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A Stackelberg Game Approach to Mixed $\mathcal{H}_2\,/\,\mathcal{H}_\infty$Control

Marc Jungers * **Emmanuel Trélat** ** **Hisham Abou-Kandil** ***

* *CRAN, Nancy-Université, CNRS,*
2 avenue de la Forêt de Haye F-54516 Vandoeuvre-les-Nancy
(e-mail: marc.jungers@cran.uhp-nancy.fr).
** *Université d'Orléans, UFR Sciences, Fédération Denis Poisson,*
Mathématiques, Laboratoire MAPMO, UMR 6628, Route de Chartres,
BP 6759, 45067 Orléans Cedex 2, France
(e-mail: emmanuel.trelat@univ-orleans.fr)
*** *SATIE, ENS Cachan, CNRS, UniverSud,*
61 av du President Wilson, F-94230 Cachan, France
(e-mail: abou-kandil@satie.ens-cachan.fr).

Abstract: The $\mathcal{H}_2\,/\,\mathcal{H}_\infty$ robust control problem is formulated as a Stackelberg differential game where the leader minimizes an $\mathcal{H}_2$ criterion while the follower deals with the $\mathcal{H}_\infty$ constraint. For a closed loop information structure in the game, the necessary conditions to solve such a constrained optimization problem are derived for the finite time horizon case. It is shown that such an approach leads to a singular control and the Stackelberg strategy degenerates due to the omnipotence of the leader. Using conjugate times theory, we prove that the derived necessary conditions are also sufficient.

1. INTRODUCTION

Robust $\mathcal{H}_2\,/\,\mathcal{H}_\infty$ control problem has been treated extensively in recent years to achieve a compromise between $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norm specifications Bernstein and Haddad [1989], Zhou et al. [1994], Doyle et al. [1994]. In fact, a predefined level for the $\mathcal{H}_\infty$-norm cannot be guaranteed by a pure $\mathcal{H}_2$-control. Several approaches have been proposed to solve the mixed $\mathcal{H}_2\,/\,\mathcal{H}_\infty$ control problem. This includes non-standard Riccati equations Bernstein and Haddad [1989], Youla parametrization Scherer [1995], convex optimization Khargonekar and Rotea [1991], entropy interpretation Mustafa et al. [1991]... The state feedback case was treated in Rotea and Khargonekar [1991] while a compromise between $\mathcal{H}_2$ and $\mathcal{H}_\infty$-regulators is proposed in Halder et al. [1997].

In this note, the mixed $\mathcal{H}_2\,/\,\mathcal{H}_\infty$ robust control problem is formulated as a Stackelberg differential game Başar and Olsder [1995], Starr and Ho [1969a,b], Ho [1970], Simaan and Cruz [1973a,b]. A gametheoretic approach has been already proposed to solve the $\mathcal{H}_2\,/\,\mathcal{H}_\infty$ control problem Limebeer et al. [1994], Florentino and Sales [1997], Chen and Zhou [2001] via a Nash strategy. However, due to the symmetry between players in a Nash strategy, one player is minimizing the $\mathcal{H}_2$ norm and the second one is associated with the worst case disturbance seen in terms of $\mathcal{H}_\infty$ norm.

For the Stackelberg strategy, the hierarchy between the leader and the follower leads to minimizing the $\mathcal{H}_2$-norm by the leader subject to the $\mathcal{H}_\infty$-constraint dealt with by the follower. The information bias in such a game is quite suitable to solve such a constraint optimization problem. The model used here was introduced in Zhou et al. [1994], Doyle et al. [1994].

The paper is organized as follows. The problem is formulated in Section 2. The main contribution is given in Section 3 where the Stackelberg strategy and the associated necessary conditions are derived under closed loop information structure condition. It is shown in Section 4 that the necessary conditions become sufficient using conjugate times theory. Concluding remarks make up Section 5.

2. PROBLEM STATEMENT

Consider the plant described by (Fig. 1)

$$\begin{cases} \dot{x}(t) &= Ax(t) + B_\infty w_\infty(t) + B_2 w_2(t) + Bu(t) \\ &= f(x, w_\infty, w_2, u), \\ z_\infty(t) &= C_\infty x(t) + D_\infty w_\infty(t) + D_{\infty u} u(t), \\ z_2(t) &= C_2 x(t) + D_{2u} u(t), \\ z(t) &= x(t), \end{cases} \tag{1}$$

with $x(t) \in \mathbb{R}^n$, $w_2(t) \in \mathbb{R}^{r_2}$, $w_\infty(t) \in \mathbb{R}^{r_\infty}$, $u(t) \in \mathbb{R}^r$, $z_2(t) \in \mathbb{R}^{m_2}$, and $z_\infty(t) \in \mathbb{R}^{m_\infty}$. The matrices A, B_∞, B_2, B, C_∞, C_2, D_∞, $D_{\infty u}$ and D_{2u} are constant matrices with appropriate dimensions. B_∞ is assumed of full rank.

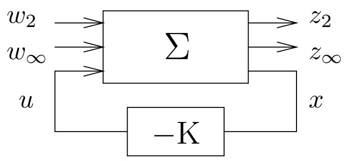

Fig. 1. System structure.

The finite horizon $[t_0, t_f]$ case is studied here (initial time t_0 and final time $t_f > t_0$). The $\mathcal{H}_2$-norm of a signal, denoted $\|.\|_{2,[t_0,t_f]}$, allows to define the induced norms $\mathcal{H}_2$ and $\mathcal{H}_\infty$ of the system. The input w_2 (respectively w_∞) and the output z_2 (resp. z_∞) define the channel for $\mathcal{H}_2$

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norm $\|z_2\|_{2,[t_0,t_f]}$ (resp. $\mathcal{H}_\infty$ norm $\sup_w \dfrac{\|z_\infty\|_{2,[t_0,t_f]}}{\|w_\infty\|_{2,[t_0,t_f]}}$). For simplicity, the feedback output z is assumed to be equal to the state x. The problem of mixed $\mathcal{H}_2\,/\,\mathcal{H}_\infty$-control design is to find a feedback control $u(t)$ stabilizing the system (1) and minimizing the $\mathcal{H}_2$-norm under the constraint that the $\mathcal{H}_\infty$-norm is less than a fixed level γ, i.e. $u(t) = -K(t)x(t)$ such that

$$\inf_{u\in\mathcal{U}} \|z_2\|_{2,[t_0,t_f]} \text{ subject to } \sup_{w_\infty\in\mathcal{W}_\infty} \frac{\|z_\infty\|_{2,[t_0,t_f]}}{\|w_\infty\|_{2,[t_0,t_f]}} < \gamma.$$

The system (1) being linear, the admissible set for the inputs u, w_2 and w_∞ are respectively $\mathcal{U} = \mathcal{L}^\infty([t_0,t_f]\times\mathbb{R}^n,\mathbb{R}^r)$, $\mathcal{W}_2 = \mathcal{L}^\infty([t_0,t_f],\mathbb{R}^{r_2})$, and $\mathcal{W}_\infty = \mathcal{L}^\infty([t_0,t_f]\times\mathbb{R}^n,\mathbb{R}^{r_\infty})$. The input w_2 is assumed to be known.

3. STACKELBERG STRATEGY

3.1 Definition

Let

$$\begin{aligned} J_2(w_\infty,w_2) &= \int_{t_0}^{t_f} L_2(x,u,w_\infty,w_2)\mathrm{d}t \\ &= \frac{1}{2}\int_{t_0}^{t_f} \left[z_2^T(t)z_2(t) + \alpha^2 w_\infty^T(t)R_\gamma w_\infty(t)\right]\mathrm{d}t, \quad (2)\end{aligned}$$

with $R_\gamma = \gamma^2 I - D_\infty^T D_\infty > 0$, for $\gamma > \overline{\sigma}(D_\infty)$, the largest singular value of D_∞, and

$$\begin{aligned} J_\infty(u,w_\infty) &= \int_{t_0}^{t_f} L_\infty(x,u,w_\infty)\mathrm{d}t \\ &= \frac{1}{2}\int_{t_0}^{t_f} \left[-z_\infty^T(t)z_\infty(t) + \gamma^2 w_\infty^T(t)w_\infty(t)\right]\mathrm{d}t, \quad (3)\end{aligned}$$

where $x(\cdot)$, $z_\infty(\cdot)$ and $z_2(\cdot)$ are solutions of (1). The criterion J_2 defined by (2) is associated with the $\mathcal{H}_2$-norm of system (1). For $\alpha \neq 0$, J_2 is convex with respect to w_∞. The criterion J_∞ defined by (3) is associated with the $\mathcal{H}_\infty$-norm of system (1). Note that, if $J_\infty > 0$, for any input $w_\infty \in \mathcal{W}_\infty$, then $\sup_{w_\infty\in\mathcal{W}_\infty} \dfrac{\|z_\infty\|_{2,[t_0,t_f]}}{\|w_\infty\|_{2,[t_0,t_f]}} < \gamma$. The infinimum of J_∞ over $w_\infty \in \mathcal{W}_\infty$ is either finite (and attained) or equal to $-\infty$, depending on the values of γ and of the final time t_f. In fact, denoting t_c the first conjugate time of the system (see Section 4), then $\inf J_\infty \geq 0$ whenever $t_f < t_c$, and $\inf J_\infty = -\infty$ whenever $t_f > t_c$.

The optimal control $u = u^*$ minimizes the $\mathcal{H}_2$-norm when $w_\infty = w_\infty^*$, the worst case input according to the $\mathcal{H}_\infty$-norm, is applied.

Stackelberg strategy is well adapted to deal with this kind of constrained minimization problem. The leader acts by choosing the control u and the follower by choosing the input w_∞. For a control $\tilde{u}$ of the leader, the rational reaction set $\mathcal{R}_\infty(\tilde{u})$ of the follower is defined by the set of the admissible input w_∞ which leads to the infinimum of $J_\infty(\tilde{u},w_\infty)$.

A Stackelberg equilibrium (u^*,w_∞^*) is defined by

$$\begin{cases} w_\infty^* \in \mathcal{R}_\infty(u^*), \\ \max\limits_{w_\infty\in\mathcal{R}_\infty(u^*)} J_2(u^*,w_\infty) \leqslant \max\limits_{w_\infty\in\mathcal{R}_\infty(u)} J_2(u,w_\infty), \end{cases} \quad (4)$$

$\forall u \in \mathcal{L}^\infty([t_0,t_f],\mathbb{R}^r)$ (see Simaan and Cruz [1973a,b]).

There are three inputs in the system u, w_2 and w_∞. u and w_∞ are considered as the two players of this non-zero sum game. The input w_2 is not a player and is considered as a disturbance. The framework corresponds to a closed-loop information structure, $u^* = u^*(x,t) \in \mathcal{U}$ and $w_\infty^* = w_\infty^*(x,t) \in \mathcal{W}_\infty$ are implicit functions of the time t and the state x (see Papavassilopoulos and Cruz [1979]).

3.2 Necessary conditions for the follower

Solving the problem from the point of view of the follower corresponds to determine its rational reaction set $\mathcal{R}_\infty(\cdot)$. This is a standard optimization problem that could be solved by applying Pontryagin's Minimum Principle. We define the Hamiltonian (see Pontryagin et al. [1962]) $H_\infty = \psi_\infty^\circ L_\infty + \psi_\infty f$, where the line vector $\psi_\infty \in \mathbb{R}^n$ is the costate vector associated with the dynamic constraint (1) and the scalar $\psi_\infty^\circ \geq 0$ with L_∞. The necessary conditions to be satisfied by the follower could be written along the solution as

$$\frac{\partial H_\infty}{\partial w_\infty}(t) = \psi_\infty^\circ \frac{\partial L_\infty}{\partial w_\infty}(t) + \psi_\infty(t)\frac{\partial f}{\partial w_\infty}(t) = 0, \quad (5)$$

$$\begin{aligned}\dot{\psi}_\infty(t) &= -\psi_\infty^\circ\left(\frac{\partial L_\infty}{\partial x}(t) + \frac{\partial L_\infty}{\partial u}(t)\frac{\partial u^*}{\partial x}(t)\right) \\ &\quad -\psi_\infty(t)\left(\frac{\partial f}{\partial x}(t) + \frac{\partial f}{\partial u}(t)\frac{\partial u^*}{\partial x}(t)\right). \quad (6)\end{aligned}$$

In addition, since the final state is free, the transversality condition leads to $\psi_\infty(t_f) = 0$. This implies that $\psi_\infty^\circ \neq 0$. Without loss of generality and for the sake of normalization we assume that $\psi_\infty^\circ = 1$.

It follows from (5) and from $\gamma > \bar{\sigma}(D_\infty)$, that R_γ is invertible and that the optimal input w_∞^* (the worst input in sense of $\mathcal{H}_\infty$-norm for an input u) is given by

$$\begin{aligned} w_\infty^*(t) &= -R_\gamma^{-1}\left[-D_\infty^T C_\infty x(t) - D_\infty^T D_{\infty u}u(t) + B_\infty^T\psi_\infty^T(t)\right] \\ &= S(x,u,\psi_\infty). \quad (7)\end{aligned}$$

We introduce the following notations

$$\begin{aligned}
W_\gamma &= I + D_\infty R_\gamma^{-1} D_\infty^T, \\
U &= D_{2u}^T D_{2u} + \alpha^2 D_{\infty u}^T D_\infty R_\gamma^{-1} D_\infty^T D_{\infty u}, \\
N &= R_\gamma + \alpha^2 D_\infty^T D_{\infty u} U^{-1} D_{\infty u}^T D_\infty, \\
\overline{B} &= B + B_\infty R_\gamma^{-1} D_\infty^T D_{\infty u}, \\
\tilde{B} &= B_\infty + \alpha^2 \overline{B} U^{-1} D_{\infty u}^T D_\infty, \\
\overline{C}_u &= D_{2u}^T C_2 + \alpha^2 D_{\infty u}^T D_\infty R_\gamma^{-1} D_\infty^T C_\infty, \\
\overline{C}_\infty &= \left(D_\infty^T D_{\infty u} U^{-1}\overline{C}_u - D_\infty^T C_\infty\right), \\
\hat{S}_\lambda &= \overline{B} U^{-1} \overline{B}^T, \\
\hat{S}_\infty &= S_\infty + \alpha^2 \overline{B} U^{-1} D_{\infty u}^T D_\infty R_\gamma^{-1} B_\infty^T, \\
S_\infty &= B_\infty R_\gamma^{-1} B_\infty^T, \\
\overline{S}_\infty &= S_\infty + \alpha^2 B_\infty R_\gamma^{-1} D_\infty^T D_{\infty u} U^{-1} D_{\infty u}^T D_\infty R_\gamma^{-1} B_\infty^T, \\
\tilde{S} &= \hat{S}_\lambda + \frac{1}{\alpha^2}\tilde{B} N^{-1} \tilde{B}^T, \\
Q &= C_2^T C_2 + \alpha^2 C_\infty^T D_\infty R_\gamma^{-1} D_\infty^T C_\infty - \overline{C}_u^T U^{-1} \overline{C}_u, \\
\tilde{Q} &= Q - \alpha^2 \overline{C}_\infty^T N^{-1} \overline{C}_\infty, \\
\overline{A} &= A + B_\infty R_\gamma^{-1} D_\infty^T C_\infty,
\end{aligned}$$

$$\hat{A} = \overline{A} - \overline{B}U^{-1}\overline{C}_u,$$
$$\tilde{A} = \hat{A} - \hat{S}_\infty \overline{S}_\infty^{-1} B_\infty R_\gamma^{-1} \overline{C}_\infty,$$

and

$$F_\infty^1(x, u, \psi_\infty) = x^T C_\infty^T W_\gamma C_\infty + u^T D_{\infty u}^T W_\gamma C_\infty - \psi_\infty \overline{A},$$
$$F_\infty^2(x, u, \psi_\infty) = x^T C_\infty^T W_\gamma D_{\infty u} + u^T D_{\infty u}^T W_\gamma D_{\infty u} - \psi_\infty \overline{B}.$$

Then

$$\dot{x}(t) = \tilde{f}(x(t), u(t), \psi_\infty(t), w_2(t)) = \overline{A}x(t) + \overline{B}u(t) - S_\infty \psi_\infty^T(t) + B_2 w_2(t), \quad (8)$$
$$x(t_0) = x_0, \quad (9)$$
$$\dot{\psi}_\infty^T = \left(F_\infty^1(x, u, \psi_\infty) + F_\infty^2(x, u, \psi_\infty) \frac{\partial u^*}{\partial x} \right)^T, \quad (10)$$
$$\psi_\infty(t_f) = 0, \quad (11)$$

and

$$\tilde{L}_\infty(x, u, \psi_\infty) = -\frac{1}{2} \left(C_\infty x + D_{\infty u} u \right)^T W_\gamma \left(C_\infty x + D_{\infty u} u \right) + \frac{1}{2} \psi_\infty S_\infty \psi_\infty^T,$$
$$\tilde{L}_2(x, u, \psi_\infty, w_2) = L_2(x, u, S(x, u, \psi_\infty), w_2).$$

3.3 Pontryagin Minimum Principle for a particular case

The minimization of J_2 subject to (8) and (10) is not a standard optimization problem. For the sake of clarity, we denote in the sequel $\frac{\partial u^*}{\partial x}$, the Jacobian of $u(t, y)$ w.r.t. the second variable, by u_y. To solve this problem from the point of view of the leader, the extended state X is introduced. X includes the state x, the costate vector ψ_∞^T, and the instantaneous cost x° (verifying $\dot{x}^\circ = \tilde{L}_2$): $X^T = \begin{pmatrix} x^T & \psi_\infty & x^\circ \end{pmatrix} \in \mathbb{R}^{2n+1}$ is solution of

$$\dot{X} = F(t, X, u, u_y^T) = \begin{pmatrix} \tilde{f} \\ F_\infty^{1T} + u_y^T F_\infty^{2T} \\ \tilde{L}_2 \end{pmatrix}, \quad (12)$$

with boundary conditions

$$x(0) = x_0, \quad \psi_\infty(t_f) = 0, \quad x^\circ(0) = 0, \quad (13)$$

where $u = u(t, h(X)) = u(t, x)$, with h the projector $h(X) = h \begin{pmatrix} x^T & \psi_\infty & x^{\circ T} \end{pmatrix}^T = x$.

It is shown below that every optimal control u^* for the optimization problem of the leader (minimizing J_2 subject to the constraints (8) and (10)) is a singular control for the system (12). This crucial fact permits to derive a Pontryagin Minimum Principle adapted to this type of problem (12). A similar approach is provided for the LQ case in Papavassilopoulos and Cruz [1979]. However the used arguments are not complete, even though the final result is correct.

We next recall the definition of the end-point mapping and of a singular control (see Lee and Markus [1967], Trélat [2005], Bonnard and Chyba [2003]).

Definition 1. The end-point mapping at time t_f of system (12) with initial state X_0 is the mapping

$$E_{X_0,t_f} : \mathcal{U} = L^\infty \left([0, t_f] \times \mathbb{R}^n, \mathbb{R}^r \right) \longrightarrow \mathbb{R}^{2n+1}, \quad u \longmapsto X_u(t_f), \quad (14)$$

where $X_u(\cdot)$ denotes the trajectory solution of (12) associated with the control u such that $X_u(t_0) = X_0$.

If the function F in (12) is of class $\mathcal{C}^p$, $p \geq 1$, then the end-point mapping E_{X_0,t_f} is also of class $\mathcal{C}^p$.

To determine the Fréchet derivative of E_{X_0,t_f}, consider a control δu such that $u + \delta u \in \mathcal{U}$ and let X be the trajectory associated with u and $X + \delta X$ with $u + \delta u$. By definition, we obtain

$$\frac{\mathrm{d}(X + \delta X)}{\mathrm{d}t} = F\big(t, X + \delta X, u(t, h(X + \delta X)) + \delta u(t, h(X + \delta X)), u_y(t, h(X + \delta X))^T + \delta u_y(t, h(X + \delta X))^T\big).$$

A Taylor series expansion leads to

$$\frac{\mathrm{d}(\delta X)}{\mathrm{d}t} = \tilde{A}\delta X + \tilde{B}\delta u + \tilde{C}\delta u_y^T, \quad (15)$$

where $\tilde{A} = \big(F_X + F_u u_y h_X + F_{u_y} u_{yy} h_X\big)$, $\tilde{B} = F_u$ and $\tilde{C} = F_{u_y}$.

Let $M(t)$ be the transition matrix associated with $\tilde{A}(t)$, i.e. the solution of the Cauchy problem

$$\dot{M}(t) = \tilde{A}(t)M(t), \qquad M(0) = I. \quad (16)$$

Then,

$$\delta X(t_f) = M(t_f) \int_0^{t_f} M^{-1}(s) \Big(\tilde{B}(s)\delta u(s) + \tilde{C}(s)\delta u_y^T(s) \Big) \mathrm{d}s, \quad (17)$$

and the next result follows.

Lemma 2. The Fréchet derivative of E_{X_0,t_f} at a point $u \in \mathcal{U}$ is given by

$$\mathrm{d}E_{X_0,t_f}(u) \cdot \delta u = M(t_f) \int_0^{t_f} M^{-1}(s) \Big(\tilde{B}(s)\delta u(s) + \tilde{C}(s)\delta u_y^T(s) \Big) \mathrm{d}s. \quad (18)$$

Definition 3. Let u be in $\mathcal{U}$, the control u is said to be singular on $[0, t_f]$ if the Fréchet derivative $\mathrm{d}E_{X_0,t_f}(u)$ is not surjective.

If the control u is singular, then there exists a line vector $\varphi \in \mathbb{R}^{2n+1}/\{0\}$ such that

$$\varphi \cdot \mathrm{d}E_{X_0,t_f}(u) = 0. \quad (19)$$

The line vector $p(t) = \varphi M(t_f) M^{-1}(t)$ verifies

$$\dot{p}(t) = -p(t)\tilde{A}(t), \qquad p(t_f) = \varphi. \quad (20)$$

It follows from (18), (19) and (20) that

$$\int_0^{t_f} p(t) \left(\tilde{B}(t)\delta u(t, h(X)) + \tilde{C}(t)\delta u_y^T(t, h(X)) \right) \mathrm{d}t = 0, \quad (21)$$

for every $\delta u(t, h(X))$. In particular, considering first controls $\delta u(t)$, (21) yields

$$p(t)\tilde{B}(t) = 0, \quad \text{and} \quad p(t)\tilde{C}(t) = 0, \text{a.e. on } [t_0, t_f].$$

Define the Hamiltonian $H_2(t, X, u, u_y, p) = pF(t, X, u, u_y)$. Then a singular control $u(t, h(X))$ is characterized by

$$\dot{X} = \frac{\partial H_2}{\partial p}, \qquad \dot{p} = -p\tilde{A} = -\frac{\mathrm{d}H_2}{\mathrm{d}X},$$
$$\frac{\partial H_2}{\partial u} = p(t)\tilde{B}(t) = 0, \qquad \frac{\partial H_2}{\partial u_y} = p(t)\tilde{C}(t) = 0. \quad (22)$$

This Hamiltonian characterization is next used to derive necessary conditions for the leader.

3.4 Necessary conditions for the leader

Lemma 4. If the control u^* is optimal for the problem defined by (8) - (10) and (2), then it is singular on $[0, t_f]$ for the extended system (12).

Proof of Lemma 4. Let X be the trajectory solution of the system (12), associated with a control u issued from $X_0 = (x_0^T, \psi_{\infty,0}^T, 0)^T$. If u is optimal for J_2, the final state $X(t_f)$ lies at the boundary of $E_{X_0,t_f}(\mathcal{U})$. Hence the end-point mapping E_{X_0,t_f} is not open at u, and it follows from the Implicit Functions Theorem that the control u is singular for system (12) on $[0, t_f]$.

The Hamiltonian H_2 associated with J_2 subject to the constraint (12) can be rewritten as

$$H_2 = \lambda_1 \tilde{f} + \lambda_2 \left(F_\infty^1 + F_\infty^2 u_x\right)^T + \lambda^\circ \tilde{L}_2, \tag{23}$$

by setting $p(t) = (\lambda_1(t), \lambda_2(t), \lambda^\circ(t))$, with $\lambda_1(t) \in \mathbb{R}^n$, $\lambda_2(t) \in \mathbb{R}^n$ (line vectors) and $\lambda^\circ(t) \in \mathbb{R}$. The Hamiltonian characterization of a singular control leads to

$$\frac{\partial H_2}{\partial u} = \lambda_1 \frac{\partial \tilde{f}}{\partial u} + \lambda_2 \left(\frac{\partial F_\infty^1}{\partial u} + \frac{\partial F_\infty^2}{\partial u} u_y\right)^T + \lambda^\circ \frac{\partial \tilde{L}_2}{\partial u} = 0, \tag{24}$$

$$\frac{\partial H_2}{\partial u_y} = \lambda_2^T F_\infty^2 = 0, \tag{25}$$

$$\dot{\lambda}_1 = -\lambda_1 \frac{\partial \tilde{f}}{\partial x} - \lambda_2 \left(\frac{\partial F_\infty^1}{\partial x} + \frac{\partial F_\infty^2}{\partial x} u_y\right)^T - \lambda^\circ \frac{\partial \tilde{L}_2}{\partial x}, \tag{26}$$

$$\dot{\lambda}_2 = -\lambda_1 \frac{\partial \tilde{f}}{\partial \psi_\infty} - \lambda_2 \left(\frac{\partial F_\infty^1}{\partial \psi_\infty} + \frac{\partial F_\infty^2}{\partial \psi_\infty} u_y\right)^T - \lambda^\circ \frac{\partial \tilde{L}_2}{\partial \psi_\infty}, \tag{27}$$

$$\dot{\lambda}^\circ = 0. \tag{28}$$

From (28), $\lambda^\circ(t) = \lambda^\circ$ is constant. According to the Pontryagin Minimum Principle, we assume that $\lambda^\circ \geq 0$.

3.5 Transversality conditions

Since the initial state $x(0) = x_0$ and the final costate line vector $\psi_\infty(t_f) = 0$ are fixed, the extended costate line vector $(\lambda_1, \lambda_2, \lambda^\circ)$ must verify the transversality conditions

$$\lambda_2(0) = 0, \quad \lambda_1(t_f) = 0. \tag{29}$$

(see for example [Trélat, 2005, page 104] for more details)

3.6 Degenerate Stackelberg strategy

From (25), we infer that $\lambda_2 \equiv 0$ or $F_\infty^2 \equiv 0$ (or both).

Proposition 5. If the matrix

$$\frac{\partial F_\infty^2}{\partial u} = D_{\infty u}^T \left(I + D_\infty R_\gamma^{-1} D_\infty^T\right) D_{\infty u} = D_{\infty u}^T W_\gamma D_{\infty u} \tag{30}$$

is invertible, then $\lambda_2 \equiv 0$. In this case, the Stackelberg strategy degenerates, due to the omnipotence of the leader.

Proof of Proposition 5. By contradiction, assume that $\lambda_2 \neq 0$. Then, $F_\infty^2 \equiv 0$. Since $\dfrac{\partial F_\infty^2}{\partial u}$ invertible, it follows from the Implicit Functions Theorem that, locally around the trajectory $u = u(t, x, \psi_\infty)$.

Hence, system (8) - (10) writes

$$\dot{x} = \tilde{f}(x, \psi_\infty, u(t, x, \psi_\infty)), \quad \dot{\psi}_\infty = F_\infty^1(x, \psi_\infty, u(t, x, \psi_\infty)). \tag{31}$$

The dynamics and the criterion J_2 are both independent of u_y. Hence, every control u_y is optimal, which contradicts (24).

The fact that $\lambda_2 \equiv 0$ means that the leader does not take into account the rational response of the follower represented by the evolution of the costate vector ψ_∞ to minimize his own criterion J_2. The Stackelberg strategy with a closed-loop information structure seems to lose globally its hierarchical structure. In fact the condition (30) indicates that, if the criterion of the follower depends on u, then the leader is able to impose to the follower a desired control. Even though the hierarchy seems to disappear, the leader is omnipotent with respect to the follower. To a certain extent, this could justify using Nash strategy in Limebeer et al. [1994] for a mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problems.

3.7 Computation of the optimal control

Since the costate vector $(\lambda_1(t_f), \lambda_2(t_f), \lambda^\circ) = (0, 0, \lambda^\circ)$ must be nontrivial, up to normalizing, we next assume $\lambda^\circ = 1$.

From (24), we deduce the expression of the optimal control

$$u^* = -U^{-1}\overline{C}_u x - U^{-1}\overline{B}^T \lambda_1^T + \alpha^2 U^{-1} D_{\infty u}^T D_\infty R_\gamma^{-1} B_\infty^T \psi_\infty^T. \tag{32}$$

Plugging this expression into the dynamics (8) yields

$$\tilde{f} = \hat{A}x - \hat{S}_\lambda \lambda_1^T - \hat{S}_\infty \psi_\infty^T + B_2 w_2. \tag{33}$$

According to (26), we obtain

$$\dot{\lambda}_1^T = -\hat{A}^T \lambda_1^T - Qx - \alpha^2 \overline{C}_\infty^T R_\gamma^{-1} B_\infty^T \psi_\infty^T = g^T(x, \lambda_1, \psi_\infty). \tag{34}$$

The evolution of ψ_∞ (10) reads now

$$\dot{\psi}_\infty = \tilde{F}_\infty^1(x, \lambda_1, \psi_\infty) + \tilde{F}_\infty^2(x, \lambda_1, \psi_\infty) u_y, \tag{35}$$

with $\tilde{F}_\infty^1(x, \lambda_1, \psi_\infty) = F_\infty^1(x, u^*(x, \lambda_1, \psi_\infty), \psi_\infty)$, and $\tilde{F}_\infty^2(x, \lambda_1, \psi_\infty) = F_\infty^2(x, u^*(x, \lambda_1, \psi_\infty), \psi_\infty)$.

Since $\lambda_2 = 0$, the relation (27) yields the constraint

$$\hat{S}_\infty^T \lambda_1^T - \alpha^2 \overline{S}_\infty \psi_\infty^T - \alpha^2 B_\infty R_\gamma^{-1} \overline{C}_\infty x = 0. \tag{36}$$

Remark 6. If $\alpha = 0$, then the necessary condition (36) becomes $\lambda_1 \hat{S}_\infty = 0$. In particular, taking into account the transversality condition (29), the first and second derivatives of this relation at $t = t_f$ yield

$$\begin{cases} x^T(t_f) Q \hat{S}_\infty = 0, \\ x^T(t_f)(Q\hat{A} - \hat{A}Q\hat{S}_\infty) = w_2^T(t_f) B_2 Q \hat{S}_\infty. \end{cases} \tag{37}$$

These conditions are additional constraints. The relation (37) is a relation at time t_f between the exogeneous input w_2 and the state x. This necessary condition is not generally verified all the more so since w_2 is in general considered as a disturbance. In conclusion, the case $\alpha = 0$ does not lead to a relevant solution for the mixed

$\mathcal{H}_2\,/\,\mathcal{H}_\infty$ problem. This result justifies the additional term $\alpha^2 w_\infty^T R_\gamma w_\infty$ in the criterion J_2 associated with the $\mathcal{H}_2$-norm, which yields convexity with respect to the control u whenever $\alpha \neq 0$. In the sequel, we assume that $\alpha \neq 0$.

Differentiating (36) with respect to t, it is clear that (36) is equivalent to the following relations

$$x^T(t_f)\overline{C}_\infty = 0, \tag{38}$$

$$\tilde{F}^2_\infty \frac{\partial u^*}{\partial x} B_\infty R_\gamma^{-1} N R_\gamma^{-1} B_\infty^T = v = \Big(\frac{1}{\alpha^2} g\hat{S}_\infty - \tilde{F}^1_\infty - \tilde{f}^T \overline{C}_\infty^T\Big). \tag{39}$$

The relation (38) implies that every x_0 is not necessarily the starting point of an optimal trajectory. The initial state x_0 of an optimal trajectory must belong to a r_∞-codim subspace of $\mathbb{R}^n$, where $r_\infty = \text{rank}\ \overline{C}_\infty$. The constraint (39) leads to

$$\tilde{F}^2_\infty \frac{\partial u^*}{\partial x} B_\infty = v B_\infty \big(B_\infty^T B_\infty\big)^{-1} R_\gamma N^{-1} R_\gamma, \tag{40}$$

and hence

$$\tilde{F}^2_\infty \frac{\partial u^*}{\partial x} \in v B_\infty \big(B_\infty^T B_\infty\big)^{-1} R_\gamma N^{-1} R_\gamma \big(B_\infty^T B_\infty\big)^{-1} B_\infty^T + \big(\text{Ker} B_\infty^T\big)^T. \tag{41}$$

Even though the optimal trajectory is unique, the expression for $\frac{\partial u^*}{\partial x}$ in not unique.

Remark 7. w_2 is not the action of one player, but a disturbance. Contrary to $\frac{\partial u^*}{\partial x}$ and $\tilde{f}$, the control $u^*(t,x)$ does not depend on this input w_2.

Remark 8. To facilitate the research of the optimal control, a restricted class of $u^*(t,y)$ can be imposed. By choosing an affine representation (see Papavassilopoulos and Cruz [1979]) of $u^*(t,y)$

$$u^*(t,y) = u_y\ \big(y - x(t)\big) + u(t), \tag{42}$$

it is possible to avoid the exact computation of $\frac{\partial u^*}{\partial x}$ on the optimal trajectory $x(t)$.

3.8 Solving by Riccati equation

From (36), and by assuming that B_∞ is of full rank,

$$\psi_\infty = \Big(\frac{1}{\alpha^2}\lambda_1 \tilde{B} - x^T \overline{C}_\infty^T\Big) N^{-1} R_\gamma (B_\infty^T B_\infty)^{-1} B_\infty^T. \tag{43}$$

Plugging this relation into (33) and (34), we obtain

$$\dot{x}(t) = \tilde{A}(t)x(t) - \tilde{S}(t)\lambda_1^T(t) + B_2 w_2(t), \tag{44}$$

$$\dot{\lambda}_1^T(t) = -\tilde{Q}(t)x(t) - \tilde{A}^T(t)\lambda_1^T(t). \tag{45}$$

Similarly to LQ problems, it is possible to express $\lambda_1(t)$ in the form

$$\lambda_1^T(t) = K_1(t)x(t) + h_1(t). \tag{46}$$

Indeed, it is clear that if the matrix $K_1(t) \in \mathbb{R}^{n\times n}$ and the column vector $h_1(t) \in \mathbb{R}^n$ verify

$$\dot{K}_1(t) = -K_1(t)\tilde{A}(t) - \tilde{A}(t)^T K_1(t) - \tilde{Q}(t) + K_1(t)\tilde{S}(t)K_1(t), \tag{47}$$

$$\dot{h}_1(t) = -K_1(t)\tilde{S}(t)h_1(t) + \tilde{A}^T(t)h_1(t) + K_1(t)B_2 w_2(t), \tag{48}$$

with boundary conditions

$$K_1(t_f) = 0, \qquad h_1(t_f) = 0, \tag{49}$$

then $\lambda_1(t)$ defined by (46) solves the differential equation (45) and the boundary condition (29). Equation (47) is a standard Riccati equation, which can be linearized using Radon's Lemma (see Abou-Kandil et al. [2003]). For a given input w_2, both equations (47) and (48) can be solved by backward integrating from final conditions (49).

4. SUFFICIENT CONDITIONS

In order to obtain sufficient conditions for this problem, some well known facts of conjugate times theory are next recalled (see for example [Bonnard et al., 2006, chapter 9] for more details).

Definition 9. The variational system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\delta x\\ \delta\lambda_1{}^T\end{pmatrix} = \begin{bmatrix}\tilde{A} & -\tilde{S}\\ -\tilde{Q} & -\tilde{A}^T\end{bmatrix}\begin{pmatrix}\delta x\\ \delta\lambda_1{}^T\end{pmatrix} \tag{50}$$

is called Jacobi's equation. The Jacobi's field $J(t) = (\delta x^T(t), \delta\lambda_1(t))$ is a nontrivial solution of (50).

The transition matrix associated with (50) is denoted $\phi(t)$, and

$$\begin{pmatrix}\delta x(t)\\ \delta\lambda_1^T(t)\end{pmatrix} = \phi(t)\begin{pmatrix}\delta x(0)\\ \delta\lambda_1^T(0)\end{pmatrix} = \begin{bmatrix}\phi_1(t) & \phi_2(t)\\ \phi_3(t) & \phi_4(t)\end{bmatrix}\begin{pmatrix}\delta x(0)\\ \delta\lambda_1^T(0)\end{pmatrix}. \tag{51}$$

Definition 10. The first conjugate time t_c is the first positive time for which there exists a Jacobi field such that $\delta x(0) = \delta x(t_c) = 0$.

This is equivalent to rank $\phi_2(t_c) < n$.

The following results are standard in LQ theory (see [Bonnard et al., 2006, chapter 9]).

Proposition 11. The first conjugate time t_c corresponds to the first finite escape time of the Riccati equation (47).

Proof of Proposition 11. The solution of the Riccati equation (47) is given by

$$K_1(t) = \phi_4(t)\phi_2^{-1}(t). \tag{52}$$

The first conjugate time t_c is the first time at which $\phi_2(t_c)$ is not invertible, that is, $\|K_1(t)\| \to +\infty$, when $t \to t_c$.

Proposition 12. The solutions of Pontryagin Minimum Principle are optimal before their first conjugate time. The control (32) with $\lambda_1(t)$ given by the Riccati equation (47) is optimal if and only if this equation admits a well defined solution on $[0, t_f]$.

Thanks to these results the necessary conditions are also sufficient. Before the first conjugate time, the optimal control -if it exists- is unique. Actually, if $\tilde{Q}$ is nonnegative, then the following additional properties hold.

Proposition 13. If $\tilde{Q} \geq 0$, then the solution $K_1(t)$ of (47) is symmetric and nonnegative.

Proof of Proposition 13. See [Abou-Kandil et al., 2003, Theorem 4.1.6], observing that $K_1(t_f) = 0$, $\tilde{S} \geq 0$ and $\tilde{Q} \geq 0$.

Proposition 14. If $\tilde{Q} \geq 0$, then $t_c = +\infty$.

Proof of Proposition 14. It is sufficient to apply [Abou-Kandil et al., 2003, Corollary 3.6.7, Example 3.6.8], observing $K_1(t_f) = 0$, $\tilde{Q} \geq 0$ and $\tilde{S} \geq 0$.

Proposition 15. If $\tilde{Q} \geq 0$, and if $K_1(t)$ converges to a limit K_1^∞ when $t \to +\infty$, then $\left(\tilde{A} - \tilde{S}K_1^\infty\right)$ is stable.

Proof of Proposition 15. Taking the limit in (47) leads to the Lyapunov equation

$$K_1^\infty(\tilde{A}-\tilde{S}K_1^\infty)+(\tilde{A}-\tilde{S}K_1^\infty)^T K_1^\infty = -\tilde{Q}-K_1^\infty\tilde{S}K_1^\infty < 0. \tag{53}$$

The result follows because K_1^∞ is symmetric and nonnegative.

In general, we do not know whether $\tilde{Q}$ is nonnegative or not. In the scalar case however we are able to prove the following result.

Proposition 16. In the scalar case, $r = n = 1$, $m_\infty = m_2 = 1$ and $r_\infty = r_2 = 1$, $\tilde{Q}$ is nonnegative.

Proof of Proposition 16. Let $\beta = D_{\infty u}^T D_\infty$ and $\eta = D_{2u}$, the matrix $\tilde{Q}$ writes

$$\tilde{Q} = \left[\, C_2^T \;\; C_\infty^T D_\infty R_\gamma^{-1} \,\right] M \begin{bmatrix} C_2 \\ R_\gamma^{-1} D_\infty^T C_\infty \end{bmatrix}, \tag{54}$$

where

$$M = \frac{2\alpha^4\beta^2}{\left(\eta^2 + \dfrac{\alpha^2\beta^2}{R_\gamma}\right)\left(\eta^2 + 2\dfrac{\alpha^2\beta^2}{R_\gamma}\right)} \begin{bmatrix} \dfrac{\beta}{R_\gamma} \\ -\eta \end{bmatrix} \begin{bmatrix} \dfrac{\beta}{R_\gamma} & -\eta \end{bmatrix}.$$

5. CONCLUSION

This paper analyzes the mixed $\mathcal{H}_2\,/\,\mathcal{H}_\infty$ control for a multi-channel system. The framework used is the Stackelberg strategy with a closed loop information structure. This strategy is well adapted to manage several criteria with different hierarchical roles. Necessary conditions are provided and lead to a differential Riccati equation. It is emphasized that the Stackelberg strategy globally degenerates, due to the omnipotence of the leader. Using conjugate times theory, sufficient conditions are given in terms of finite escape time for the solution of the Riccati equation.

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