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Linear Parameter Varying Control for Sampled-data Multi-rate Systems *

D. Antunes * **C. Silvestre** * **R. Cunha** *

* *Instituto Superior Técnico, Institute for Systems and Robotics*
Av. Rovisco Pais, 1, 1049-001 Lisboa, Portugal
{dantunes,cjs,rita}@isr.ist.utl.pt

Abstract: A new methodology for the design and implementation of linear parameter varying (LVP) controllers for multi-rate sampled-data systems is presented and its stability properties are analyzed. A controller structure is first proposed for the regulation of multi-rate systems with more measured outputs than inputs. This structure is specially suited for a gain-scheduling implementation that verifies an important property known as the linearization property. The proposed solution guarantees local stability of the feedback interconnection of the nonlinear multi-rate system and the LPV controller about individual equilibria, and ultimate boundedness of a conveniently defined closed-loop error in response to slowly varying exogenous inputs. An example is presented that illustrates the applicability of the proposed solution.

1. INTRODUCTION

This paper addresses the controller design problem for nonlinear sampled-data multi-rate systems, providing a linear-parameter varying (LPV) solution. We consider a continuous-time plant with multiple input and output channels, connected to a digital controller via multiple sample and hold devices running at different, a priori fixed rates. In contrast to other approaches which are based on controller emulation or direct discrete-time design for the numerical discretization of the plant, the proposed solution directly takes into account the sample and hold mechanism in the design phase.

Over the last decades the design of control laws for multi-rate systems has received considerable attention since different rates in sensor measurements and between sensors and actuators are often present in control problems. Typical cases arise from hardware restrictions, for example, due to the fact that the discrete-analog (D/A) converters are generally faster than the analog-discrete (A/D) converters or from the use of camera-based sensors, which might require larger sampling periods than other sensors and actuators (Fujimoto and Hori (2001)). Another important application arises in the design of integrated guidance and control systems for vehicular applications (Antunes et al. (2007), Antunes et al. (2008)), where the linear positioning sensors (e.g. Global Navigation Satellite System (GNSS)) are usually available at a lower rate than the remaining sensors. See also Tomizuka (2004) for other applications. The peculiarities of the multi-rate problems render non-trivial the generalization of standard results in single-rate digital control to the multi-rate case. The subject of linear multi-rate control, which is intimately

related to periodic systems theory, has been the object of extensive research. See, for example Bittanti and Bolzern (1985), Lall and Dullerud (2001), and Colaneri (1991), Scattolini and Schiavoni (1993), where a solution to the output regulation problem for multi-rate square systems was presented. More recent work has focused on the control of nonlinear multi-rate systems, see for example Polushin and Marquez (2004).

Closely related to gain-scheduling, Linear Parameter Varying controller design constitutes nowadays a powerful tool for tackling difficult nonlinear problems. As explained in detail in Rugh and Shamma (2000), the standard procedure for designing an LPV controller involves the selection of scheduling variables or parameters; linearization of the non-linear plant about the equilibrium manifold; synthesis of controllers for the family of plant linearizations, which typically involves linear controller design for a given set of equilibrium points; and implementation of the controller. The implementation must be such that the controller verifies the linearization property: At each equilibrium point, the nonlinear LPV controller must linearize to the linear controller designed for that equilibrium.

In this paper we tackle the problem of designing and implementing LPV controllers for nonlinear multi-rate systems. In particular, we consider non-square plants whose additional outputs are in general required to achieve enhanced performance or to obtain system detectability. Inspired by the D-methodology presented in Kaminer et al. (1995) the proposed solution consists of a gain-scheduled controller that provides integral action and verifies the linearization property taking into account the multi-rate characteristics of the original plant. Building upon the work presented in Rugh and Shamma (2000) for the continuous time case, and in Lawrence (1997, 2001) for the single-rate sampled-data case, we guarantee *i)* local stability of the feedback interconnection of the sampled-data multi-rate system and LPV controller about individual equilibria, and *ii)* ulti-

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mate boundedness of a conveniently defined closed-loop error in response to slowly varying exogenous inputs.

The remainder of the paper is organized as follows. We introduce the problem in Section 2, propose a solution in Section 3, and analyze the resulting stability properties in Section 4. An example is gradually presented along the paper to illustrate the different steps in building a solution. Finally, Section 5 presents the concluding remarks. The notation adopted is fairly standard. The space of n-dimensional continuous-time signals $x(t)$, $x : \mathbb{R}^+ \mapsto \mathbb{R}^n$, and discrete-time signals x_k, $x : \mathbb{Z}^+ \mapsto \mathbb{R}^n$, are denoted by $\mathcal{L}(\mathbb{R}^+)$ and $l(\mathbb{Z}^+)$, respectively. Given a matrix A, $\sigma(A)$ denotes its spectral radius and $\lambda_i(A)$ an eigenvalue. For $r > 0$ and $p \in \mathbb{R}^n$, $B_r(p)$ denotes the open ball $\{x : \|x - p\| < r\}$. A function ρ is said to belong to class $\mathcal{K}$ if it is continuous, strictly increasing, and zero at zero. For the sake of brevity, most of the proofs and technical results are either omitted or only outlined in the paper, and the reader is referred to Antunes et al. (2007) for a comprehensive presentation of this material.

2. PROBLEM FORMULATION

Consider the nonlinear system
$$G := \begin{cases} \dot{x}(t) = f(x(t), u(t), w(t)) \\ y(t) = h(x(t), w(t)) \end{cases} \quad (1)$$
where f and h are twice continuously differentiable functions, $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input, and the vector $w(t) \in \mathbb{R}^{n_w}$ contains references and possibly other exogenous inputs. The vector $y(t) \in \mathbb{R}^p$ can be decomposed as $y(t) = [y_m(t)^T \ y_r(t)^T]^T = [h_m(x(t), w(t))^T \ h_r(x(t), w(t))^T]^T$ where $y_m(t) \in \mathbb{R}^{n_{y_m}}$ is a vector of measured outputs available for feedback and $y_r(t) \in \mathbb{R}^{n_{y_r}}$ is a vector of tracking outputs, which we assume to have the same dimensions as the control input, $n_{y_r} = m$. This vector is required to track the reference $r(t)$ with zero steady state error, i.e., the error vector defined as $e(t) := y_r(t) - r(t)$ must satisfy $e(t) = 0$ at steady-state. Some of the components of $y_r(t)$ may be included in $y_m(t)$ as well.

2.1 Linearization family

We assume that there exists a unique family of equilibrium points for G of the form
$$\Sigma := \{(x_0, u_0, w_0) : f(x_0, u_0, w_0) = 0, \ y_{r0} = h_r(x_0, w_0) = r_0\}$$
which can be parameterized by a vector $\alpha_0 \in \Xi \in \mathbb{R}^s$, s.t.
$$\Sigma = \{(x_0, u_0, w_0) = a(\alpha_0), \alpha_0 \in \Xi\} \quad (2)$$
where a is a continuously differentiable function. We further assume that there exists a continuously differentiable function v such that $\alpha_0 = v(y_0, w_0)$. By applying the function v to the measured values of y and w, we obtain the variable
$$\alpha = v(y, w), \quad (3)$$
which is usually referred to as the scheduling variable.

Linearizing the nonlinear system G about the equilibrium manifold Σ parameterized by α_0 yields the family of linear systems

$$G_l(\alpha_0) := \begin{bmatrix} \dot{x}_\delta(t) \\ y_\delta(t) \end{bmatrix} = \begin{bmatrix} A(\alpha_0) & B_1(\alpha_0) & B_2(\alpha_0) \\ C_2(\alpha_0) & D_{21}(\alpha_0) & 0 \end{bmatrix} \begin{bmatrix} x_\delta(t) \\ w_\delta(t) \\ u_\delta(t) \end{bmatrix} \quad (4)$$

where, e.g. $A(\alpha_0) = \frac{\partial f}{\partial x}(a(\alpha_0))$ and $x_\delta(t) = x(t) - x_0$.

2.2 Multi-rate sensors and actuators

We consider that the sample and hold devices that interface the discrete-time controller and the continuous-time plant operate at different rates. Associated with each sampler S_i, corresponding to the ith component of $y(t)$, there is a sequence of sampling times $\{\sigma_1^i, \sigma_2^i, ..\}$ that verify $0 < \sigma_j^i < \sigma_{j+1}^i$. Similarly, associated with each holder H_i, corresponding to the ith component of $u(t)$, there is a sequence of hold times $\{\tau_1^i, \tau_2^i, ..\}$ that verify $0 < \tau_j^i < \tau_{j+1}^i$. We assume that the sample and hold operations are periodic and that their periods are related by rational numbers. Thus we can define a sequence of equally spaced time instants $\{t_0, t_1, ...\}$, $t_{k+1} - t_k = t_s, k \in \mathbb{Z}^+$, such that for every sampling time σ_j^i and hold time τ_j^i there exists a k_1 and a k_2 for which $\sigma_j^i = t_{k_1}$ and $\tau_j^i = t_{k_2}$. In addition, we introduce the matrix $\Gamma_k = \mathrm{diag}(g_1(k), ..., g_p(k))$, where $g_i(k) = 1$ if $\sigma_j^i = t_k$ for some j and $g_i(k) = 0$ otherwise, and the matrix $\Omega_k := \mathrm{diag}(r_1(k), ..., r_m(k))$, where $r_i(k) = 1$ if $\tau_j^i = t_k$ for some j and $r_i(k) = 0$ otherwise. Due to the periodic nature of the sample and hold devices we have $\Gamma_k = \Gamma_{k+h}$ and $\Omega_k = \Omega_{k+h}$, for some positive integer h which denotes the period. We further assume that every output is sampled and every input is updated at least once in a period.

The multi-rate sample and hold operators can then be written as
$$\begin{array}{ll} S : \mathcal{L}(\mathbb{R}^+) \mapsto l(\mathbb{Z}^+) & H : l(\mathbb{Z}^+) \mapsto \mathcal{L}(\mathbb{R}^+) \\ S = \Gamma_d S_{t_s} & H = H_{t_s} \Omega_d \end{array} \quad (5)$$
where the operators $\Omega_d : l(\mathbb{Z}^+) \mapsto l(\mathbb{Z}^+)$, $H_{t_s} : l(\mathbb{Z}^+) \mapsto \mathcal{L}(\mathbb{R}^+)$, $S_{t_s} : \mathcal{L}(\mathbb{R}^+) \mapsto l(\mathbb{Z}^+)$ and $\Gamma_d : l(\mathbb{Z}^+) \mapsto l(\mathbb{Z}^+)$ are given by
$$\Omega_d : \xi_{k+1} = (I - \Omega_k)\xi_k + \Omega_k u_k, \ \xi_0 = 0 \qquad H_{t_s} u(t) = \tilde{u}_k$$
$$\tilde{u}_k = (I - \Omega_k)\xi_k + \Omega_k u_k \qquad t \in [t_k, t_{k+1}[$$
$$S_{t_s} : \tilde{y}_k = \begin{bmatrix} \tilde{y}_{mk} \\ \tilde{y}_{rk} \end{bmatrix} = y(t_k) \qquad \Gamma_d : y_k = \begin{bmatrix} \Gamma_{mk} & 0 \\ 0 & \Gamma_{rk} \end{bmatrix} \begin{bmatrix} \tilde{y}_{mk} \\ \tilde{y}_{rk} \end{bmatrix}$$
and $\Gamma_k = \begin{bmatrix} \Gamma_{mk} & 0 \\ 0 & \Gamma_{rk} \end{bmatrix}$ is partitioned according to the output decomposition $y^T = [y_m^T \ y_r^T]$. Similarly to $y(t)$, the values of $r(t)$ sampled at the instants t_k are denoted by $\tilde{r}_k$ and to take into account the multi-rate nature of the outputs we define $r_k = \Gamma_{rk}\tilde{r}_k$. We can then introduce the error variables $\tilde{e}_k = \tilde{y}_{rk} - \tilde{r}_k$ and $e_k = y_{rk} - r_k$.

2.3 Problem Statement

Given this setup the problem addressed in this paper can be stated as follows:

Problem 1. 1) For a fixed operating point α_0, find a possibly time-varying discrete-time linear controller $C(\alpha_0)$: $[y_{\delta mk}, e_k] \mapsto u_{\delta k}$

$$C(\alpha_0) = \left\{ \begin{bmatrix} x_{\delta k+1}^c \\ u_{\delta k} \end{bmatrix} = \begin{bmatrix} A_k^c(\alpha_0) & B_{1k}^c(\alpha_0) & B_{2k}^c(\alpha_0) \\ C_k^c(\alpha_0) & D_{1k}^c(\alpha_0) & D_{2k}^c(\alpha_0) \end{bmatrix} \begin{bmatrix} x_{\delta k}^c \\ y_{\delta mk} \\ e_k \end{bmatrix} \right. \quad (6)$$

for the linearization of the nonlinear plant (4) with multi-rate interface (5) that stabilizes the closed loop system and achieves zero steady-state error for e_k, where $u_{\delta k} = u_k - u_0$, $y_{\delta mk} = y_{mk} - \Gamma_{mk} y_{m0}$, and $y_{m0} = h_m(x_0, w_0)$.

2) Based on the family of linear controllers $C(\alpha_0)$, implement a discrete-time controller K, possibly nonlinear and time-varying, that verifies the linearization property, which will be defined shortly, and takes the form

$$K = \begin{cases} x_{k+1}^c = & f_c(x_k^c, y_{mk}, e_k, \alpha_k, k) \\ u_k = & h_c(x_k^c, y_{mk}, e_k, \alpha_k, k) \end{cases}, \qquad (7)$$

where $x_k^c \in \mathbb{R}^{n_K}$ and given its dependence on the scheduling variable α_k sampled at time t_k, K is referred to as a gain-scheduled controller. By linearization property we formally mean that if we consider a family of equilibrium points Σ^c for the controller compatible with the family of equilibrium points Σ defined in (2), such that

$$\Sigma^c := \{ x_0^c : x_0^c = f_c(x_0^c, y_{m0}, 0, \alpha_0, k), u_0 = h_c(x_0^c, y_{m0}, 0, \alpha_0, k),$$
$$(x_0, u_0, w_0) = a(\alpha_0), \ \alpha_0 \in \Xi \}$$

the controller K linearizes to $C(\alpha_0)$, at each equilibrium point α_0, that is, for example, $\frac{\partial f_c}{\partial x^c}(\alpha_0, k) = A_k^c(\alpha_0)$ and $\frac{\partial h_c}{\partial y_m}(\alpha_0, k) = D_{1k}^c(\alpha_0)$.

The proposed solution for part 2) has an LPV structure and we will show that under mild assumptions the linearization property just described is sufficient to guarantee local stability about each equilibrium point of the feedback interconnection of the non-linear plant and LPV controller with multi-rate interface.

3. PROPOSED SOLUTION

3.1 Regulator structure

In this section we focus on solving part 1) of the problem statement. To this end, we consider a simple linear system of the form

$$G_L = \begin{cases} \dot{x}(t) = Ax(t) + B_2 u(t) \\ y(t) = \begin{bmatrix} y_m(t) \\ y_r(t) \end{bmatrix} = C_2 x(t) = \begin{bmatrix} C_m \\ C_r \end{bmatrix} x(t) \end{cases}, \qquad (8)$$

where $u(t)$ and $y_r(t) \in \mathbb{R}^m$. It is well-known that zero-error output regulation for constant references can be achieved by incorporating in the controller structure a number of integrators equal to the number of regulated outputs (Francis and Wonham (1976)). Regulation for constant references is not tied in with linearity and is achieved even in the presence of model uncertainties that do not affect closed loop stability (Khalil (2000)). For the single-rate case, integral action is typically applied to the regulated errors directly (Khalil (2000), Kaminer et al. (1995)). For the multi-rate case, the controller is in general required to be time-varying, and as presented in Colaneri et al. (1991); Scattolini and Schiavoni (1993) for square systems, the integrators should be placed at the plant's input. Note that due to the time-varying characteristics of the controller, directly integrating the errors would produce a non-constant signal at the plant's input. However, in many applications the number of available outputs need be greater than the dimension of the actuation vector to achieve enhanced performance or to obtain system detectability, as will be illustrated shortly by an example.

Motivated by this discussion we propose the controller structure depicted in Fig. 1 for the regulation of non-square systems. In the figure C_I and C_D correspond to discrete time linear periodic integrators and differentiators, respectively, that can be written as

$$C_I = \begin{cases} x_{k+1}^I = x_k^I + \Omega_k u_k^I \\ y_k^I = x_k^I + \Omega_k u_k^I \end{cases} \qquad (9)$$

$$C_D = \begin{cases} x_{k+1}^D = (I - \Gamma_{mk}) x_k^D + \Gamma_{mk} u_k^D \\ y_k^D = -\Gamma_{mk} x_k^D + \Gamma_{mk} u_k^D \end{cases} \qquad (10)$$

At equilibrium, the constant values of the non regulated outputs are differentiated thereby obviating the need to feedforward these values, whose accurate determination is in general precluded by the presence of model uncertainties. It is straightforward to derive expressions for $\Gamma_D = C_D \Gamma_d$, $\Omega_I = \Omega_d C_I$ and G_d which results from discretizing G_L and is given by

$$G_d = \begin{cases} x_{k+1} = A_d x_k + B_d \tilde{u}_k \\ \tilde{y}_k = \begin{bmatrix} \tilde{y}_{mk} \\ \tilde{y}_{rk} \end{bmatrix} = C_d x_k = \begin{bmatrix} C_{dm} \\ C_{dr} \end{bmatrix} x_k \end{cases}, \qquad (11)$$

where $A_d = e^{At_s}$, $B_d = \int_0^{t_s} e^{A\tau} d\tau B_2$ and $C_d = C_2$. Consider the system G_a seen by the controller and corresponding to the series connection of Ω_I, G_d and Γ_D, $G_a = \Gamma_D G_d \Omega_I$. The next result guarantees that, under mild assumptions, this augmented system G_a preserves the detectability and stabilizability properties of the original plant G_d. Note that G_a is a linear time-varying periodic system. For the definitions of stabilizability and detectability for these systems see Bittanti and Bolzern (1985).

Lemma 2. Assume that the following conditions hold:
i) (A_d, B_d) is stabilizable and (A_d, C_d) is detectable.
ii) If there exists $\lambda_k(A_d) = \lambda_k$ such that $\|\lambda_k\| = 1$ then $\lambda_k^h \neq 1$, and if there exists a pair $\lambda_i(A_d) = \lambda_i$, $\lambda_j(A_d) = \lambda_j$ such that $\lambda_i \neq \lambda_j$, $\|\lambda_i\| \geq 1$ and $\|\lambda_j\| \geq 1$ then $\lambda_j^h \neq \lambda_i^h$.
iii) There are no transmission zeros at $z = 1$ from the input of G_d to the regulated output, i.e.

$$\begin{bmatrix} A_d - I & B_d \\ C_{dr} & 0 \end{bmatrix} \qquad (12)$$

is full rank.
Then, the periodic system $G_a = \Gamma_D G_d \Omega_I$ is detectable and stabilizable.

Under the stated assumptions, the stabilizability and detectability of G_a ensures that there exists an asymptotic stabilizing controller C_K for G_a (Colaneri (1991)). It is also straightforward to show that this structure achieves zero-output regulation for $e(t)$ as stated in the next result.
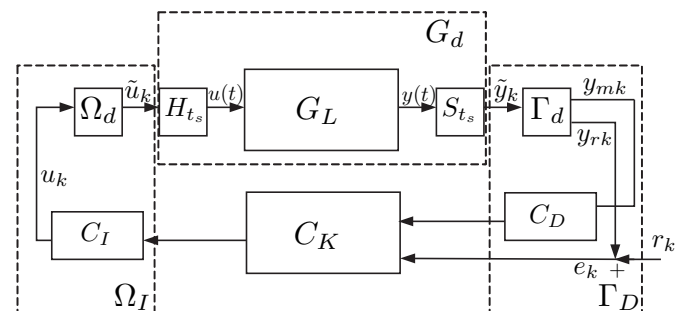

Fig. 1. Regulator structure for non-square systems

The proof follows from the use of integral action and the asymptotic stability of the feedback interconnection.

Lemma 3. Consider the feedback interconnection of G_a and C_K and suppose the controller C_K asymptotically stabilizes the resulting closed loop system. Then, zero-error output regulation for $y_r(t)$ is achieved even in the presence of plant uncertainty, provided that closed-loop stability is preserved.

Going back to the original problem stated in Section 2.3, we can conclude from Lemma 3 that a stabilizing controller of the form

$$C(\alpha_0) = \begin{cases} \begin{bmatrix} x^K_{\delta k+1} \\ y^K_{\delta k} \end{bmatrix} = \begin{bmatrix} A^K_k(\alpha_0) & B^K_{1k}(\alpha_0) & B^K_{2k}(\alpha_0) \\ C^K_k(\alpha_0) & D^K_{1k}(\alpha_0) & D^K_{2k}(\alpha_0) \end{bmatrix} \begin{bmatrix} x^K_{\delta k} \\ y^D_{\delta k} \\ e_{\delta k} \end{bmatrix} \\ \begin{bmatrix} x^D_{\delta k+1} \\ y^D_{\delta k} \end{bmatrix} = \begin{bmatrix} I - \Gamma_{mk} & \Gamma_{mk} \\ -\Gamma_{mk} & \Gamma_{mk} \end{bmatrix} \begin{bmatrix} x^D_{\delta k} \\ y_{\delta mk} \end{bmatrix} \\ \begin{bmatrix} x^I_{\delta k+1} \\ u_{\delta k} \end{bmatrix} = \begin{bmatrix} I & \Omega_k \\ I & \Omega_k \end{bmatrix} \begin{bmatrix} x^I_{\delta k} \\ y^K_{\delta k} \end{bmatrix} \end{cases}$$

(13)

where the first subsystem is a realization for $C_K(\alpha_0)$, provides a solution for part 1) of the problem statement, for fixed α_0. We assume that the design phase produces a family of controllers $C(\alpha_0)$ such that its parameters are continuously differentiable functions of α_0. The next example illustrates some of the concepts introduced so far.

Example 1. Suppose the nonlinear system (1) is given by

$$G = \begin{cases} \dot{x}_1(t) = -x_1(t) - x_2(t) \\ \dot{x}_2(t) = -x_2(t) + u(t) \\ \dot{x}_3(t) = -x_2(t) + x_2^3(t) + 0.5x_3(t) + u(t) \end{cases}. \quad (14)$$

The output $y_r(t) = x_1(t)$ is required to track the reference $r(t)$ with zero steady-state error. Considering $\alpha_0 = r_0$, the equilibrium manifold (2) is given by

$$\Sigma = \{x_{10} = -x_{20} = -u_0 = \alpha_0, \; x_{30} = 2\alpha_0^3, \; \alpha_0 \in \mathbb{R}\}.$$

The linearization family $G_l(\alpha_0)$, described by (4), can also be easily obtained, where for example

$$A(\alpha_0) = \begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & -1+3\alpha_0^2 & 0.5 \end{bmatrix}.$$

Consider the problem of designing a linear controller for $G_l(\alpha_0)$ with $\alpha_0 = 0$. Notice that if we consider the output to be $y_r = x_1$ this linear system is non-minimum phase. Furthermore, it is straightforward to check that $G_l(0)$ is not detectable from the regulated output $y_r = x_1$. Hence we assume that x_3 is also available for feedback and set $y_m = [x_1 \; x_3]^T$, which implies that the system becomes non-square. The sampling and updating periods for y_{m1}, y_{m2}, and u are set to $t_{sy_1} = 0.25$, $t_{sy_2} = 0.1$ and $t_{su} = 0.05$, respectively. According to the framework of Section 2.2 we have $t_s = 0.05$, $h = 10$, and the h-periodic matrices Γ_{mk}, Γ_{rk}, Ω_k are determined by

$$\Gamma_{mk} = \begin{bmatrix} \sigma_k^1 & 0 \\ 0 & \sigma_k^2 \end{bmatrix}, \; \Gamma_{rk} = \sigma_k^3, \; \Omega_k = \tau_k^1 = 1, \forall_k$$

$$\sigma_k^2 = \begin{cases} 1 & k \text{ odd,} \\ 0 & \text{otherwise} \end{cases}, \quad \sigma_k^1 = \sigma_k^3 = \begin{cases} 1 & k = 1, 6 \\ 0 & \text{otherwise} \end{cases}.$$

The discretization of $G_l(0)$ verifies all the conditions of Lemma 2 and therefore a linear controller with the structure (13) can be synthesized for this system. The stabilizing controller C_K in (13) is obtained using the standard H_2

output-feedback synthesis solution for periodic systems. The performance of this controller will be evaluated in simulation and compared with that of a gain-scheduled controller in Section 4, where it will be shown that zero-steady state error is obtained for $y_r = x_1$.

3.2 Gain-scheduled implementation

Having designed the parameterized family of linear controllers $C(\alpha_0)$ as described in (13), suppose we implement the gain-scheduled nonlinear controller K as follows

$$K = \begin{cases} \begin{bmatrix} x^K_{k+1} \\ y^K_k \end{bmatrix} = \begin{bmatrix} A^K_k(\alpha_k) & B^K_{1k}(\alpha_k) & B^K_{2k}(\alpha_k) \\ C^K_k(\alpha_k) & D^K_{1k}(\alpha_k) & D^K_{2k}(\alpha_k) \end{bmatrix} \begin{bmatrix} x^K_k \\ y^D_k \\ e_k \end{bmatrix} \\ \begin{bmatrix} x^D_{k+1} \\ y^D_k \end{bmatrix} = \begin{bmatrix} I - \Gamma_{mk} & \Gamma_{mk} \\ -\Gamma_{mk} & \Gamma_{mk} \end{bmatrix} \begin{bmatrix} x^D_k \\ y_{mk} \end{bmatrix} \\ \begin{bmatrix} x^I_{k+1} \\ u_k \end{bmatrix} = \begin{bmatrix} I & \Omega_k \\ I & \Omega_k \end{bmatrix} \begin{bmatrix} x^I_k \\ y^K_k \end{bmatrix} \\ \alpha_k = g(\tilde{y}_k, w_k) \\ \begin{bmatrix} x^Y_{k+1} \\ \tilde{y}_k \end{bmatrix} = \begin{bmatrix} I - \Gamma_k & \Gamma_k \\ I - \Gamma_k & \Gamma_k \end{bmatrix} \begin{bmatrix} x^Y_k \\ y_k \end{bmatrix} \quad \text{(15.A)} \end{cases}$$

(15)

which is an LPV controller. Notice that α_k, which was considered to be a constant design parameter during the design process, now becomes a scheduling variable computed on-line from the plant outputs and exogenous variables. Due to the multi-rate nature of the output, the system described by (15.A) is used to perform a hold operation on the output y_k so that the scheduling variable α_k is computed, at each iteration, according to the last sampled value of the output. The exogenous vector is assumed to be available at each sampling instant, so that $w_k = w(t_k)$. Notice that the non-linear controller proposed in (15) conforms to the general description of K given in (7) with $x^c_k = \left[(x^K_{k+1})^T \; (x^D_{k+1})^T \; (x^I_{k+1})^T \right]^T$. Moreover as we will show in Section 4.1, it verifies the linearization property and therefore constitutes a solution to part 2) of the problem statement.

As a final remark, when the multi-rate set-up particularizes to the single-rate case, there is a close relation between the method presented herein and the velocity implementation (Kaminer et al. (1995)), which is a method to implement gain-scheduled controllers for the single-rate case. See Antunes et al. (2007) for the details.

4. STABILITY PROPERTIES

In this section we show that the linearization property holds for the gain-scheduled implementation (15) and establish the results of local stability and ultimate boundedness in response to exogenous inputs for the feedback interconnection of the nonlinear multi-rate system and the proposed controller. These two last results can be obtained using the theoretical framework of jump systems and building upon the work presented in Lawrence (1997, 2001) for the single-rate sampled-data case.

In what follows, the feedback interconnection of the nonlinear system G, described by (1), and gain-scheduled controller K, described by (15), with multi-rate sampled-data interface (5) is denoted by $F_{nl} := \mathcal{F}(SGH, K)$. Similarly, for each α_0, the feedback interconnection of

the linearized system $G_l(\alpha_0)$, described by (4), and the designed controller $C(\alpha_0)$, described by (13), with multi-rate sampled-data interface (5) is denoted by $F_l(\alpha_0) := \mathcal{F}(SG_l(\alpha_0)H, C(\alpha_0))$.

4.1 Linearization property

The required linearization property for the gain-scheduled controller implementation (15) is stated in the next result.

Theorem 4. Suppose for each parameter vector $\alpha_0 \in \Xi$, $F_l(\alpha_0)$ is asymptotically stable. Then F_{nl} admits an unique equilibrium point associated with α_0 and the linearization of the gain-scheduled controller K about this equilibrium coincides with the designed controller $C(\alpha_0)$.

4.2 Local stability at each operating point

In order to establish local stability for F_{nl} about each equilibrium point, we start by considering the generic feedback interconnection of a continuous-time plant and a discrete-time periodically time-varying controller

$$
\begin{aligned}
\dot{x}(t) &= f(x(t), u(t)) & z_{k+1} &= a(z_k, y_k, k) \\
y(t) &= h(x(t)), \ t \geq 0 & u_k &= c(z_k, y_k, k), \ k \geq 0
\end{aligned}
\quad (16)
$$

with standard sample and hold interface

$$
u(t) = u_k, \ t \in [t_k, t_{k+1}[, \ t_{k+1} - t_k = t_s \quad y_k = y(t_k) \quad (17)
$$

where functions f and h are continously differentiable, and a and c are continously differentiable in z and y and h-periodic in k, e.g., $a(.,.,k) = a(.,.,k+h)$. Assuming that the closed-loop system has a single equilibrium point at the origin $f(0,0) = 0$, $h(0) = 0$, $a(0,0,k) = 0$, $c(0,0,k) = 0$ and introducing the vector

$$
x_s(t) = [x(t)^T \ z_{k+1}^T \ u_k^T]^T, \ t \in [t_k, t_{k+1}[,
$$

local exponential stability for this interconnection about the origin equilibrium can be defined as

$$
\exists_{r_0, C, \gamma} : \forall_{t_0 \geq 0, x_s(t_0) \in B_{r_0}(0)} \|x_s(t)\| \leq Ce^{-\gamma(t-t_0)}\|x_s(t_0)\|, \\
t \geq t_0.
$$

Consider also the feedback interconnection of the linearized system and linearized controller

$$
\begin{aligned}
\dot{x}_\delta(t) &= Ax_\delta(t) + Bu_\delta(t) & z_{\delta k+1} &= N_k z_{\delta k} + M_k y_{\delta k} \\
y_\delta(t) &= Cx_\delta(t), \quad t \geq 0 & u_{\delta k} &= L_k z_{\delta k} + K_k y_{\delta k}, \ k \geq 0
\end{aligned}
\quad (18)
$$

with sample and hold interface similar to that defined in (17), and where, for example, $B = \frac{\partial f}{\partial u}(0,0)$, $N_k = \frac{\partial a}{\partial z}(0,0,k)$ and $K_k = \frac{\partial c}{\partial y}(0,0,k)$. Notice that N_k, M_k, L_k, K_k are h-periodic matrices. The next theorem establishes the stability relation between systems (16) and (18).

Theorem 5. The following statements are equivalent:
i) The system described by (16) is exponentially stable about the origin.
ii) The linearized system (18) is exponentially stable.
iii) For $A_d = e^{At_s}$ and $B_d = \int_0^{t_s} e^{A\tau} d\tau B$,

$$
\sigma(\prod_{k=1}^h \begin{bmatrix} A_d + B_d K_k C & B_d L_k \\ M_k C & N_k \end{bmatrix}) < 1 \quad (19)
$$

The local stability property for F_{nl} is given by the following corollary.

Corollary 6. Suppose for each parameter vector $\alpha_0 \in \Xi$, $F_l(\alpha_0)$ is asymptotically (exponentially) stable. Then F_{nl}

is locally exponentially stable about each equilibrium point associated with α_0.

Proof. About each equilibrium point, characterized by constant α_0 and $w(t) = w_0$, the nonlinear system G can be rewritten as $\dot{x}_\delta(t) = \bar{f}(x_\delta(t), u_\delta(t))$, $y_\delta(t) = \bar{h}(x_\delta(t))$, where $\bar{f}(x_\delta(t), u_\delta(t)) := f(x_0 + x_\delta(t), u_0 + x_\delta(t), w_0)$, and $\bar{h}(x_\delta(t)) := h(x_0 + x_\delta(t))$ and a similar redefinition can be applied to K. Moreover, simple block manipulations show that $F_{nl} = \mathcal{F}(SGH, K) = \mathcal{F}(\Gamma_d S_{t_s} GH_{t_s} \Omega_d, K) = \mathcal{F}(S_{t_s} GH_{t_s}, \Omega_d K\Gamma_d)$. Hence, F_{nl} conforms to (16)-(17), where G is the continuous-time system and $\Omega_d K\Gamma_d$ is the discrete-time controller connected by standard sample and hold interface. According to (4) and the key linearization property of K, the linearizations of G and $\Omega_d K\Gamma_d$ are given by $G_l(\alpha_0)$ and $\Omega_d C(\alpha_0)\Gamma_d$, respectively. By Theorem 5 we can conclude that the asymptotic stability of $F_l(\alpha_0) = \mathcal{F}(S_{t_s} G_l(\alpha_0) H_{t_s}, \Omega_d C(\alpha_0)\Gamma_d)$ implies local exponential stability of F_{nl} about each equilibrium point.

4.3 Ultimate boundedness for slowly varying inputs

In this section we restrict the scheduling variable to depend solely on the exogenous inputs $\alpha(t) = w(t)$ so that we can impose a bound on its time-derivative. Then, building upon the work presented in Lawrence (1997, 2001) for single-rate sampled-data systems, it is possible to address the multi-rate case and show that for any initial condition starting near the equilibrium manifold described by Σ and Σ^c, the error between the state of F_{nl} and the corresponding equilibrium value parameterized by $w(t)$ is ultimately bounded, with an ultimate bound that depends on the time-derivative of $w(t)$. Defining

$$
x_s(t) := [x^T(t) \ (x_{k+1}^c)^T \ u_k^T]^T, \ t \in [t_k, t_{k+1}[,
$$

as the state of F_{nl} and

$$
x_{s0}(\alpha_0) := [x_0^T(\alpha_0) \ (x_0^c(\alpha_0))^T \ u_0^T(\alpha_0)]^T,
$$

as the corresponding parameterized equilibrium points, we can establish the following result.

Theorem 7. Suppose for each parameter vector $\alpha_0 \in \Xi$, $F_l(\alpha_0)$ is asymptotically stable. Then there exist positive constants δ_1, δ_2, k and γ, and a class $\mathcal{K}$ function $b(.)$ such that the following property holds. If, for any $t_0 \geq 0$, a continuously differentiable exogenous input $w(t) = \alpha(t)$ satisfies $w(t) \in \Xi, t \geq t_0$

$$
\|x_s(t_0) - x_{s0}(w(t_0))\| < \delta_1 \ \text{and} \ \nu := \sup_{t \geq t_0} \|\dot{w}(t)\| < \delta_2
$$

then there exists a $t_1 \geq t_0$ such that

$$
\|x_s(t) - x_{s0}(w(t))\| \leq ke^{-\gamma(t-t_0)}\|x_s(t_0) - x_{s0}(w(t_0))\|, \\
t_0 \leq t < t_1
$$

$$
\|x_s(t) - x_{s0}(w(t))\| \leq b(\nu), \ t \geq t_1.
$$

We return to Example 1 to illustrate these stability properties.

Example 1. (cont.) We design a gain-scheduling controller for the multi-rate system considered in Example 1, using the following methodology: *i)* the parameter space is discretized according to $\alpha_{0l} = -0.9 + 0.1l, \ l \in \{0,...,18\}$; *ii)* for each value α_{0l}, a controller with the structure (13) is computed using the standard H_2 output-feedback synthesis solution for periodic systems, yielding a finite set of controllers; and *iii)* the controller coefficients are

interpolated by quadratic parameter dependent functions. The scheduling variable is set to $\alpha = r$. Figure 2 shows the response of the closed-loop system output $y_r(t)$ to an input $r(t)$ consisting of a sequence of steps, obtained with both the gain-scheduled and linear controllers. This simulation
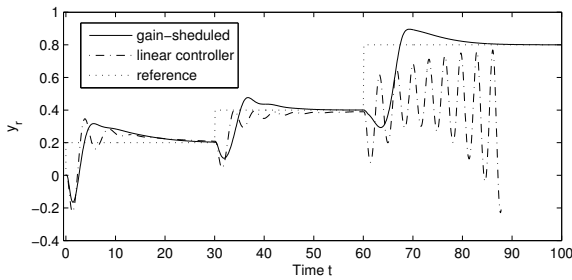

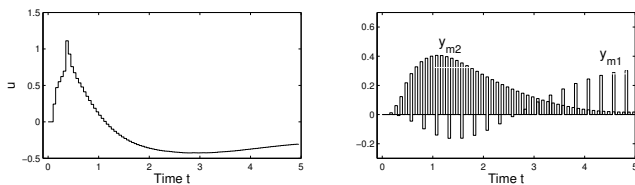Fig. 2. Gain-scheduled and linear controller

Fig. 3. Actuation and multi-rate inputs y_{m1} and y_{m2}

justifies the use of parameter varying controllers since the linear controller leads to instability. The actuation and the multi-rate inputs y_{m1} and y_{m2} of the gain-scheduled controller are shown in Fig. 3 for a short period of time. Notice that zero steady-state error is obtained for y_r. In Fig. 4, the responses to a slow and a fast ramp input in $r(t)$ obtained with the gain-scheduled controller are shown. One can see that the deviation between the output and the corresponding equilibrium value (which coincides with the value of the reference) depends on the rate of variation of the reference which is in agreement with Theorem 7.
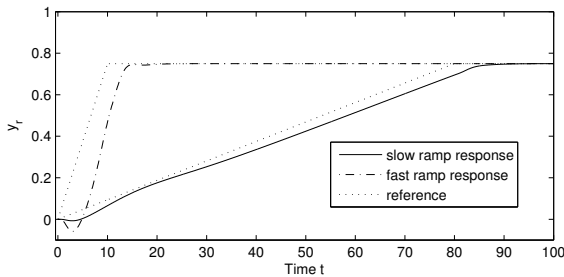

Fig. 4. Slow and fast ramp response

5. CONCLUSIONS

A methodology for the design and implementation of linear parameter varying controllers for multi-rate sampled-data systems was presented and its stability properties were analyzed. The proposed solution is based on a controller structure with integral action that for single-rate systems particularizes to the velocity implementation. The key linearization property of this structure allows for guaranteeing local stability of the sampled-data feedback interconnection of the non-linear system and proposed gain-scheduled controller about constant operating points, and ultimate boundedness of a conveniently defined error when slowly varying exogenous inputs are applied to the closed-loop system.

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