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Stability and Control of Systems with Uncertain Time-Varying Sampling Period and Time Delay

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Abstract: This paper addresses stability and control issues of systems with uncertain and time-varying sampling period and time-delay. These arise e.g. in embedded or networked control applications where limited computation or communication resources have to be scheduled. The uncertain and time-varying sampling period and time delay are transformed into polytopic and additive norm-bounded uncertainties in the discretized system description. Control design and stability analysis methods are given in the form of LMIs applying switched parameter-dependent quadratic Lyapunov functions. A reduction algorithm is proposed in order to decrease the amount of LMIs necessary for stability analysis. The control design and stability analysis methods as well as the reduction algorithm are illustrated by an example.

Keywords: Switched discrete and hybrid systems, Time-varying systems, Systems with time-delays, Uncertainty descriptions, LMIs, Networked embedded control systems.

1. INTRODUCTION

Control systems are increasingly implemented with limited computation and communication resources. Scheduling of the computation capacity and communication bandwidth is necessary to efficiently utilize these resources. Conventional sampled-data control theory assuming constant sampling periods and negligible computation as well as transmission latencies is no longer applicable since scheduling leads to time-varying sampling periods and time-varying time delays. Instead, implementation-aware stability analysis and control design methods are required for these embedded and networked control systems.

Embedded and networked control systems have received considerable attention during the last years. Recent surveys are given in Hristu-Varsakelis and Levine (2005) and Hespanha et al. (2007). Time-varying sampling periods and time delays have been addressed in several ways. If the sampling period and time delay can be measured, a jitter compensation can be applied as shown in Martí et al. (2001). Considering the time-varying sampling period and time delay from a stochastic perspective as in Nilsson (1998) yields mean square stability and LQG performance but no robust stability and performance as depicted by Årzén et al. (2003). A method guaranteeing robust stability and performance based on a common Lyapunov function has been proposed by Schinkel et al. (2002) for time-varying sampling period and by Izák et al. (2007) for both time-varying sampling period and time delay. The sampling period and time delay were elements of a finite set. Recently, Hetel et al. (2006, 2007) suggested to express systems with either uncertain time-varying sampling period or time delay bounded on an interval in a polytopic form and to apply switched parameter-dependent quadratic Lyapunov functions for stability analysis and control design.

In this paper stability and control of systems with both time-varying sampling period and time delay are studied. The time-varying sampling period and time delay are considered to be uncertain but bounded on switched intervals. Switching of the intervals allows structuring the uncertainty. Timing knowledge obtained by analysis of the embedded or networked control system can be exactly mapped onto this uncertainty description, whereby conservatism can be reduced. Moreover control algorithms with changing demand for sampling period and computation time as well as networks with different load conditions can be addressed directly by switching the intervals. The resulting system with switched interval uncertainty is reformulated as a system with switched polytopic and additive norm-bounded uncertainty. Contrary to the papers above both time-varying sampling period and time delay are regarded utilizing Minkowski addition of polytopes. Linear matrix inequalities (LMIs) for stability analysis and design of stabilizing switched state feedback controllers are derived based on switched parameter-dependend Lyapunov functions. The number of LMIs required for stability analysis increases rapidly with the number of vertices of the polytopes since time-varying sampling period and time delay are considered simultaneously. In some cases the problem even becomes computationally intractable. Therefore a novel algorithm for efficient reduction of the number of vertices is proposed.

The paper is organized as follows. In section 2 the discretization of a linear continuous-time plant with an input delay is performed. It is shown how a switched interval uncertainty of the sampling period and time delay can be expressed in the terms of a Taylor series expansion and transformed into a discrete-time state equation with polytopic and additive norm-bounded uncertainty. A switched full state feedback control law is adopted in section 3 and a method for its design is given for systems with polytopic uncertainty only. Since the control design method does not guarantee the stability of the closed loop for the systems with both polytopic and additive norm-bounded uncertainty a stability analysis method is introduced. The stability analysis method leads to a considerable amount of LMIs which can be reduced by an algorithm proposed in section 4. Both the control design and stability analysis methods with the reduction algorithm are illustrated by an example in section 5. Section 6 concludes the paper.

2. MODELING

2.1 Plant description

The plant is described by the continuous-time state equation

$$\dot{\boldsymbol{x}}_c(t) = \boldsymbol{A}_c\boldsymbol{x}_c(t) + \boldsymbol{B}_c\boldsymbol{u}(t-\tau(t)) \tag{1}$$

where $\boldsymbol{x}_c(t) \in \mathbb{R}^n$ is the state vector, $\boldsymbol{u}_c(t-\tau(t)) \in \mathbb{R}^m$ is the input vector with time-varying input delay $\tau(t)$ and $\boldsymbol{A}_c \in \mathbb{R}^{n\times n}$ and $\boldsymbol{B}_c \in \mathbb{R}^{n\times m}$ are the system and input matrix respectively. The time-varying delay which is actually caused by jitter in the controller execution or network transmission time is included in the plant dynamics. Discretization with time-varying sampling period h_k and time delay $\tau_k \le h_k$ applying zero order hold leads to the augmented discrete-time state equation

$$\boldsymbol{x}(k+1) = \boldsymbol{A}_k\boldsymbol{x}(k) + \boldsymbol{B}_k\boldsymbol{u}(k) \tag{2}$$

where

$$\boldsymbol{x}(k) = \begin{pmatrix} \boldsymbol{x}_c(k) \\ \boldsymbol{u}(k-1) \end{pmatrix} \tag{3a}$$

$$\boldsymbol{A}_k = \begin{pmatrix} \boldsymbol{\Phi}(h_k) & \boldsymbol{\Gamma}_1(h_k,\tau_k) \\ \boldsymbol{0} & \boldsymbol{0} \end{pmatrix},\ \boldsymbol{B}_k = \begin{pmatrix} \boldsymbol{\Gamma}_0(h_k,\tau_k) \\ \boldsymbol{I} \end{pmatrix} \tag{3b}$$

with

$$\boldsymbol{\Phi}(h_k) = \mathrm{e}^{\boldsymbol{A}_c h_k} \tag{4a}$$

$$\boldsymbol{\Gamma}_0(h_k,\tau_k) = \int_0^{h_k-\tau_k} \mathrm{e}^{\boldsymbol{A}_c s}\, ds\boldsymbol{B}_c \tag{4b}$$

$$\boldsymbol{\Gamma}_1(h_k,\tau_k) = \mathrm{e}^{\boldsymbol{A}_c(h_k-\tau_k)} \int_0^{\tau_k} \mathrm{e}^{\boldsymbol{A}_c s}\, ds\boldsymbol{B}_c. \tag{4c}$$

This representation is adapted from a representation of time delay systems given in Åström and Wittenmark (1990). To enable further transformations the actuation period $g_k = h_k - \tau_k$ is introduced, yielding

$$\boldsymbol{\Phi}(h_k) = \mathrm{e}^{\boldsymbol{A}_c h_k} \tag{5a}$$

$$\boldsymbol{\Gamma}_0(h_k,\tau_k) = \int_0^{g_k} \mathrm{e}^{\boldsymbol{A}_c s}\, ds\boldsymbol{B}_c \tag{5b}$$

$$\boldsymbol{\Gamma}_1(h_k,\tau_k) = \int_0^{\tau_k} \mathrm{e}^{\boldsymbol{A}_c(s+g_k)}\, ds\boldsymbol{B}_c. \tag{5c}$$

2.2 Uncertainty description

The time-varying sampling period h_k and time delay τ_k and thus also the actuation period g_k are assumed to be uncertain but bounded on intervals. The intervals are considered to be switched where the switching index $\sigma =$

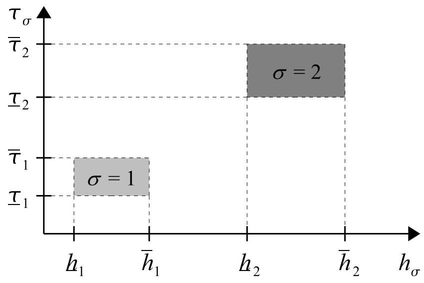

Fig. 1. Switched interval uncertainty

$\sigma(k) \in \{1,\dots,L\}$ for all $k \in \mathbb{Z}^+$ indicates the active mode, i.e.

$$0 < \underline{h}_\sigma \le h_\sigma \le \overline{h}_\sigma \tag{6a}$$

$$0 < \underline{\tau}_\sigma \le \tau_\sigma \le \overline{\tau}_\sigma \le \overline{h}_\sigma \tag{6b}$$

$$0 < \underline{g}_\sigma \le g_\sigma \le \overline{g}_\sigma \tag{6c}$$

with $\underline{g}_\sigma = \underline{h}_\sigma - \overline{\tau}_\sigma$ and $\overline{g}_\sigma = \overline{h}_\sigma - \underline{\tau}_\sigma$. An illustration of the switched interval uncertainty is given in Fig. 1. The switching index is supposed to be known at sampling instant k in the following. A short description of the uncertainty is given by the intervals

$$(h_\sigma,\tau_\sigma) \in \mathcal{I}_\sigma = [\underline{h}_\sigma,\overline{h}_\sigma] \times [\underline{\tau}_\sigma,\overline{\tau}_\sigma] \subset \mathbb{R}^2\,. \tag{7}$$

Switching of the intervals leads to switched system and input matrices (5). A convenient notation for switched matrices is given by

$$\boldsymbol{M}_\sigma = \sum_{l=1}^{L} \xi_l(k)\,\boldsymbol{M}_l \tag{8}$$

with the switching parameters

$$\xi_l(k) = \begin{cases} 1 \text{ if } \sigma(k) = l \\ 0 \text{ otherwise} \end{cases}, \quad \sum_{l=1}^{L} \xi_l(k) = 1\,. \tag{9}$$

Moreover the switched interval uncertainty affects the system and input matrices (5) of the discretized plant dynamics in a nonlinear manner. Therefore in the following a polytopic formulation with additive norm-bounded uncertainty allowing control design and stability analysis via LMIs is derived, extending the method from Hetel et al. (2006, 2007).

2.3 Polytopic formulation with norm-bounded uncertainty

Taylor series expansion

The matrix exponentials contained in the system and input matrices (5) are expanded in Taylor series, yielding

$$\boldsymbol{\Phi}_\sigma = \sum_{q=0}^{\infty} \frac{{\boldsymbol{A}_c}^q}{q!} h_\sigma^q = \sum_{q=0}^{M} \frac{{\boldsymbol{A}_c}^q}{q!} h_\sigma^q + \Delta\boldsymbol{\Phi}_\sigma \tag{10a}$$

$$\boldsymbol{\Gamma}_{0\sigma} = \int_0^{g_\sigma} \sum_{n=0}^{\infty} \frac{{\boldsymbol{A}_c}^n}{n!} s^n ds\boldsymbol{B}_c \qquad | \; q = n+1 \tag{10b}$$

$$= \left(\sum_{q=1}^{\infty} \frac{{\boldsymbol{A}_c}^{q-1}}{q!} g_\sigma^q\right)\boldsymbol{B}_c = \left(\sum_{q=1}^{N} \frac{{\boldsymbol{A}_c}^{q-1}}{q!} g_\sigma^q\right)\boldsymbol{B}_c + \Delta\boldsymbol{\Gamma}_{0\sigma}$$

$$\boldsymbol{\Gamma}_{1\sigma} = \int_0^{\tau_\sigma} \sum_{n=0}^{\infty} \frac{{\boldsymbol{A}_c}^n}{n!} (s+g_\sigma)^n ds\boldsymbol{B}_c \qquad | \; q = n+1 \tag{10c}$$

$$= \left(\sum_{q=1}^{\infty} \frac{\boldsymbol{A}_c{}^{q-1}}{q!} h_\sigma^q - \sum_{q=1}^{\infty} \frac{\boldsymbol{A}_c{}^{q-1}}{q!} g_\sigma^q \right) \boldsymbol{B}_c$$

$$= \left(\sum_{q=1}^{M} \frac{\boldsymbol{A}_c{}^{q-1}}{q!} h_\sigma^q - \sum_{q=1}^{N} \frac{\boldsymbol{A}_c{}^{q-1}}{q!} g_\sigma^q \right) \boldsymbol{B}_c + \Delta\boldsymbol{\Gamma}_{1\sigma}$$

where the remainder is computed as

$$\Delta\boldsymbol{\Phi}_\sigma = \mathrm{e}^{\boldsymbol{A}_c h_\sigma} - \sum_{q=0}^{M} \frac{\boldsymbol{A}_c{}^q}{q!} h_\sigma^q \tag{11a}$$

$$\Delta\boldsymbol{\Gamma}_{0\sigma} = \left(\int_0^{g_\sigma} \mathrm{e}^{\boldsymbol{A}_c s}\, ds - \sum_{q=1}^{N} \frac{\boldsymbol{A}_c{}^{q-1}}{q!} g_\sigma^q \right) \boldsymbol{B}_c \tag{11b}$$

$$\Delta\boldsymbol{\Gamma}_{1\sigma} = \left(\int_0^{\tau_\sigma} \mathrm{e}^{\boldsymbol{A}_c (s+g_\sigma)}\, ds \right. \tag{11c}$$

$$\left. - \sum_{q=1}^{M} \frac{\boldsymbol{A}_c{}^{q-1}}{q!} h_\sigma^q + \sum_{q=1}^{N} \frac{\boldsymbol{A}_c{}^{q-1}}{q!} g_\sigma^q \right) \boldsymbol{B}_c \,.$$

Partitioning the Taylor series in an Mth-/Nth-order approximation and a remainder allows a polytopic formulation of the approximation part.

Polytopic formulation

The polytopic formulation of the system and input matrices (10) is given by

$$\boldsymbol{\Phi}_\sigma = \sum_{i=1}^{M+1} \mu_i(k)\, \boldsymbol{U}_{\sigma i}^2 + \Delta\boldsymbol{\Phi}_\sigma \tag{12a}$$

$$\boldsymbol{\Gamma}_{0\sigma} = \sum_{j=1}^{N+1} \nu_j(k)\, \boldsymbol{U}_{\sigma j}^0 + \Delta\boldsymbol{\Gamma}_{0\sigma} \tag{12b}$$

$$\boldsymbol{\Gamma}_{1\sigma} = \sum_{i=1}^{M+1} \mu_i(k)\, \boldsymbol{U}_{\sigma i}^1 + \sum_{j=1}^{N+1} -\nu_j(k)\, \boldsymbol{U}_{\sigma j}^0 + \Delta\boldsymbol{\Gamma}_{1\sigma} \tag{12c}$$

$$= \sum_{i=1}^{M+1} \sum_{i=1}^{M+1} \mu_i(k)\, \nu_j(k)\, (\boldsymbol{U}_{\sigma i}^1 - \boldsymbol{U}_{\sigma j}^0) + \Delta\boldsymbol{\Gamma}_{1\sigma}$$

with the uncertain parameters

$$\mu_i(k) = \mu_i\left(h_{\sigma(k)}\right) \geq 0, \qquad \sum_{i=1}^{M+1} \mu_i(k) = 1 \tag{13a}$$

$$\nu_j(k) = \nu_j\left(g_{\sigma(k)}\right) \geq 0, \qquad \sum_{i=1}^{N+1} \nu_j(k) = 1. \tag{13b}$$

Equation (12c) comprises a Minkowski addition of two polytopes. The time-varying vertices of the polytopes are

$$\boldsymbol{U}_{\sigma j}^0 = \left(\frac{\boldsymbol{A}_c{}^{N-1}}{N!} \;\; \frac{\boldsymbol{A}_c{}^{N-2}}{(N-1)!} \;\ldots\; \frac{\boldsymbol{A}_c}{2!} \;\; \boldsymbol{I} \;\; \boldsymbol{0} \right) \boldsymbol{\phi}_j(N, g_\sigma) \boldsymbol{B}_c \tag{14a}$$

$$\boldsymbol{U}_{\sigma i}^1 = \left(\frac{\boldsymbol{A}_c{}^{M-1}}{M!} \;\; \frac{\boldsymbol{A}_c{}^{M-2}}{(M-1)!} \;\ldots\; \frac{\boldsymbol{A}_c}{2!} \;\; \boldsymbol{I} \;\; \boldsymbol{0} \right) \boldsymbol{\phi}_i(M, h_\sigma) \boldsymbol{B}_c \tag{14b}$$

$$\boldsymbol{U}_{\sigma i}^2 = \left(\frac{\boldsymbol{A}_c{}^{M}}{M!} \;\; \frac{\boldsymbol{A}_c{}^{M-1}}{(M-1)!} \;\ldots\; \frac{\boldsymbol{A}_c{}^2}{2!} \;\; \boldsymbol{A}_c \;\; \boldsymbol{I} \right) \boldsymbol{\phi}_i(M, h_\sigma) \tag{14c}$$

where $\boldsymbol{\phi}_i(P, \rho)$ with $\underline{\rho} \leq \rho \leq \overline{\rho}$ is defined as

$$\boldsymbol{\phi}_1(P, \rho) = \left(\underline{\rho}^P \boldsymbol{I} \quad \underline{\rho}^{P-1} \boldsymbol{I} \;\ldots\; \underline{\rho}^2 \boldsymbol{I} \quad \underline{\rho} \boldsymbol{I} \quad \boldsymbol{I} \right)^T \tag{15a}$$

$$\boldsymbol{\phi}_2(P, \rho) = \left(\underline{\rho}^P \boldsymbol{I} \quad \underline{\rho}^{P-1} \boldsymbol{I} \;\ldots\; \underline{\rho}^2 \boldsymbol{I} \quad \overline{\rho} \boldsymbol{I} \quad \boldsymbol{I} \right)^T \tag{15b}$$

$$\vdots$$

$$\boldsymbol{\phi}_{P+1}(P, \rho) = \left(\overline{\rho}^P \boldsymbol{I} \quad \overline{\rho}^{P-1} \boldsymbol{I} \;\ldots\; \overline{\rho}^2 \boldsymbol{I} \quad \overline{\rho} \boldsymbol{I} \quad \boldsymbol{I} \right)^T . \tag{15c}$$

The Taylor series expansion (10) and the polytopic formulation (12) of the plant are equivalent since there always exists a mapping between $h_{\sigma(k)}$, $g_{\sigma(k)}$ and $\mu_i(k)$, $\nu_j(k)$.

Substituting the polytopic matrices (12) into the augmented discrete-time state equation (2) of the plant, extracting the sums and considering switching of $\sigma(k)$, the switched augmented polytopic discrete-time state equation with additive norm-bounded uncertainty is obtained:

$$\boldsymbol{x}(k+1) = \left(\sum_{l=1}^{L} \sum_{i=1}^{M+1} \sum_{j=1}^{N+1} \xi_l(k)\, \mu_i(k)\, \nu_j(k)\, \boldsymbol{A}_{lij} + \Delta\boldsymbol{A}_l \right) \boldsymbol{x}(k)$$

$$+ \left(\sum_{l=1}^{L} \sum_{j=1}^{N+1} \xi_l(k)\, \nu_j(k)\, \boldsymbol{B}_{lj} + \Delta\boldsymbol{B}_l \right) \boldsymbol{u}(k) \tag{16}$$

where

$$\boldsymbol{A}_{lij} = \begin{pmatrix} \boldsymbol{U}_{li}^2 & \boldsymbol{U}_{li}^1 - \boldsymbol{U}_{lj}^0 \\ \boldsymbol{0} & \boldsymbol{0} \end{pmatrix}, \quad \boldsymbol{B}_{lj} = \begin{pmatrix} \boldsymbol{U}_{lj}^0 \\ \boldsymbol{I} \end{pmatrix} \tag{17a}$$

$$\Delta\boldsymbol{A}_l = \begin{pmatrix} \Delta\boldsymbol{\Phi}_l & \Delta\boldsymbol{\Gamma}_{1l} \\ \boldsymbol{0} & \boldsymbol{0} \end{pmatrix}, \quad \Delta\boldsymbol{B}_l = \begin{pmatrix} \Delta\boldsymbol{\Gamma}_{0l} \\ \boldsymbol{0} \end{pmatrix}. \tag{17b}$$

The matrices in (17a) represent vertices of a switched polytopic uncertainty, the matrices in (17b) form vertices of a switched additive norm-bounded uncertainty since the time-varying sampling period and time delay are bounded on $\mathcal{I}_\sigma$.

In the following section switched state feedback control is introduced. Control design and stability analysis methods are formulated in terms of LMIs.

3. CONTROL AND STABILITY

3.1 Closed-loop description

Consider the switched state feedback control law

$$\boldsymbol{u}(k) = \boldsymbol{K}_\sigma \boldsymbol{x}(k) = (\boldsymbol{K}_{x\sigma} \;\; \boldsymbol{K}_{u\sigma}) \begin{pmatrix} \boldsymbol{x}_c(k) \\ \boldsymbol{u}(k-1) \end{pmatrix} \tag{18}$$

where the switched feedback matrix is described by

$$\boldsymbol{K}_\sigma = \sum_{l=1}^{L} \xi_l(k)\, \boldsymbol{K}_l \,, \tag{19}$$

changing actively according to the active mode $\sigma(k)$.

Substituting (18) into (16) leads to the closed-loop state equation

$$\boldsymbol{x}(k+1) = \left(\sum_{l=1}^{L} \sum_{i=1}^{M+1} \sum_{j=1}^{N+1} \xi_l(k)\, \mu_i(k)\, \nu_j(k)\, \boldsymbol{H}_{lij} + \boldsymbol{\Theta}_l \right) \boldsymbol{x}(k) \tag{20}$$

where

$$\boldsymbol{H}_{lij} = \begin{pmatrix} \boldsymbol{U}_{li}^2 + \boldsymbol{U}_{lj}^0 \boldsymbol{K}_{xl} & \boldsymbol{U}_{li}^1 - \boldsymbol{U}_{lj}^0 + \boldsymbol{U}_{lj}^0 \boldsymbol{K}_{ul} \\ \boldsymbol{K}_{xl} & \boldsymbol{K}_{ul} \end{pmatrix} \tag{21a}$$

$$\boldsymbol{\Theta}_l = \begin{pmatrix} \Delta\boldsymbol{\Phi}_l + \Delta\boldsymbol{\Gamma}_{0l}\boldsymbol{K}_{xl} & \Delta\boldsymbol{\Gamma}_{1l} + \Delta\boldsymbol{\Gamma}_{0l}\boldsymbol{K}_{ul} \\ \boldsymbol{0} & \boldsymbol{0} \end{pmatrix}. \quad (21b)$$

The closed-loop state equation contains three independent time-varying parameters: $\xi_l(k)$ for switching, $\mu_i(k)$ for variations of the sampling period and $\nu_j(k)$ for variations of the actuation period. The matrices $\boldsymbol{H}_{lij} \in \mathbb{R}^{(n+m)\times(n+m)}$ again represent vertices of a switched polytopic uncertainty, the matrices $\boldsymbol{\Theta}_l \in \mathbb{R}^{(n+m)\times(n+m)}$ constitute vertices of a switched additive norm-bounded uncertainty. The norm in mode σ is upper-bounded by

$$\|\boldsymbol{\Theta}_\sigma\|_2 \le \gamma_\sigma^{\frac{1}{2}}\,, \qquad \gamma_\sigma = \sup_{\mathcal{I}_\sigma}\left\{\max_i\left(\lambda_i\left(\boldsymbol{\Theta}_\sigma^T\boldsymbol{\Theta}_\sigma\right)\right)\right\}. \quad (22)$$

The upper bound γ_σ depends on the feedback matrix $\boldsymbol{K}_\sigma$. Therefore the control design requires two steps: First a controller is designed neglecting the norm-bounded uncertainty, then this controller is analyzed for stability regarding the norm-bounded uncertainty. Contrary to previous works, the following theorems for control design and stability analysis consider all three time-varying parameters mentioned above.

3.2 Control synthesis

Theorem 1. The closed-loop system (20) without the norm-bounded uncertainty $\boldsymbol{\Theta}_l = \boldsymbol{0}$ is stabilizable by the switched state feedback control law (18) if there exist positive definite symmetric matrices $\boldsymbol{S}_{lij} \in \mathbb{R}^{(n+m)\times(n+m)}$ and matrices $\boldsymbol{G}_l \in \mathbb{R}^{(n+m)\times(n+m)}$ and $\boldsymbol{R}_l \in \mathbb{R}^{m\times(n+m)}$ such that the LMIs

$$\begin{pmatrix} \boldsymbol{G}_l + \boldsymbol{G}_l^T - \boldsymbol{S}_{lij} & \boldsymbol{G}_l^T\boldsymbol{A}_{lij}^T + \boldsymbol{R}_l^T\boldsymbol{B}_{lj}^T \\ \boldsymbol{A}_{lij}\boldsymbol{G}_l + \boldsymbol{B}_{lj}\boldsymbol{R}_l & \boldsymbol{S}_{rpq} \end{pmatrix} > 0 \quad (23)$$

are feasible $\forall l, r = 1, 2, \ldots, L$, $\forall i, p = 1, 2, \ldots, M+1$ and $\forall j, q = 1, 2, \ldots, N+1$. The switched feedback matrices are given by $\boldsymbol{K}_l = \boldsymbol{R}_l\boldsymbol{G}_l^{-1}$.

Proof. The proof follows from the proof of Theorem 1 in Hetel et al. (2006) by reformulation for all three time-varying parameters, refer also to the proof of the forthcoming Theorem 2. □

Remark. If the switching index $\sigma(k)$ is not available at sampling instant k, a static state feedback control law with the constant feedback matrix $\boldsymbol{K}$ can be applied, imposing the additional constraints $\boldsymbol{G}_l = \boldsymbol{G}$ and $\boldsymbol{R}_l = \boldsymbol{R}$ for all $l = 1, 2, \ldots, L$.

3.3 Stability analysis

Theorem 2. If there exist positive definite symmetric matrices $\boldsymbol{S}_{lij} \in \mathbb{R}^{(n+m)\times(n+m)}$ and symmetric matrices $\boldsymbol{G}_l \in \mathbb{R}^{(n+m)\times(n+m)}$ for all $l = 1, 2, \ldots, L$; $i = 1, 2, \ldots, M+1$ and $j = 1, 2, \ldots, N+1$ such that LMIs

$$\begin{pmatrix} \gamma_l\boldsymbol{I} - \boldsymbol{S}_{lij} & \boldsymbol{H}_{lij}\boldsymbol{G}_l & \boldsymbol{H}_{lij}\boldsymbol{G}_l \\ \boldsymbol{G}_l\boldsymbol{H}_{lij}^T & \boldsymbol{S}_{rpq} - 2\boldsymbol{G}_l & \boldsymbol{0} \\ \boldsymbol{G}_l\boldsymbol{H}_{lij}^T & \boldsymbol{0} & 2\boldsymbol{G}_l - \boldsymbol{I} \end{pmatrix} < 0 \quad (24)$$

are feasible $\forall l, r = 1, 2, \ldots, L$, $\forall i, p = 1, 2, \ldots, M+1$ and $\forall j, q = 1, 2, \ldots, N+1$ with $\|\boldsymbol{\Theta}_\sigma\|_2^2 \le \gamma_\sigma\ \forall\sigma \in \{1, \ldots, L\}$, then the switched closed loop (20) is stable for all tuples $(h_\sigma, \tau_\sigma) \in \mathcal{I}_\sigma$ with $\sigma \in \{1, \ldots, L\}$.

Proof. Following the proof given in Hetel et al. (2007) the LMIs (24) can be rewritten into the form of Lyapunov inequalities

$$\left(\boldsymbol{\mathcal{H}} + \overline{\boldsymbol{\Theta}}\right)\boldsymbol{\mathcal{S}}_+\left(\boldsymbol{\mathcal{H}} + \overline{\boldsymbol{\Theta}}\right)^T - \boldsymbol{\mathcal{S}} + \overline{\boldsymbol{\Theta}}_\gamma < 0 \quad (25)$$

with

$$\boldsymbol{\mathcal{H}} = \sum_{l=1}^{L}\xi_l(k)\,\boldsymbol{\mathcal{H}}_l\,, \quad \boldsymbol{\mathcal{H}}_\sigma = \sum_{i=1}^{M+1}\sum_{j=1}^{N+1}\mu_i(k)\,\nu_j(k)\,\boldsymbol{H}_{\sigma ij}\,,$$

$$\boldsymbol{\mathcal{S}} = \sum_{l=1}^{L}\sum_{i=1}^{M+1}\sum_{j=1}^{N+1}\xi_l(k)\,\mu_i(k)\,\nu_j(k)\,\boldsymbol{S}_{lij}\,,$$

$$\boldsymbol{\mathcal{S}}_+ = \sum_{l=1}^{L}\sum_{i=1}^{M+1}\sum_{j=1}^{N+1}\xi_l(k+1)\mu_i(k+1)\nu_j(k+1)\boldsymbol{S}_{lij}\,,$$

$$\overline{\boldsymbol{\Theta}}_\gamma = \sum_{l=1}^{L}\xi_l(k)\left(\gamma_l\boldsymbol{I} - \boldsymbol{\Theta}_l\boldsymbol{\Theta}_l^T\right), \quad \overline{\boldsymbol{\Theta}} = \sum_{l=1}^{L}\xi_l(k)\,\boldsymbol{\Theta}_l\,.$$

Requiring positive definiteness of $\overline{\boldsymbol{\Theta}}_\gamma$ is equivalent to the condition $\|\boldsymbol{\Theta}_\sigma\|_2^2 \le \gamma_\sigma\ \forall\sigma \in \{1, \ldots, L\}$. The polytopes $\boldsymbol{\mathcal{S}}$ and $\boldsymbol{\mathcal{S}}_+$ define a switched parameter-dependent quadratic Lyapunov function $\mathcal{V}(\boldsymbol{x}(k)) = (\boldsymbol{x}(k))^T\boldsymbol{\mathcal{S}}\boldsymbol{x}(k)$ at time instant k and $k+1$ respectively. The uncertain closed-loop dynamic with polytopic uncertainty is given by $\boldsymbol{\mathcal{H}}_\sigma$ and its switched version by $\boldsymbol{\mathcal{H}}$. The switched additive norm-bounded uncertainty is denoted as $\overline{\boldsymbol{\Theta}}$.

Our expanded form of the theorem and the proof provides γ_σ for each mode σ instead of one global γ as used in the original paper. □

If the feasibility problem (24) is not solvable, one can reduce the intervals $\mathcal{I}_\sigma$ in which the controller should provide stability or increase M, N which squeeze down the upper bound of the uncertainties γ_σ and approximate the original system more precisely. However, the number of LMIs required to be evaluated $((N+1)(M+1)L)^2 + (N+1)(M+1)L$ grows rapidly with increase of the approximation precision. It should also be noticed that Theorem 2 gives only a sufficient condition for the closed-loop stability. Therefore one should not overestimate the approximation precision and rather change the controller than let the number of LMIs increase excessively.

In the following section an algorithm for reducing the amount of LMIs is proposed which is essential to preserve computational tractability for high M and N.

4. REDUCING THE COMPUTATIONAL COMPLEXITY

A different consideration of the closed-loop system (20) is adopted firstly. The closed-loop system in a single mode σ equals a linear difference inclusion (LDI) with some convex hull $\boldsymbol{\Omega}_\sigma$ generated around the polytope $\boldsymbol{\mathcal{H}}_\sigma$ due to the norm-bounded uncertainty $\boldsymbol{\Theta}_\sigma$:

$$\boldsymbol{x}(k+1) \in (\boldsymbol{\mathcal{H}}_\sigma + \boldsymbol{\Theta}_\sigma)\,\boldsymbol{x}(k) \text{ with } (\boldsymbol{\mathcal{H}}_\sigma + \boldsymbol{\Theta}_\sigma) \in \boldsymbol{\Omega}_\sigma. \quad (26)$$

For the theory on polytopic and norm-bounded LDIs see e.g. Boyd et al. (1994), or more general Filippov (1988). The computational complexity of Theorem 2 grows quadratically with the number of vertices in each polytope $\boldsymbol{\mathcal{H}}_\sigma$ since the LMIs are constructed as pairwise combinations of all vertices in all polytopes. The key idea of the proposed algorithm is to reduce the number of vertices in each polytope while deliberately increasing the uncertainty bound to $\hat{\gamma}_\sigma = r_\sigma \cdot \gamma_\sigma$ with $1 \leq r_\sigma \in \mathbb{R}$. The new LDI

$$\boldsymbol{x}(k+1) \in (\widehat{\boldsymbol{\mathcal{H}}}_\sigma + \widehat{\boldsymbol{\Theta}}_\sigma)\boldsymbol{x}(k) \quad \text{with } (\widehat{\boldsymbol{\mathcal{H}}}_\sigma + \widehat{\boldsymbol{\Theta}}_\sigma) \in \widehat{\boldsymbol{\Omega}}_\sigma \tag{27}$$

has less vertices in the polytope $\widehat{\boldsymbol{\mathcal{H}}}_\sigma$ but larger convex domain $\widehat{\boldsymbol{\Omega}}_\sigma$ due to the increased upper uncertainty bound $\widehat{\boldsymbol{\Theta}}_\sigma$. After the reduction for each mode $\sigma \in \{1, \ldots, L\}$,

$$\|\boldsymbol{\Theta}_\sigma\|_2^2 = \gamma_\sigma \leq \hat{\gamma}_\sigma = \|\widehat{\boldsymbol{\Theta}}_\sigma\|_2^2\,, \quad \boldsymbol{\mathcal{H}}_\sigma \supseteq \widehat{\boldsymbol{\mathcal{H}}}_\sigma\,, \quad \boldsymbol{\Omega}_\sigma \subseteq \widehat{\boldsymbol{\Omega}}_\sigma\,,$$

holds and therefore $\bigcup_{\sigma=1}^{L} \boldsymbol{\Omega}_\sigma \subseteq \bigcup_{\sigma=1}^{L} \widehat{\boldsymbol{\Omega}}_\sigma$. If the stability of the new system with switched uncertainty

$$\boldsymbol{x}(k+1) \in (\widehat{\boldsymbol{\mathcal{H}}} + \widehat{\boldsymbol{\Theta}})\boldsymbol{x}(k) \quad \text{with} \tag{28}$$
$$\widehat{\boldsymbol{\mathcal{H}}} = \sum_{l=1}^{L} \xi_l(k)\, \widehat{\boldsymbol{\mathcal{H}}}_l\,, \quad \widehat{\boldsymbol{\Theta}} = \sum_{l=1}^{L} \xi_l(k)\, \widehat{\boldsymbol{\Theta}}_l$$

can be proved according to Theorem 2 (which requires lower number of LMIs), the stability of the original switched polytopic and additive norm-bounbed uncertain system (20) is guaranteed.

The reduction algorithm is formulated as follows:

Algorithm 1 Reducing the vertices in $\boldsymbol{\mathcal{H}}$

Require: r_σ and γ_σ
  for each mode σ set $i = 1$ **do**
    for $j = 1$ to $N{-}1$ **do**
      if $\boldsymbol{H}_{\sigma ij}$ has already been deleted **then**
        continue with the next for loop
      end if
      if it is possible to find $\alpha \in [0, 1]$ and some maximized index b such that for $a, b, c \in \mathbb{N}$ inequality $\|(1-\alpha)\boldsymbol{H}_{\sigma ia} + \alpha \boldsymbol{H}_{\sigma ib} - \boldsymbol{H}_{\sigma ic}\|_2^2 \leq (r_\sigma - 1)\gamma_\sigma$ holds $\forall c$ where $a = j < c < b \leq N{+}1$ **then**
        delete all $\boldsymbol{H}_{\sigma ic}$ where $a < c < b$ and $1 \leq i \leq M{+}1$
      end if
    end for
  end for
  for each mode σ set $j = 1$ **do**
    repeat the above procedure while iterating over the index i
  end for

Considering equation (21a) it is evident that $\boldsymbol{H}_{\sigma ic} - \boldsymbol{H}_{\sigma ia}$ and $\boldsymbol{H}_{\sigma ib} - \boldsymbol{H}_{\sigma ia}$ do not depend on index i. Due to this fact the main "for loop" in each mode σ is run with fixed $i = 1$ and $j = 1$ respectively. We will also omit the indices i, σ and j, σ respectively in $\boldsymbol{H}_{\sigma ij}$ in the next paragraph.

The algorithm searches for a scalar $\alpha \in [0, 1]$ for which both the equality and the inequality

$$(1-\alpha)\boldsymbol{H}_a + \alpha \boldsymbol{H}_b - \boldsymbol{H}_c = \boldsymbol{\Delta} \tag{29a}$$
$$\|\boldsymbol{\Delta}\|_2^2 \leq (\hat{\gamma}_\sigma - \gamma_\sigma) \tag{29b}$$

hold for some given $\boldsymbol{H}_a, \boldsymbol{H}_b, \boldsymbol{H}_c \in \boldsymbol{\mathcal{H}}_\sigma$. Then the vertex $\boldsymbol{H}_c$ can be omitted in the new polytope $\widehat{\boldsymbol{\mathcal{H}}}_\sigma$ because it is replaced by the convex combination of $\boldsymbol{H}_a, \boldsymbol{H}_b$ and the new resulting uncertainty bound $\|\boldsymbol{\Delta}\|_2 + \gamma_\sigma^{\frac{1}{2}}$ is lower than predefined $\hat{\gamma}_\sigma^{\frac{1}{2}}$. This is also illustrated in Fig. 2. In order to find such α for which $\|\boldsymbol{\Delta}\|_2^2$ is minimized we will look for α^* minimizing the squared Frobenius norm $\|\boldsymbol{\Delta}\|_F^2 = \operatorname{trace}(\boldsymbol{\Delta}^T\boldsymbol{\Delta})$ since $\|\boldsymbol{\Delta}\|_2^2 \leq \|\boldsymbol{\Delta}\|_F^2$. The advantage of minimizing the Frobenius norm is that α^* can very easily be computed analytically using the orthogonal projection of $\boldsymbol{Z}_0 = \boldsymbol{H}_c - \boldsymbol{H}_a$ onto $\boldsymbol{Z}_1 = \boldsymbol{H}_b - \boldsymbol{H}_a$ in a Hilbert space with inner product defined as $\operatorname{trace}(\boldsymbol{Z}_1^T\boldsymbol{Z}_0)$:

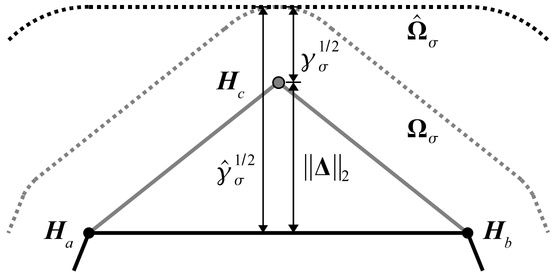

Fig. 2. Reduced polytope $\widehat{\boldsymbol{\mathcal{H}}}_\sigma$ (black) has a larger convex hull and contains the original polytope $\boldsymbol{\mathcal{H}}_\sigma$ (gray).

$$\alpha^* = \operatorname{trace}\left(\boldsymbol{Z}_1^T\boldsymbol{Z}_0\right)/\operatorname{trace}\left(\boldsymbol{Z}_1^T\boldsymbol{Z}_1\right). \tag{30}$$

α^* is then substituted into (29a) in order to obtain the value of $\|\boldsymbol{\Delta}\|_2^2$ and (29b) is checked.

In the next section an example is presented which demonstrates the control design and stability analysis together with the reduction algorithm for the stability analysis part.

5. EXAMPLE

Consider the stable continuous-time plant given by the matrices

$$\boldsymbol{A}_c = \begin{pmatrix} 0 & 1 \\ -100 & -0.1 \end{pmatrix}; \quad \boldsymbol{B}_c = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{31}$$

The controller of the plant has to operate in two modes with predefined intervals for sampling period and time delay:

$$\sigma = 1: \quad h_1 \in [30, 50]\text{ms}\,, \quad \tau_1 \in [8,\ 16]\text{ms} \tag{32a}$$
$$\sigma = 2: \quad h_2 \in [90, 110]\text{ms}\,, \quad \tau_2 \in [40, 60]\text{ms}. \tag{32b}$$

Using Theorem 1 which considers only the polytopic uncertainty, the stabilizing controller has been designed for $M = 3$, $N = 4$ as

$$\boldsymbol{K}_1 = \begin{pmatrix} -6.575 \\ -0.153 \\ -0.041 \end{pmatrix}; \quad \boldsymbol{K}_2 = \begin{pmatrix} -4.785 \\ -0.482 \\ -0.275 \end{pmatrix}. \tag{33}$$

It is also possible to obtain different stabilizing controllers for some lower values of M, N as denoted in Table 1. As far as only a stabilizing controller for the switched uncertain system with polytopic uncertainty is designed no exact statements can be made about the stability of the switched uncertain system with polytopic and additive norm-bounded uncertainty. However, one can expect that if the upper bounds of the uncertainty norms γ_σ are "sufficiently small" the designed controller will also pass the stability analysis. For controller (33) stability is proved for $M = 5$ and $N = 5$. During the stability analysis of controller (33) the proposed reduction algorithm has been used. The scaling parameters are chosen as $r_1 = r_2 = r$ for the sake of simplicity. The stability analysis has been made for gradually increasing M and N where the scaling parameter r has been adjusted in a way to keep the number of LMIs at approximately 4000. After the execution of the reduction algorithm the LMIs were built from the remaining vertices $\boldsymbol{H}_{lij}$ according to Theorem 2 and

M	N	LMIs	feasible design	feasible analysis
2	2	324	yes	for $M=5$, $N=5$
3	3	1056	yes	for $M=5$, $N=5$
3	**4**	**1640**	**yes**	**for $M=5$, $N=5$**
4	4	2550	yes	for $M=5$, $N=5$
5	5	5256	yes	for $M=5$, $N=5$

Table 1. Feasibility of the control design and necessary M, N for feasible stability analysis.

M	N	r	$\hat{\gamma}_1$	$\hat{\gamma}_2$	LMIs	feasible
3	4	1.0	$2.30\,\mathrm{e}-5$	$2.93\,\mathrm{e}-2$	1640	no
4	4	1.0	$5.75\,\mathrm{e}-6$	$1.68\,\mathrm{e}-2$	2550	no
4	**5**	**1.0**	$\mathbf{5.77\,e-6}$	$\mathbf{1.68\,e-2}$	**3660**	**no**
5	5	1.0	$5.12\,\mathrm{e}-9$	$4.23\,\mathrm{e}-5$	5256	yes
5	**5**	**2.2**	$\mathbf{1.13\,e-8}$	$\mathbf{9.32\,e-5}$	**3660**	**yes**
5	5	2.5	$1.28\,\mathrm{e}-8$	$1.06\,\mathrm{e}-4$	3660	no
5	6	1.0	$1.35\,\mathrm{e}-9$	$3.56\,\mathrm{e}-5$	7140	yes
5	6	2.1	$2.83\,\mathrm{e}-9$	$7.48\,\mathrm{e}-5$	4422	yes
5	6	2.7	$3.64\,\mathrm{e}-9$	$9.62\,\mathrm{e}-5$	4422	no

Table 2. Stability analysis for controller (33). Feasibility, count of LMIs and upper bounds of uncertainties depend on M, N and r.

checked for feasibility. The results of the stability analysis for different settings of M, N and r are given in Table 2. For $M = 5$, $N = 5$ with reduction ($r = 2.2$) up to 30.3 % of the LMIs could be eliminated. Compared to $M = 4$, $N = 5$ without reduction ($r = 1.0$) the same number of LMIs (3660) is used for stability analysis. However, the upper bounds of the additive uncertainties $\hat{\gamma}_1$ and $\hat{\gamma}_2$ are considerably lower when applying the reduction algorithm. Hence a higher approximation precision is achieved while keeping the number of LMIs low without increasing the conservatism substantially, making the application of the reduction algorithm highly beneficial.

In this example the control design and stability analysis LMIs were implemented in YALMIP (Löfberg (2004)) and solved using the SeDuMi solver (Sturm (2001)).

6. CONCLUSIONS AND FUTURE WORK

In this paper control design and stability analysis methods are described for systems with uncertain and time-varying sampling period and time delay. The time-varying sampling period and time delay are assumed to be uncertain but bounded on switched intervals. This means that they jump arbitrarily between some finite number of modes and in each mode they can vary within some predefined bounded interval. The active mode is assumed to be known in real time. The value of the sampling period and time delay within the particular bounded interval is unknown.

Firstly an LMI approach has been proposed for the design of a stabilizing feedback control law in Theorem 1. Only a polytopic description of the interval uncertainties has been considered in this step while neglecting the additive norm-bounded uncertainty. Secondly the stability of the closed loop has been analyzed taking into account both polytopic and additive norm-bounded uncertainties in Theorem 2. In order to reduce the amount of LMIs the new reducing Algorithm 1 has been proposed. The stability condition given in Theorem 2 is only sufficient and the usage of the reduction algorithm increase its conservatism in an adjustable manner. It is worth to notice that the stability analysis method and reduction algorithm can be applied to an arbitrarily designed switched state feedback control.

Future work will focus on instability analysis to gain better insight into the stability properties of systems with switched interval uncertainty. Furthermore the reduction algorithm could be improved by posing the polytope and convex hull fitting as an optimization problem. Finally control design guaranteeing robust performance for systems with switched interval uncertainty will be studied.

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