathrm{P}_\Delta\{\Delta \in \mathbb{D}|\ \Delta \in \mathbb{B}(\mathcal{X})\} \le \epsilon, \qquad (9)$$

then $\mathcal{X}$ is said to be an ϵ-level POA set.

Note that the boundedness of the POA set is guaranteed by (8) under Assumption 1(d). The meaning of (9) is illustrated in Fig. 1(a).

Now, suppose that $\mathcal{X}$ is a bounded convex polyhedron. Then, $\mathbb{B}(\mathcal{X})$ is measurable for the following reason. Define

$$\begin{aligned}
\mathbb{B}_i(\mathcal{X}) &:= \{\Delta \in \mathbb{D}|\ \mathcal{X} \not\subseteq \mathcal{K}_i(\Delta)\},\ i \in \mathbb{Z}_+, \\
\mathbb{B}_\infty(\mathcal{X}) &:= \bigcup_{i\in\mathbb{Z}_+} \mathbb{B}_i(\mathcal{X}).
\end{aligned}$$

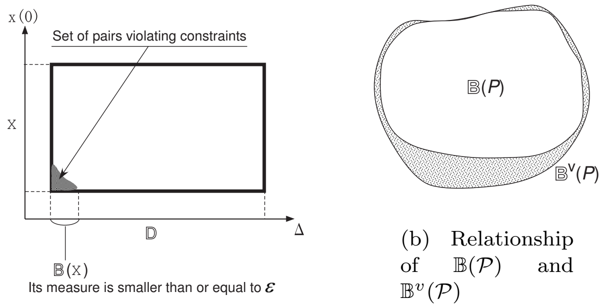
By the definitions of $\mathbb{B}_\infty(\mathcal{X})$ and $\mathcal{K}_\infty(\Delta)$, we have

$$\mathbb{B}_\infty(\mathcal{X}) = \{\Delta \in \mathbb{D}|\ \mathcal{X} \not\subseteq \mathcal{K}_\infty(\Delta)\}.$$

In addition, the definition of $\mathbb{B}(\mathcal{X})$ implies that $\mathbb{B}(\mathcal{X}) = \{\Delta \in \mathbb{D}|\ \mathcal{X} \not\subseteq \mathcal{S}(\Delta)\}$. We thus obtain $\mathbb{B}(\mathcal{X}) = \mathbb{B}_\infty(\mathcal{X})$ from the equation (4). Since $\mathcal{X}$ is a bounded convex polyhedron, $\mathbb{B}_i(\mathcal{X}) \in \mathcal{D}$, $i \in \mathbb{Z}_+$ under Assumption 1 (a). By the definition of σ-algebra, we have $\mathbb{B}_\infty(\mathcal{X}) \in \mathcal{D}$.

The objective of this paper is to construct as large a polyhedral ϵ-level POA set as possible for a prescribed ϵ. For this purpose, we introduce the following function.

Definition 7. (Violation function). The violation function $v(\Delta,\mathcal{P})$ is defined by



(a) ϵ-level POA set

(b) Relationship of $\mathbb{B}(\mathcal{P})$ and $\mathbb{B}^v(\mathcal{P})$

Fig. 1. Definitions of POA set, $\mathbb{B}(\mathcal{P})$ and $\mathbb{B}^v(\mathcal{P})$

$$v(\Delta,\mathcal{P}) := \begin{cases} 1,\ \text{if } \ \Delta \notin \mathbb{D}_\mathrm{E} & (10) \\ 1,\ \text{else if } \mathcal{P} \not\subseteq \mathcal{S}(\Delta) & (11) \\ 0,\ \text{otherwise} & (12) \end{cases}$$

$$\mathbb{D}_\mathrm{E} := \{\Delta \in \mathbb{D}|\ \beta_j(\Delta) < 1\ \forall j \in \{1,\cdots,n_c\}\}$$

for any $\Delta \in \mathbb{D}$ and bounded convex polyhedron $\mathcal{P} \subset \mathbf{R}^n$.

Note that the function $v(\Delta,\mathcal{P})$ with a fixed $\mathcal{P}$ is a measurable function of $\Delta \in \mathbb{D}$. In addition, given Δ and $\mathcal{P}$, we can compute $v(\Delta,\mathcal{P})$ since $\mathcal{S}(\Delta)$ is finitely determined and a convex polyhedron if $\Delta \in \mathbb{D}_\mathrm{E}$.

Proposition 8. Suppose that a bounded polyhedron $\mathcal{P} \subseteq \mathcal{S}(\mathbf{0})$ satisfies

$$\begin{aligned}
&\mathrm{P}_\Delta\{\Delta \in \mathbb{D}|\ \Delta \in \mathbb{B}^v(\mathcal{P})\} \le \epsilon, \qquad (13)\\
&\mathbb{B}^v(\mathcal{P}) := \{\Delta \in \mathbb{D}|\ v(\Delta,\mathcal{P}) = 1\}.
\end{aligned}$$

Then, (9) holds with $\mathcal{X} = \mathcal{P}$, and hence $\mathcal{P}$ is an ϵ-level POA set.

The converse of Proposition 8 is not always true (The relationship of $\mathbb{B}(\mathcal{P})$ and $\mathbb{B}^v(\mathcal{P})$ is illustrated in Fig. 1(b).), because $\Delta \notin \mathbb{D}_\mathrm{E}$ assures neither $\mathcal{S}(\Delta) = \emptyset$ nor $\Delta \in \mathbb{B}(\mathcal{P})$. However, if $A(\Delta)$ has no eigenvalues on the unit disk, the conservatism can be arbitrarily reduced by making σ small. In the disturbance free case with the assumption of robust stability (Hatanaka and Takaba [2008a]), the violation function $v(\mathcal{P},\Delta)$ can be chosen so that $\mathcal{B}^v(\mathcal{P}) = \mathcal{B}(\mathcal{P})$. In contrast, this paper cannot adopt this definition, since such a function prohibits efficient computation of the POA set of this paper.

Remark 9. It is also possible to handle independent and identically distributed stochastic disturbances, where each $w(t)$, $t \in \mathbb{Z}_+$ is assumed to be a random parameter on a probability space $(\mathbb{W},\mathcal{W},\mathrm{P}_w)$. However, we do not include it in this paper so as not to confuse the readers, since the stochastic disturbance case is essentially different from the deterministic one. The main difference is that the determination of the probability space to measure the risk of constraint violations is not trivial similarly to the problem of Hatanaka and Takaba [2008b]. The idea of Hatanaka and Takaba [2008b] is thus used instead of Hatanaka and Takaba [2008a] in order to compute the POA set in the stochastic disturbance case.

3.2 Computation of A POA Set

In this subsection, we present a computational procedure of the ϵ-level POA set. The algorithm iteratively updates a polyhedral set $\mathcal{P}_k \subset \mathbf{R}^n$, which is a candidate of the POA set, similarly to Hatanaka and Takaba [2008a]. Before executing the procedures, we prepare a confidence parameter $\delta \in (0,1)$ as well as $\epsilon \in (0,1)$. Let the initial polyhedron $\mathcal{P}_0$ be the nominal MOA set $\mathcal{S}(\mathbf{0})$, which is computed by the method due to Gilbert and Kolmanovsky [1999].

Suppose that, after the k-th iteration, we have a polyhedral set $\mathcal{P}_k \subset \mathbf{R}^n$. We first draw N_k random samples $\mathbb{D}_k^{\mathrm{f}} := \{\Delta_k^{(l)}, l \in \{1, \cdots, N_k\}\}$ according to the probability measure $\mathrm{P}_{\mathbf{\Delta}}$. If there exists $\Delta_k^{(l)} \in \mathbb{D}_k^{\mathrm{f}}$ and $j \in \{1, \cdots, n_c\}$ satisfying $\beta_j(\Delta_k^{(l)}) > 1$, then we conclude $\mathcal{S} = \emptyset$. Though this conclusion is not always true, the false conclusion can be almost avoided as long as Σ is robustly stable. Of course, if $A(\Delta_k^{(l)})$ is unstable, the conclusion is true. If such $\Delta_k^{(l)} \in \mathbb{D}_k^{\mathrm{f}}$ and $j \in \{1, \cdots, n_c\}$ do not exist, the set $\mathcal{P}_k$ is updated according to the rule

$$\mathcal{P}_{k+1} := \mathcal{P}_k \cap \mathcal{S}(\mathbb{D}_k^{\mathrm{f}}), \tag{14}$$

where $\mathcal{S}(\mathbb{D}')$ is defined for any subset $\mathbb{D}' \subseteq \mathbb{D}$ as $\mathcal{S}(\mathbb{D}') := \cap_{\Delta \in \mathbb{D}'} \mathcal{S}(\Delta)$. In addition, after the counter k reaches an integer $\bar{k}$, we perform the additional operation

$$\mathcal{P}_k := \alpha \mathcal{P}_k \tag{15}$$

before executing (14) in order to assure the finite termination of the algorithm, where $\alpha \in (0,1)$ and $\bar{k} \in \mathbb{Z}_+$ are prescribed numbers.

Let the termination condition of the algorithm be

$$v(\Delta_k^{(j)}, \mathcal{P}_k) = 0 \ \forall j \in \{1, \cdots, N_k\}, \tag{16}$$

which is almost the same as the standard sequential randomized algorithm of Oishi [2007]. The equation (16) is equivalent to

$$\mathcal{P}_k \subseteq \mathcal{S}(\mathbb{D}_k^{\mathrm{f}}). \tag{17}$$

In summary, the following algorithm provides the POA set with high probability or the conclusion of $\mathcal{S} = \emptyset$. If Algorithm 1 outputs CONCLUDE$= -1$, then we conclude that $\mathcal{S} = \emptyset$. In contrast, if Algorithm 1 outputs CONCLUDE$= 1$, then we conclude that the resulting set $\mathcal{P}$ is an ϵ-level POA set. Note that the strategy to alleviate the computational effort shown in Hatanaka and Takaba [2008a] can be applied to Algorithm 1, too.

Theorem 10.

- Algorithm 1 terminates in a finite number of iterations.
- Suppose that the algorithm terminates at the k_T-th iteration without finding $l \in \{1, \cdots, N_k\}$ and $j \in \{1, \cdots, n_c\}$ satisfying $\beta_l(\Delta^{(l)}) \geq 1$.
  - · If N_k is an integer satisfying

$$N_k \geq \frac{\log \frac{\pi^2 (k+1)^2}{6\delta}}{\log \frac{1}{1-\epsilon}}, \tag{18}$$

    the resulting set $\mathcal{P}_k$ is an ϵ-level POA set with probability greater than $1 - \delta$.
  - · If $k_T < \bar{k}$, then the resulting set $\mathcal{P}$ satisfies $\mathcal{P} \supseteq \mathcal{S}$. Otherwise, $\mathcal{P}$ satisfies $\mathcal{P} \supseteq \alpha^{k_T - \bar{k} + 1} \mathcal{S}$.

Algorithm 1 Algorithm for computing a POA set

Parameters: $\alpha \in (0,1)$, $\bar{k} \in \mathbb{Z}_+$, $\epsilon \in (0,1)$, $\delta \in (0,1)$
Compute $\mathcal{P}_0 = \mathcal{S}(\mathbf{0})$ and set $k := 0$ and CONCLUDE$:= 0$.
while CONCLUDE $= 0$ **do**
  Draw N_k samples $\{\Delta_k^{(l)}, l \in \{1, \cdots, N_k\}\}$ from $\mathbb{D}$ according to the probability measure P_Δ.
  if there exist $l \in \{1, \cdots, N_k\}$ and $j \in \{1, \cdots, n_c\}$ such that $\beta_j(\Delta_k^{(l)}) \geq 1$ **then**
    CONCLUDE$= -1$
  else
    if (17) is satisfied **then**
      let $\mathcal{P} := \mathcal{P}_k$ and CONCLUDE$= 1$.
    else
      if $k \geq \bar{k}$ **then**
        perform the operation (15).
      end if
      Update the convex polyhedron $\mathcal{P}_k$ based on the rule (14). Set $k := k + 1$.
    end if
  end if
end while

Proof . We prove only the first statement since the second one can be proven in almost the same way as the disturbance free case (Hatanaka and Takaba [2008a]).

If a sample is chosen from $\mathbb{D} \setminus \mathbb{D}_E$, then this algorithm terminates and we have CONCLUDE$= -1$. It is thus sufficient to prove that Algorithm 1 terminates even if samples are continually drawn from the set $\mathbb{D}_{\mathrm{E}}$. From the proof of Proposition 4, $\mathbf{0} \in \mathrm{int}\,(\mathcal{S}(\mathbb{D}_{\mathrm{E}}))$ holds true. It follows from (14), (15) and $\mathcal{P}_{k+1} \subseteq \mathcal{P}_k$ that $\mathcal{P}_k \subseteq \alpha^{(k-\bar{k})} \mathcal{P}_{\bar{k}}$ after k reaches $\bar{k}$. Thus, there is a finite $k \geq \bar{k}$ satisfying $\alpha^{(k-\bar{k})} \mathcal{P}_{\bar{k}} \subseteq \mathcal{S}(\mathbb{D}_{\mathrm{E}})$. For such a k, we have $\mathcal{P}_k \subseteq \mathcal{S}(\mathbb{D}_{\mathrm{E}}) \subseteq \mathcal{S}(\mathbb{D}_k^{\mathrm{f}})$. This implies the termination of Algorithm 1, and completes the proof. □

The last statement of Theorem 10 provides us the size of the resulting POA set *a posteriori*. This allows us to eventually get a sufficiently large POA set by reseting the parameters α, $\bar{k}$ and re-performing Algorithm 1.

In Algorithm 1, we have to solve a lot of linear programming problems to remove redundant conditions (Hatanaka and Takaba [2008a]). The computational effort of Algorithm 1 thus becomes large as the dimension of the state increases, since the number of optimization variables of the linear programs is equal to the number of the states. Though its increase rate is in polynomial order, it should be noted that the numerical error may become unignorable. A conceivable realistic technique to avoid this problem is scaling the state vector as stated in Kvasnica et al. [2006].

3.3 Computation of $\mathcal{S}(\mathbb{D}_k^{\mathrm{f}})$

Algorithm 1 requires that the set $\mathcal{S}(\mathbf{0}) \cap \mathcal{S}(\Delta)$ can be efficiently computed every time Δ is fixed. Hereafter, we show a procedure for computing this set. Notice that we can assume $\Delta \in \mathbb{D}_{\mathrm{E}}$ since Algorithm 1 concludes $\mathcal{S} = \emptyset$ otherwise.

We prepare an ellipsoid $\Omega(P,1)$ satisfying $\mathcal{S}(\mathbf{0}) \subset \Omega(P,1)$ at the beginning of Algorithm 1. Then, the zero input response $A^t(\Delta)x(0)$ should be contained in the ellipsoid $\Omega(P, \|A^t(\Delta)\|)$ if $x(0) \in \mathcal{S}(\mathbf{0})$.

Lemma 11. Let $\mathcal{X}(\Delta)$ and $T(\Delta)$ denote

$$\mathcal{X}(\Delta) := \{x \in \mathbb{R}^n \mid |c_j(0; x, \Delta, \mathbf{0}_\mathrm{f})| \le 1 - \beta_j(\Delta) \quad \forall j \in \{1, \cdots, n_c\}\},$$

$$T(\Delta) := \min_{t \in \mathbb{Z}_+} t \text{ subject to } \|A^t(\Delta)\| \le 1.$$

Then, for a given $i \in \mathbb{Z}_+$, we have $\Omega(P, \|A^t(\Delta)\|) \subset \mathcal{X}(\Delta)$ for any $t \ge i$ if

$$\Omega(P, \|A^t(\Delta)\|) \subset \mathcal{X}(\Delta) \quad \forall t \in \{i, \cdots, i + T(\Delta)\}. \tag{19}$$

Now, we define $\bar{i}_\mathrm{o}(\Delta) := \min_{i \in \mathbb{Z}_+} i$ subject to (19) . Then, $\bar{i}_\mathrm{o}(\Delta)$ is a finite value because $\Delta \in \mathbb{D}_\mathrm{E}$, that is, $A(\Delta)$ is stable and $\mathbf{0} \in \mathrm{int}(\mathcal{X}(\Delta))$.

Corollary 12. If $\Sigma(\Delta)$ is initialized in any element of $\mathcal{S}(\mathbf{0}) \subset \Omega(P,1)$, then the constraint is satisfied after the time $\bar{i}_\mathrm{o}(\Delta)$. Namely, we obtain $\mathcal{S}(\mathbf{0}) \cap \mathcal{S}(\Delta) = \mathcal{S}(\mathbf{0}) \cap \mathcal{K}_{\bar{i}_\mathrm{o}(\Delta)}(\Delta)$.

By the definition of $\bar{i}_\mathrm{o}(\Delta)$, we can compute it only by algebraic calculations. Additionally, the set $\mathcal{K}_{\bar{i}_\mathrm{o}(\Delta)}(\Delta)$ is easily computed (Gilbert and Kolmanovsky [1999]). It is thus possible to compute $\mathcal{S}(\mathbf{0}) \cap \mathcal{S}(\Delta)$ efficiently every time Δ is fixed.

3.4 Application of A POA Set to Control Design

The POA set is available instead of the MOA set for some constrained control schemes such as controller switching strategies (Hirata and Fujita [1998, 2000]) and reference governors (Gilbert et al. [1995], Gilbert and Kolmanovsky [1999]). Though the direct use of the POA set is difficult in standard model predictive control strategies (Goodwin et al. [2004]), it is possible to use it in the dual or triple mode predictive control scheme as in Imsland et al. [2006]. It should be noted that the POA and even MOA set for systems with time invariant uncertainties is not positively invariant. We thus have to incorporate some additional procedure as in Gilbert and Kolmanovsky [1999] in order to assure stability. An application result of the reference governor due to Gilbert and Kolmanovsky [1999] is shown in Hatanaka and Takaba [2008a].

4. NUMERICAL SIMULATION

Consider the uncertain system

$$x(t+1) = A(\Delta)x(t) + Bw(t),$$
$$c(t) = C(\Delta)x(t),$$
$$A(\Delta) = \frac{1}{10}\begin{bmatrix} 0 & 10 \\ -8+\Delta_1 & 9+\Delta_1^2 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix},$$
$$C(\Delta) = \begin{bmatrix} \cos(\Delta_2) & -\sin(\Delta_2) \\ \sin(\Delta_2) & \cos(\Delta_2) \end{bmatrix}.$$

Assume that the uncertainty Δ is uniformly distributed over $\mathbb{D} := \{\Delta \in \mathbf{R}^2 \mid |\Delta_1| \le 1, |\Delta_2| \le \pi\}$, and that the disturbance $w(t)$ belongs to the set $\mathbb{W} = \{w \in \mathbf{R} \mid |w| \le w_m\}$ at each time $t \in \mathbb{Z}_+$. In this example, we set $\sigma = 0.01$.

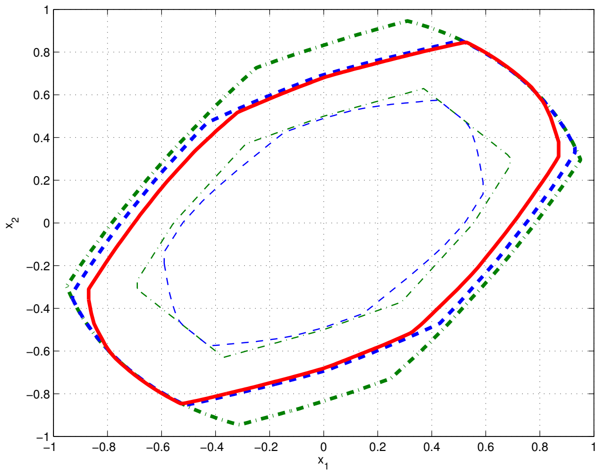


Fig. 2. POA and MOA sets for various w_m's

Fig. 2 shows the ϵ-level POA sets for various w_m obtained by Algorithm 1 with $\epsilon = 10^{-2}$, $\delta = 10^{-5}$, $\alpha = 0.999$ and $\bar{k} = 10$ (thick lines), and the estimated MOA set in the deterministic framework due to Blanchini [1994] (thin lines). Note that the deterministic method is applied by overbounding the uncertainty. The red solid lines represent the case of $w_m = 0.5$, the blue dashed lines $w_m = 0.2$, the green dash-dotted lines $w_m = 0$, where the deterministic method concludes that the MOA set is empty in the case of $w_m = 0.5$. Of course, the robustly safe region becomes small as w_m increases both in the deterministic and probabilistic frameworks. We see from this figure that the present method avoids overbounding the uncertainty, and provides a larger safe region than the deterministic one by accepting a certain risk level of constraint violations.

In Figs. 3 and 4, responses of c for 100 random uncertainties and disturbances with $w_m = 0.2$ from the initial states $x_1 = [0.94 \ 0.30]^\top, x_2 = [0.4 \ -0.5]^\top$ in the POA set and $x_3 = [0.58 \ 0.13]^\top, x_4 = [0.15, -0.4]^\top$ in the deterministic MOA set. The constraints were never violated in these trials even if the initial state is chosen from the POA set. These figures also illustrate that the system with the initial states in the POA set operates near the constraint boundaries, while that in the deterministic MOA set far from. This numerical example suggests that the POA set enables us to achieve quite safe and non-conservative responses and hence a high performance in control of uncertain constrained systems, by accepting a certain risk level of constraint violations.

5. CONCLUSIONS

In this paper, we have extended the notion of an ϵ-level POA set to uncertain systems subject to disturbances. The probabilistic scheme has a greater flexibility in handling uncertainties compared with the previous works where uncertainties are not taken into account at all or the worst case uncertainty scenario is assumed. Though our approach accepts a certain risk level of constraint violations, it is sufficiently useful in many real situations at least when we consider systems not suffering severely from infrequent constraint violations e.g. input and output saturations.

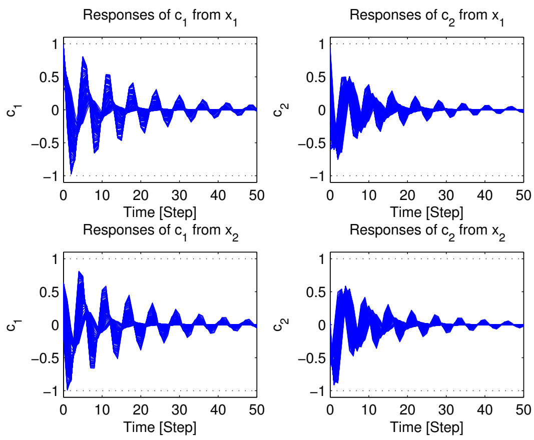


Fig. 3. Responses from the initial states in the POA set for random uncertainties and disturbances

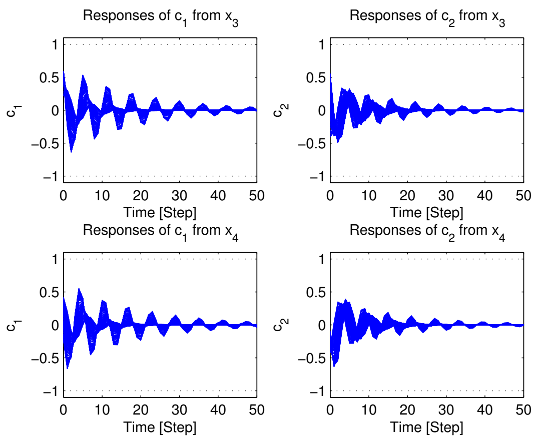


Fig. 4. Responses from the initial states in the deterministic MOA set for random uncertainties and disturbances

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