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French Champagne and Belgian Chocolate Problems in Simultaneous Stabilization of Linear Systems

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Abstract: This paper considers two open problems associated with simultaneous stabilization of linear systems, namely French champagne problem and Belgian chocolate problem. Based on the recent development in automated inequality-type theorem proving, the explicit bounds which guarantee the existence of stabilizing controllers with fixed order are determined. In addition, two conjectures concerning the Belgian chocolate problem are formulated. Some numerical examples are worked out.

Keywords: linear systems, simultaneous stabilization, French champagne problem, Belgian chocolate problem, inequality-type theorem, automated theorem proving

1. INTRODUCTION

Simultaneous stabilization of linear systems is a fundamental issue in system and control theory, and is of theoretical as well as practical significance. Simultaneous stabilization problem focuses on the following:

Let $p_1, p_2, \cdots, p_k$ be k scalar linear time-invariant systems. Under what condition does there exist a fixed controller c that is stabilizing for each $p_i (i = 1, \cdots, k)$?

When $k = 1$, this problem is reduced to the stabilization of a single system and there always exists a stabilizing controller for a single system provided no unstable pole-zero cancellations occur. Meanwhile, once a stabilizing controller of a single system is found, it is easy to parameterize the set of all stabilizing controllers of this system. This parametrization is known as Youla-Kucera parametrization discovered by Youla et al. (1976) and Kucera (1979), respectively.

If $k = 2$, according to the Youla-Kucera parametrization, it is possible to rephrase simultaneous stabilization of two systems into strong stabilization (stabilization with a stable controller) of a single systems. This relationship was discovered for scalar systems by Saeks and Murray (1982), and for multi-variable systems by Vidyasagar and Viswanadham (1982). For the strong stabilization problem of a single system, Youla et al. (1974) established an elegant criterion: a system is stabilizable by a stable controller if and only if it has an even number of real unstable poles between each pair of real unstable zeros! This remarkable and easily testable condition is called PIP(Parity Interlacing Property).

For the case of simultaneously stabilizing $k \geq 3$ systems, it is much more complicated than one expected. Vidyasagar and Viswanadham (1982) indicated that it is possible to transform simultaneous stabilization of k systems to strong stabilization of corresponding $k-1$ systems. Blondel et al. (1994) proved that simultaneous stabilization of k systems is equivalent to bistable stabilization of associated $k - 2$ systems. Bistable stabilization means stabilization with a stable and inverse-stable controller. Such a controller is called bistable controller or unit controller. Although many necessary or sufficient conditions for simultaneous stabilization of three or more systems were obtained in recent years, easily testable necessary and sufficient conditions have not been found yet. Blondel and Gevers (1994) showed that the simultaneous stabilization of three systems is not rationally decidable, i.e. it is not possible to find tractable necessary and sufficient conditions for simultaneous stabilization of three systems that involve only a combination of finite arithmetical operations (addition, substraction, multiplication and division), logical operations (and, or) and sign test operations (equal to, greater than, greater than or equal to, less than, less than or equal to) on the coefficients of the three systems!

To illustrate the complexity of the simultaneous stabilization problem of three systems, a specific simultaneous stabilization problem called French Champagne Problem (FCP) was proposed in (Blondel et al. 1993) and a bottle of good French champagne was offered for its solution.

For the equivalence between simultaneous stabilization of three systems and bistable stabilization of a single associ-

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ated system, a difficult problem called Belgian Chocolate Problem (BCP) was proposed in (Blondel 1994). In addition, concerning the general case, another one, namely Generalized Belgian Chocolate Problem (GBCP), was considered in (Blondel 1994). Blondel promised a kilogram of Belgian chocolate for the solution to each of these two problems.

Patel (1999) gave the FCP a negative answer and solved it completely. Furthermore, a more general simultaneous stabilization problem, the Generalized French Champagne Problem (GFCP), was proposed in (Patel 1999). For the BCP, the positive answer was given in (Patel et al. 2002) and (Burke et al. 2006), respectively. Although many numerical improving results were reported in (Patel et al. 2002), (Burke et al. 2006) and (Chang et al. 2007), the GBCP is still open up to now.

Apparently, when the degree of the controller is fixed, the controller design problem of simultaneous stabilization can be transformed in essence to the problem of how to solve a set of algebraic inequalities.

In the early 1950's, Tarski (1951) published the well-known work on the decidability of this kind of problems. Tarski's decision algorithm is of theoretical significance only, since it can not be used to verify any non-trivial algebraic or geometric propositions in practice due to its very high computational complexity. The Cylindrical Algebraic Decomposition (CAD) algorithm proposed by Collins et al. (1984) and subsequently improved by him and his collaborators is the first practical decision algorithm and can be used to verify non-trivial algebraic or geometric propositions on computer. Although, as a generic program for automated theorem proving, its computational complexity was still very high, the CAD and its improved variations have become one of the major tools for solving this kind of problems.

Wu (1978) proposed a new decision procedure for proving geometry theorems. As an important progress in automated theorem proving, Wu's method is very efficient for mechanically proving elementary geometry theorem of equality type. The success of Wu's method inspired the research of algebraic approach to automated theorem proving. However, automated inequality proving has been a difficult topic in the area of automated reasoning for many years since the relevant algorithms depend on real algebra and real geometry. In 1996, Yang et al. introduced a powerful tool, the Complete Discrimination Systems (CDS) of polynomials, for automated reasoning in real algebra. By means of CDS, many inequality-type theorem from various applications have been proved or disproved.

Recently, combining discriminant sequences of polynomials with Wu's method as well as Partial Cylindrical Algebraic Decomposition, Yang et al. (1999 and 2001) proposed some algorithms which are able to discover new inequalities. These algorithms are complete for an extensive class of problems involving inequalities and are applicable to the controller design in simultaneous stabilization. Based on these algorithms, two generic programs called Discoverer and Bottema respectively were implemented as Maple packages. Some of the following results are obtained by calling Discoverer and Bottema.

In this paper, we investigate the GFCP and GBCP under the conditions that the degree of the stabilizers is fixed beforehand. Based on the recent development in automated inequality-type theorem proving, the explicit bounds which guarantee the existence of stabilizers with fixed order are determined. In addition, two conjectures concerning the GBCP are formulated.

2. FRENCH CHAMPAGNE PROBLEM AND BELGIAN CHOCOLATE PROBLEM

In this paper, all polynomials are of real coefficients. We denote by P the set of polynomials, P^n the set of n-th order polynomials where n is an non-negative integer, H the set of Hurwitz stable polynomials (all roots lie within left half of the complex plane), H^n the set of n-th order Hurwitz polynomials, MH^n the set of monic n-th order Hurwitz polynomials. The variable of polynomials is s.

2.1 French champagne problem

The following problem is the well-known French Champagne Problem (FCP) of simultaneous stabilization.

French Champagne Problem (FCP) Does there exist a controller that simultaneous stabilizes the following three plants:

$$p_1(s) = \frac{2}{17}\frac{s-1}{s+1}, \quad p_2(s) = \frac{(s-1)^2}{(9s-8)(s+1)}, \quad p_3(s) = 0?$$

Patel (1999) solved this problem by showing that there does not exist a stabilizing controller. In addition, a more general simultaneous stabilization problem, the Generalized French Champagne Problem (GFCP), was proposed by Patel (1999), which contains the FCP as a special case.

Generalized French Champagne Problem (GFCP) What is the range of real δ for the existence of a controller that simultaneously stabilizes the following three plants:

$$p_1(s) = \frac{2\delta(s-1)}{s+1}, \quad p_2(s) = \frac{2\delta(s-1)^2}{((1+\delta)s-(1-\delta))(s+1)},$$

$p_3(s) = 0$?

For simplicity of the presentation, let us give the following equivalent statement of this problem.

Problem statement 1 (GFCP') Let $a_1(s) = s+1$, $b_1(s) = 2\delta(s-1)$, $a_2(s) = ((1+\delta)s-(1-\delta))(s+1)$ and $b_2(s) = 2\delta(s-1)^2$. What is the range of real δ for which there exist stable polynomials $x(s) \in H^n$ and polynomials $y(s) \in P^m$ such that $a_i(s)x(s) + b_i(s)y(s) \in H$ $(i = 1, 2)$?

Obviously, the GFCP focuses on determining the range of δ when there exists a simultaneously stabilizer while the FCP asks whether $\delta = \frac{1}{17}$ is in the range. The following theorem in (Guan et al. 2007) gives a completely theoretical solution to the GFCP.

Theorem 1 (Guan et al. 2007) The necessary and sufficient condition for the existence of a controller that simultaneously stabilizes the following three plants:

$$p_1(s) = \frac{2\delta(s-1)}{s+1}, \quad p_2(s) = \frac{2\delta(s-1)^2}{((1+\delta)s-(1-\delta))(s+1)},$$

$p_3(s) = 0$

is $\delta = 0$ or $|\delta| > \dfrac{1}{16}$.

Although the GFCP was resolved theoretically by Guan et al. (2007), it is still difficult in practice to construct the simultaneously stabilizing controller.

In the follows, some controller examples for the GFCP will be provided by applying Discoverer or Bottema (Yang et al. 1999 and 2001). Without loss of generality, the denominator polynomials of the controllers are supposed to be monic. In addition, for convenience, only the case $\delta > 0$ is considered.

Here, the real rational transfer functions are not necessarily proper, that is to say, the degree of numerator may be greater than that of denominator. Meanwhile, the existence questions are prior to the properness issues for the fact proven by Blondel (1994), i.e. if k plants are simultaneously stabilizble by a non-proper controller, they are also simultaneously stabilizble by a proper controller. Indeed, due to the roots of a polynomial continuously depend on its coefficients, the proper controllers can always be found when the corresponding non-proper controllers exist.

Example 1 (Guan et al. 2007) Let $a_1(s) = s+1$, $b_1(s) = 2\delta(s-1)$, $a_2(s) = ((1+\delta)s - (1-\delta))(s+1)$ and $b_2(s) = 2\delta(s-1)^2$, $\delta > 0$. There exist stable polynomials $x(s) \in MH^i (i = 0, 1, 2)$ and polynomials $y(s) \in P^0$ such that $a_j(s)x(s) + b_j(s)y(s) \in H$ $(j = 1, 2)$ only for $\delta > \frac{1}{2}$, whereas there do not exist such polynomials for $\delta \le \frac{1}{2}$. For a given $\delta > \frac{1}{2}$, for example, $\delta = \frac{3}{4}$, $c(s) = y_0$ is a desired controller if and only if $\frac{1}{6} < y_0 < \frac{1}{2}$.

Remark 1 Example 1 shows that: 1) When the degree of the controller is restricted, the explicit bounds of δ can be obtained by using the packages developed by Yang et al. (1999 and 2001). 2) Moreover, when δ is fixed, the range of the parameters of controllers can be obtained. Due to the completeness of the algorithms, the conditions obtained are both necessary and sufficient.

Example 2 (Guan et al. 2007) Let $a_1(s) = s+1$, $b_1(s) = 2\delta(s-1)$, $a_2(s) = ((1+\delta)s - (1-\delta))(s+1)$ and $b_2(s) = 2\delta(s-1)^2$, $\delta > 0$. There exist stable polynomials $x(s) \in MH^i (i = 0, 1)$ and polynomials $y(s) \in P^1$ such that $a_j(s)x(s)+b_j(s)y(s) \in H$ $(j = 1, 2)$ for $\delta > \frac{1}{4}$, whereas there do not exist such polynomials for $\delta \le \frac{1}{4}$. For a given $\delta > \frac{1}{4}$, say, $\delta = \frac{1}{2}$, $c(s) = y_1 s + y_0$ is a requested controller, where $(y_1, y_0) \in \{(\frac{1}{10}, \frac{51}{100}), (\frac{3}{5}, \frac{3}{5}), (\frac{4}{5}, \frac{3}{5})\}$. Moreover, to get a proper controller, by the continuous dependence for roots of polynomials on their coefficients, it is known that if $\varepsilon > 0$ is sufficiently small, e.g. $\varepsilon = \frac{1}{10}$, $\tilde{c}(s) = \frac{y_1 s + y_0}{\varepsilon s + 1}$ is a desired proper controller, where $(y_1, y_0) \in \{(\frac{4}{5}, \frac{3}{5})\}$.

Example 3 (Guan et al. 2007) Let $a_1(s) = s+1$, $b_1(s) = 2\delta(s-1)$, $a_2(s) = ((1+\delta)s - (1-\delta))(s+1)$ and $b_2(s) = 2\delta(s-1)^2$, $\delta > 0$. There exist stable polynomials $x(s) \in MH^0$ and polynomials $y(s) \in P^2$ such that $a_i(s)x(s) + b_i(s)y(s) \in H$ $(i = 1, 2)$ for $\delta > \frac{1}{6}$, whereas there do not exist such polynomials for $\delta \le \frac{1}{6}$. When $\delta = \frac{10}{59}$, for example, $c(s) = y_2 s^2 + y_1 s + y_0$ is a desired controller, where $(y_2, y_1, y_0) \in \{(\frac{97}{50}, \frac{39001}{10000}, \frac{245001}{100000}), (\frac{19501}{10000}, \frac{39003}{10000}, \frac{4900299}{2000000})\}$. Moreover, to get a proper controller, by the continuous dependence for roots of polynomials on their coefficients, it is known that if $\varepsilon > 0$ is sufficiently small, e.g. $\varepsilon = 10^{-7}$, $\tilde{c}(s) = \frac{y_2 s^2 + y_1 s + y_0}{\varepsilon s^2 + \varepsilon s + 1}$ is a desired proper controller, where $(y_2, y_1, y_0) \in \{(\frac{97}{50}, \frac{39001}{10000}, \frac{245001}{100000})\}$.

Remark 2 The controllers obtained in the above examples are the sample points picked out from the cells of some appropriate decomposition of the parametric space which satisfy the requirement of simultaneous stabilization.

Remark 3 From the computational experiments, it seems that the improvement on the bound of δ in the GFCP mainly depends on the increase on the order of numerator polynomials of the stabilizing controller.

Example 4 (Guan et al. 2007) Let $a_1(s) = s+1$, $b_1(s) = 2\delta(s-1)$, $a_2(s) = ((1+\delta)s - (1-\delta))(s+1)$ and $b_2(s) = 2\delta(s-1)^2$, $\delta > 0$. There do not exist stable polynomials $x(s) \in MH^0$ and polynomials $y(s) \in P^3$ such that $a_i(s)x(s) + b_i(s)y(s) \in H$ $(i = 1, 2)$ for $\delta \le \frac{1}{8}$. However, when $\delta = \frac{1}{7}$, for example, $c(s) = y_3 s^3 + y_2 s^2 + y_1 s + y_0$ is a desired controller, where $(y_3, y_2, y_1, y_0) \in \{(\frac{26}{25}, \frac{376}{125}, \frac{50001}{10000}, \frac{300001}{100000}), (\frac{11407}{10000}, \frac{121}{40}, \frac{50001}{10000}, \frac{300001}{100000})\}$. Moreover, to get a proper controller, by the continuous dependence for roots of polynomials on their coefficients, it is known that if $\varepsilon^2 > \varepsilon_1$ for sufficiently small $\varepsilon_1 > 0$ and $\varepsilon > 0$, e.g. $\varepsilon = 10^{-7}$ and $\varepsilon_1 = 10^{-15}$, $\tilde{c}(s) = \frac{y_3 s^3 + y_2 s^2 + y_1 s + y_0}{\varepsilon_1 s^3 + \varepsilon s^2 + \varepsilon s + 1}$ is a desired proper controller.

Remark 5 The value of δ appeared in Example 4 is a improvement over the minimal bound proposed in (Patel et al. 2002).

2.2 Belgian chocolate problem

The following problems were proposed in (Blondel 1994).

Belgian Chocolate Problem (BCP) Can the continuous-time second-order system

$$p(s) = \frac{s^2 - 1}{s^2 - 1.8s + 1}$$

be stabilized by a stable and inverse stable controller?

Generalized Belgian Chocolate Problem (GBCP) For what values of real δ is the scalar linear system

$$p(s) = \frac{s^2 - 1}{s^2 - 2\delta s + 1}$$

stabilizable by a stable controller whose inverse is also stable?

We give the equivalent statement of the GBCP as follows.

Problem statement 2 (GBCP') Let $a(s) = s^2 - 2\delta s + 1$, $b(s) = s^2 - 1$. What is the range of real δ for which there exist stable polynomials $x(s) \in H^n$ and $y(s) \in H^m$ such that $a(s)x(s) + b(s)y(s) \in H$?

Apparently, the GBCP concerns the range of δ when there exists a bistable controller while the BCP asks whether $\delta = 0.9$ is in this range.

For the BCP, many bistable controllers were obtained in (Patel et al. 2002), (Burke et al. 2006) and (Chang et al. 2007), respectively. Yet the GBCP is still open up to now.

Note that stabilization of GBCP is impossible for $\delta = 1$ since then an unstable pole-zero cancellation occurs in $p(s)$. Conversely, stabilization is easy for $\delta < 0.5$ (Burke et al. 2006). According to Blondel (1994), there exists a number $\delta^* < 1$ such that stabilization is possible for all $\delta < \delta^*$, but impossible for $\delta \geq \delta^*$. That is to say, there exists a critical value which splits the stabilizable and unstabilizable parameter region. Thus the GBCP reduces to determining δ^*. A theoretical bound for δ^* was first given in (Rupp 1994) and improved further in (Blondel et al. 1995). These bounds were given for an equivalent problem in the z-domain. The corresponding result in the s-domain is given below. It is known that δ^* lies in the following range:

$$0.7615941559557649 < \delta^* < 0.9999800001999982$$

According to (Patel et al. 2002), we know that $\delta^* > 0.937$. In addition, (Burke et al. 2006) improved the results by finding a bistable stabilizer when $\delta = 0.94375$ and predicted that $\delta^* > 0.951$. Recently, some improving numerical results were reported in (Chang et al. 2007). Now, it is known that $\delta^* > 0.973974$.

In this section, we concern the GBCP under the conditions that the degree of the stabilizers is fixed beforehand. Thus we hope to indicate the interesting relationships between the explicit bounds of δ and the degree of the controllers.

We firstly prove the following proposition:

Proposition 1 (He et al. 2007) Let $a(s) = s^2 - 2\delta s + 1$, $b(s) = s^2 - 1$, $\delta > 0$. There do not exist $x(s) \in H^0$ and $y(s) \in H^n (n = 0, 1, 2, \ldots)$, such that $a(s)x(s) + b(s)y(s) \in H$.

Proof: See Appendix A.

In the follows, we will provide some controller examples for the GBCP by applying Discoverer or Bottema. Here we also suppose that the denominator polynomials of the controllers are monic.

Example 5 Let $a(s) = s^2 - 2\delta s + 1$, $b(s) = s^2 - 1$, $\delta \in [0, 1]$. There exist stable polynomials $x(s) \in MH^1$ and $y(s) \in H^i (i = 0, 1)$, such that $a(s)x(s) + b(s)y(s) \in H$ only for $\delta < \frac{\sqrt{2}}{2}$, whereas there do not exist such polynomials for $\delta \geq \frac{\sqrt{2}}{2}$. For a given $\delta < \frac{\sqrt{2}}{2}$, say, $\delta = \frac{7}{10}$, $c(s) = \frac{y_0}{s + x_0}$ is a desired controller, where $(x_0, y_0) \in \{(\frac{71}{100}, \frac{7099}{10000})\}$.

Example 6 Let $a(s) = s^2 - 2\delta s + 1$, $b(s) = s^2 - 1$, $\delta \in [0, 1]$. There exist stable polynomials $x(s) \in MH^2$ and $y(s) \in H^i (i = 0, 1, 2)$, such that $a(s)x(s) + b(s)y(s) \in H$ only for $\delta < \frac{\sqrt{3}}{2}$, whereas there do not exist such polynomials for $\delta \geq \frac{\sqrt{3}}{2}$. For a given $\delta < \frac{\sqrt{3}}{2}$, say, $\delta = \frac{85}{100}$, $c(s) = \frac{y_0}{s^2 + x_1 s + x_0}$ is a desired controller, where $(x_1, x_0, y_0) \in \{(\frac{9}{5}, \frac{132}{125}, \frac{10559}{10000})\}$.

Example 7 Let $a(s) = s^2 - 2\delta s + 1$, $b(s) = s^2 - 1$, $\delta \in [0, 1]$. There exist stable polynomials $x(s) \in MH^3$ and $y(s) \in H^0$, such that $a(s)x(s) + b(s)y(s) \in H$ only for $\delta < \frac{\sqrt{2+\sqrt{2}}}{2}$, whereas there do not exist such polynomials for $\delta \geq \frac{\sqrt{2+\sqrt{2}}}{2}$, where the exact value of $k = \frac{\sqrt{2+\sqrt{2}}}{2} = 0.923879\ldots$ is the largest real root of $8k^4 - 8k^2 + 1$.

For a given $\delta < \frac{\sqrt{2+\sqrt{2}}}{2}$, for example, $\delta = \frac{923}{1000}$, $c(s) = \frac{y_0}{s^3 + x_2 s^2 + x_1 s + x_0}$ is a desired controller, where $(x_2, x_1, x_0, y_0) \in \{(\frac{37}{20}, \frac{303239}{125000}, \frac{1314133}{1000000}, \frac{26282659}{20000000})\}$.

When $\delta = 0.9$ (namely BCP), we obtain a desired controller $c(s) = \frac{y_0}{s^3 + x_2 s^2 + x_1 s + x_0}$, where $(x_2, x_1, x_0, y_0) \in \{(\frac{23}{10}, \frac{329}{100}, \frac{73}{40}, \frac{18249}{10000}), (\frac{213}{100}, \frac{301}{100}, \frac{417}{250}, \frac{2083}{1250})\}$.

Example 8 Let $a(s) = s^2 - 2\delta s + 1$, $b(s) = s^2 - 1$, $\delta \in [0, 1]$. There exist stable polynomials $x(s) \in MH^4$ and $y(s) \in H^0$, such that $a(s)x(s) + b(s)y(s) \in H$ only for $\delta < \frac{\sqrt{10+2\sqrt{5}}}{4}$, whereas there do not exist such polynomials for $\delta \geq \frac{\sqrt{10+2\sqrt{5}}}{4}$, where the exact value of $k = \frac{\sqrt{10+2\sqrt{5}}}{4} = 0.951056\ldots$ is the largest real root of $16k^4 - 20k^2 + 5$. Moreover, we present a fourth-order bistable controller for $\delta > 0.9510$ in Appendix B, which improves the results in (Burke et al. 2006).

In addition, more precise details about the distribution of δ^* and a conjecture to the GBCP can be found in Appendix C.

Remark 6 As a special case, we know that there does not exist $\delta \geq 0.96$ with $x(s) \in MH^5$ and $y(s) \in H^0$, such that $a(s)x(s) + b(s)y(s) \in H$.

2.3 A conjecture to generalized Belgian chocolate problem

Consider the cases of controllers with $x(s) \in MH^n (n = 1, 2, 3, 4)$ and $y(s) \in H^0$. Note that for the real variable $\delta \to \delta_n^*$, there exists $a(s)x(s) + b(s)y(s) \to s^{n+2}$. Meanwhile, notice that the necessary condition on the stability of real polynomial requires all coefficients of the polynomial to be of the same sign. Thus each $\delta_n^* (n = 1, 2, 3, 4)$ can also be obtained by solving a set of corresponding equations.

Furthermore, a set of necessary conditions concerning δ_n^* in the GBCP can be formulated from a generalization of these properties, and can be stated as the following conjecture:

Conjecture 1 For the case of controllers with structure $x(s) \in MH^n (n = 1, 2, \ldots)$ and $y(s) \in H^0$, each δ_n^* is equal to or less than δ_n, where δ_n is the largest real root of the following continued fractions

$$\begin{cases} f_n = -\dfrac{1}{f_{n-1}} - 2\delta \\ f_0 = -\delta \end{cases}$$

From Maple 9, we can obtain:

$\delta_1 = 0.7071\ldots$ is the largest real root of $1 - 2\delta^2$;

$\delta_2 = 0.8660\ldots$ is the largest real root of $-\delta(-3 + 4\delta^2)$;

$\delta_3 = 0.9238\ldots$ is the largest real root of $-1 + 8\delta^2 - 8\delta^4$;

$\delta_4 = 0.9510\ldots$ is the largest real root of $-\delta(5 - 20\delta^2 + 16\delta^4)$;

$\delta_5 = 0.9659\ldots$ is the largest real root of $1 - 18\delta^2 + 48\delta^4 - 32\delta^6$;

$\delta_6 = 0.9749\ldots$ is the largest real root of $-\delta(-7 + 56\delta^2 - 112\delta^4 + 64\delta^6)$;

$\delta_7 = 0.9807\ldots$ is the largest real root of $-1 + 32\delta^2 - 160\delta^4 + 256\delta^6 - 128\delta^8$;

$\delta_8 = 0.9848\ldots$ is the largest real root of $-\delta(9 - 120\delta^2 + 432\delta^4 - 576\delta^6 + 256\delta^8)$;

$\delta_9 = 0.9876\ldots$ is the largest real root of $1 - 50\delta^2 + 400\delta^4 - 1120\delta^6 + 1280\delta^8 - 512\delta^{10}$;

$\delta_{10} = 0.9898\ldots$ is the largest real root of $-\delta(-11 + 220\delta^2 - 1232\delta^4 + 2816\delta^6 - 2816\delta^8 + 1024\delta^{10})$;

$\ldots$

It is easy to see that $\delta^*_n = \delta_n$ for $n = 1, 2, 3, 4$ and $\delta^*_n < \delta_n$ for $n > 4$.

On the other hand, as n gets larger, δ_n is found to get closer and closer to 1, yet even $\delta_{90} = 0.9998510240\ldots$ is less than the theoretical upper bound of δ^* proposed in (Blondel et al. 1995), namely $\delta^*_{upper} = 0.99998000019999 82$.

Consequently, a stabilizer with fifth-order was presented for $\delta^c_5 = 0.95138549197075$ in (Chang et al. 2007), yet it is known that $\delta^c_5 < \delta_5 = 0.9659\ldots$

Similarly, a stabilizer with sixth-order was presented for $\delta^c_6 = 0.96292177890276$ in (Chang et al. 2007), yet it is known that $\delta^c_6 < \delta_6 = 0.9749\ldots$

Moreover, a stabilizer with seventh-order was reported for $\delta^c_7 = 0.96292783033099$ in (Chang et al. 2007), yet it is known that $\delta^c_7 < \delta_7 = 0.9807\ldots$

Meanwhile, a stabilizer with eighth-order was reported for $\delta^c_8 = 0.96696634493729$ in (Chang et al. 2007), yet it is known that $\delta^c_8 < \delta_8 = 0.9848\ldots$

Again, a stabilizer with ninth-order was reported for $\delta^c_9 = 0.96700163$ in (Chang et al. 2007), yet it is known that $\delta^c_9 < \delta_9 = 0.9876\ldots$

And a stabilizer with tenth-order was reported for $\delta^c_{10} = 0.97397439924082$ in (Chang et al. 2007), yet it is known that $\delta^c_{10} < \delta_{10} = 0.9898\ldots$

In fact, regarding all the stabilizing controllers found so far, the Conjecture 1 is satisfied!

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Appendix A. PROOF OF PROPOSITION 1

Proposition 1 Let $a(s) = s^2 - 2\delta s + 1$, $b(s) = s^2 - 1$, $\delta > 0$. There do not exist $x(s) \in H^0$ and $y(s) \in H^n (n = 0, 1, 2, \ldots)$, such that $a(s)x(s) + b(s)y(s) \in H$.

Proof: (By contradiction.)

Suppose that there exist Hurwitz polynomials $x(s) \in H^0$ and $y(s) \in H^n (n = 0, 1, 2, \ldots)$, such that $a(s)x(s) + b(s)y(s) \in H$.

Without loss of generality, we assume $x(s) = x_0$ and $y(s) = y_n s^n + y_{n-1}s^{n-1} + \cdots + y_1 s + y_0$, where $x_0 \neq 0$ and $y_n y_0 > 0$. Hence $a(s)x(s) + b(s)y(s) = (s^2 - 2\delta s + 1) x_0 + (s^2 - 1)(y_n s^n + y_{n-1}s^{n-1} + \cdots + y_1 s + y_0) = y_n s^{n+2} + y_{n-1}s^{n+1} + (y_{n-2} - y_n)s^n + \cdots + (x_0 + y_0 - y_2)s^2 + (-2\delta x_0 - y_1)s + (x_0 - y_0)$.

Consider $y_0 \neq 0$. There are two cases, both of which lead to contradiction.

Case 1: Assume that $y_0 > 0$, since $y(s) \in H^n$, we obviously have $y_i > 0 (i = 1, 2, \ldots n)$. If $x_0 > 0$, then $(-2\delta x_0 - y_1) < 0$. Hence $y_n > 0$, $(-2\delta x_0 - y_1) < 0$ and $a(s)x(s) + b(s)y(s) \in H$, which is a contradiction. Meanwhile, if $x_0 < 0$, then $(x_0 - y_0) < 0$. Hence $y_n > 0$, $(x_0 - y_0) < 0$ and $a(s)x(s) + b(s)y(s) \in H$, which is also a contradiction.

Case 2: Otherwise, assume that $y_0 < 0$, since $y(s) \in H^n$, we obviously have $y_i < 0 (i = 1, 2, \ldots n)$. If $x_0 > 0$, then $(x_0 - y_0) > 0$. Hence $y_n < 0$, $(x_0 - y_0) > 0$ and $a(s)x(s) + b(s)y(s) \in H$, which is a contradiction. Meanwhile, if $x_0 < 0$, then $(-2\delta x_0 - y_1) > 0$. Hence $y_n < 0$, $(-2\delta x_0 - y_1) > 0$ and $a(s)x(s) + b(s)y(s) \in H$, which is also a contradiction.

This completes the proof.

Appendix B. STABILIZER WITH $X(S) \in MH^4$ AND $Y(S) \in H^0$ FOR $\delta > 0.951$ IN GBCP

The maximal value of δ obtained here is

δ_{max}=0.95105651 62896110 29420643 88946743 89112458 55475719 38,

while the polynomials $x(s)$ and $y(s)$ are $x(s) = s^4 + x_3 s^3 + x_2 s^2 + x_1 s + x_0$, $y(s) = y_0$, where

x_0=1.6180339889 4536286946 9494993565 8952157345 86 8769824,

x_1=3.0776835375 2911962833 8955905332 3366931516 57 1678076,

x_2=2.6180339889 9793598068 1189007281 2013909496 03 0952963,

x_3=1.9021130326 7922205884 1287778934 8778224917 10 9514388,

y_0=1.6180339889 4536286946 9494993565 8952157344 86 8769824.

Recently, using the numerical algorithm based on a global optimization methodology, (Chang et al. 2007) improved the maximal value of δ to 0.97397439924082 and presented the corresponding bistable controller of tenth-order as follows:

$x(s) = s^{10} + x_9 s^9 + x_8 s^8 + x_7 s^7 + x_6 s^6 + x_5 s^5 + x_4 s^4 + x_3 s^3 + x_2 s^2 + x_1 s + x_0$, where

x_9=1.97351109136261, x_8=5.49402092964662,

x_7=8.78344232801755, x_6=11.67256448604672,

x_5=13.95449016040116, x_4=11.89912895529042,

x_3=9.19112429409894, x_2=5.75248874640322,

x_1=2.03055901420484, x_0=1.03326203778346;

and $y(s) = y_5 s^5 + y_4 s^4 + y_3 s^3 + y_2 s^2 + y_1 s + y_0$, where

y_5=0.00066128189295, y_4=3.611364710425,

y_3=0.03394722108511, y_2=3.86358782861648,

y_1=0.0178174691792, y_0=1.03326203778319.

Meanwhile, for the δ larger than 0.951, many corresponding stabilizers with order more than four were provided in (Chang et al. 2007).

Appendix C. THE DISTRIBUTION OF δ^* IN GBCP

In the following table, we present the upper bounds of δ (δ^*) obtained so far in GBCP.

$x(s)$	$y(s)$	δ^*	$x(s)$	$y(s)$	δ^*
MH^0	H^0	No Existence			
MH^1	H^0	$\frac{\sqrt{2}}{2}$	MH^1	MH^0	$\frac{1}{2}$
MH^1	H^1	$\frac{\sqrt{2}}{2}$	MH^1	MH^1	No Existence
MH^2	H^0	$\frac{\sqrt{3}}{2}$	MH^2	MH^0	$\frac{\sqrt{3}}{2}$
MH^2	H^1	$\frac{\sqrt{3}}{2}$	MH^2	MH^1	$\frac{\sqrt{2}}{2}$
MH^2	H^2	$\frac{\sqrt{3}}{2}$	MH^2	MH^2	$\frac{\sqrt{2}}{2}$
MH^3	H^0	$\frac{\sqrt{2+\sqrt{2}}}{2}$	MH^3	MH^0	δ'
MH^3	H^1	$\frac{\sqrt{2+\sqrt{2}}}{2}$	MH^3	MH^1	$\frac{\sqrt{3}}{2}$
MH^3	H^2	$\frac{\sqrt{2+\sqrt{2}}}{2}$	MH^3	MH^2	$\frac{\sqrt{3}}{2}$
MH^3	H^3	$\frac{\sqrt{2+\sqrt{2}}}{2}$	MH^3	MH^3	$\frac{\sqrt{3}}{2}$
MH^4	H^0	$\frac{\sqrt{10+2\sqrt{5}}}{4}$			
MH^4	H^1	$\frac{\sqrt{10+2\sqrt{5}}}{4}$	MH^4	MH^1	δ''
MH^4	H^2	$\frac{\sqrt{10+2\sqrt{5}}}{4}$			
MH^4	H^3	$\frac{\sqrt{10+2\sqrt{5}}}{4}$			
MH^4	H^4	$\frac{\sqrt{10+2\sqrt{5}}}{4}$			

where $\delta' = 0.554194\ldots$ is the smallest real root of $512\delta^{10} - 1536\delta^8 - 192\delta^7 + 1536\delta^6 + 384\delta^5 - 488\delta^4 - 192\delta^3 - 78\delta^2 - \delta + 54$ and $\delta'' = 0.933781\ldots$ is the largest real root of $16\delta^5 - 20\delta^3 - 2\delta^2 + 5\delta + 2$.

Based on these results, we have another conjecture about δ^* as follows:

Conjecture 2 In GBCP, assume that δ_b^* is the upper bound of δ for $x(s) \in MH^n$ and $y(s) \in H^0$, then δ_b^* is also the upper bound of δ for $x(s) \in MH^n$ and $y(s) \in H^i$ $(i = 1, \ldots n)$.

Note that the Conjecture 2 is satisfied when $n < 5$.