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Robust control in uncertain multi-inventory systems and consensus problems ⋆

D. Bauso * **L. Giarrè** ** **R. Pesenti** ***

* *Dipartimento di Ingegneria Informatica-DINFO, Università di Palermo, 90128 Palermo, Italy, (e-mail: dario.bauso@unipa.it)*
** *Dipartimento di Ingegneria dell'Automazione e dei Sistemi-DIAS, Università di Palermo, 90128 Palermo, Italy (e-mail: giarre@unipa.it)*
*** *Dipartimento di Matematica Applicata-DMA, Università di Venezia, 30123 Venezia, Italy, (e-mail: pesenti@unive.it)*

Abstract: We consider a continuous time linear multi–inventory system with unknown demands bounded within ellipsoids and controls bounded within polytopes. We address the problem of ϵ-stabilizing the inventory since this implies some reduction of the inventory costs. The main results are certain conditions under which ϵ-stabilizability is possible through a saturated linear state feedback control. The idea of this approach is similar to the consensus problem solution for a network of continuous time dynamic agents, where each agent evolves according to a first order dynamics has bounded control and it is subject to unknown but bounded disturbances. In this context, we derive conditions under which consensus can be reached. All the results are based on a Linear Matrix Inequalities (LMIs) approach and on some recent techniques for the modeling and analysis of polytopic systems with saturations.

1. INTRODUCTION

We consider a continuous time linear multi–inventory system with unknown demands bounded within ellipsoids and controls bounded within polytopes. The system is modeled as a first order one integrating the discrepancy between controls and demands at different sites (buffers). Thus, the state represents the buffer levels. We wish to study conditions under which the state can be driven within an a-priori chosen target set through a saturated linear state feedback control. Let ϵ be a maximal dimension of the target set, the above problem corresponds to ϵ-stabilizing the state.

This work is in line with some recent literature on robust optimization Adida et al. [2006], Bertsimas and Thiele [2006] and control Bauso et al. [2006-a] of inventory systems. Here as well as in Bauso et al. [2006-a] we focus on saturated linear state feedback controls since such controls arise naturally in any system with bounded controls.

The main results of this work can be summarized as follows. Initially we introduce the necessary and sufficient conditions for the ϵ-stabilizability in the form of an inclusion between convex sets. In the case where both demands and controls are bounded within polytopes, it is well known that verifying such conditions is NP-hard McCormick [1996]. In Bauso et al. [2007-b] we prove that verification becomes easy when both demands and controls are bounded within ellipsoids. The case where demands are bounded within ellipsoids and controls are bounded within polytopes is an open problem and we propose certain sufficient LMI conditions to solve it.

The same approach can be used to solve a consensus problem for a network of continuous time dynamic agents, where each agent evolves according to a first order dynamics with bounded control subject to unknown but bounded disturbances. In this context, we derive also conditions under which consensus can be reached. The solution of the consensus problem for a network of agents with Unknown but Bounded (UBB) disturbance without saturation has been described in Bauso et al. [2007-a].

All the results are based on a Linear Matrix Inequalities (LMIs) approach in line with the recent work Boukas [2006] on inventory/manufacturing systems. In particular, when addressing the politopic case, we use the same technique provided in Gomes da Silva, Jr. and Tarbouriech [2001] to rewrite the model with saturations in polytopic form. Once we do this, we can apply the LMI analysis covered in the book Boyd et al. [1994] for polytopic systems.

2. PROBLEM FORMULATION

Consider the continuous time linear multi–inventory system

$$\dot{x}(t) = Bu(t) - w(t), \qquad (1)$$

where $x(t) \in \mathbb{R}^n$ is a vector whose components are the buffer levels, $u(t) \in \mathbb{R}^m$ is the controlled flow vector, $B \in \mathbb{Q}^{n\times m}$, with $m \geq n$ and $rank(B) = n$ is the controlled process matrix and $w(t) \in \mathbb{R}^n$ is the unknown demand. To model backlog $x(t)$ may be less than zero. Demands are bounded within ellipsoids, i.e.,

$$w(t) \in \mathcal{W} = \{w \in \mathbb{R}^n : w^T R_w w \leq 1\}, \qquad (2)$$

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Fig. 1. Graph with one node and two arcs.

while controls are bounded within polytopes

$$u(t) \in \mathcal{U} = \{u \in \mathbb{R}^m : u^- \leq u \leq u^+\} \quad (3)$$

with assigned u^+, u^-.

The ellipsoidal region in constraints (2) describes some coupling effect on demand uncertainty. For any positive definite matrix $P \in \mathbb{R}^{n\times n}$, define the function $V(x) = x^T Px$ and the ellipsoidal target set $\Pi = \{x \in \mathbb{R}^n : V(x) \leq 1\}$. In addition, for any matrix $K \in \mathbb{R}^{n\times n}$, define as saturated linear state feedback control any policy

$$u = sat_{[u^-,u^+]}(-Kx) \quad (4)$$

where, for any vectors a, b and ζ of same dimensions, the sat operator is defined as

$$sat_{[a,b]}(\zeta) = \begin{cases} b, & \text{if } \zeta > b, \\ \zeta, & \text{if } a \leq \zeta \leq b, \\ a, & \text{if } \zeta < a, \end{cases}$$

the above inequalities holding component-wise.

Problem 1. (ϵ-stabilizing) Given system (1), find conditions on the positive definite matrix $P \in \mathbb{R}^{n\times n}$, under which there exists a saturated linear state feedback control $u = sat(-Kx)$ such that it is possible to drive the state $x(t)$ within the target set Π.

Solving the above problem corresponds to ϵ-stabilizing the state x where the relation between ϵ and Π is

$$\epsilon := \max_x \{\|x\|_\infty : x \in \Pi\}. \quad (5)$$

Example 1. Throughout this paper we consider, as illustrative example, the graph with one node and two arcs depicted in Fig. 1. The incidence matrix is $B = [1 \quad 1]$. The continuous time dynamics is

$$\dot{x}(t) = \underbrace{[1 \quad 1]}_{B} \underbrace{\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}}_{u} - w = u_1(t) + u_2(t) - w(t),$$

with demand bounded in the ellipsoid

$$w^2 \leq 1$$

and with the following polytopic constraints on the control u

$$-2 \leq u_1 \leq 3, \; -2 \leq u_2 \leq 1. \quad (6)$$

Finally, the target set is the sphere of unitary radius $\Pi = \{x \in \mathbb{R} : x^2 \leq 1\}$.

In the following we discuss for which initial state the system is certainly ϵ-stabilizable through a (pure) linear state feedback control; hence we show that if we saturated the previous linear policy the system is ϵ-stabilizable for any initial state.

3. POLYTOPIC SYSTEM

System (1) is ϵ-stabilizable if and only if for all $w \in \mathcal{W}$, there exists $u \in int\{\mathcal{U}\}$ such that $Bu = w$ (see, e.g., Blanchini et al. [1997]). For the short of notation, the previous condition is usually expressed as

$$B\mathcal{U} \supset \mathcal{W}. \quad (7)$$

Deciding whether (7) holds is NP-hard, when $\mathcal{U}$ and $\mathcal{W}$ are polytopes. In Bauso et al. [2007-b], we prove that verifying (7) becomes easy when both $\mathcal{U}$ and $\mathcal{W}$ are ellipsoids.

In this paper controls u are subject to the polytopic constraints (3). In the following, among the saturated linear state feedback control (4) we prove that we can solve Problem 1 using controls of type $u = sat(-kHx)$, with $k \in \mathbb{R}$ and $H \in \mathbb{R}^n$ s.t. $BH = I$. More specifically, we choose the control

$$u_i = sat_{[u_i^-,u_i^+]}(-kH_{i\bullet}x), \quad (8)$$

where $H_{i\bullet}$ denotes the ith row of H. Henceforth we omit the indexes of the sat function.

Under the control $u = sat(-kHx)$, the closed loop dynamics becomes

$$\dot{x} = Bsat(-kHx) - w. \quad (9)$$

Our idea is to rewrite the above dynamics in the following polytopic form

$$\dot{x} = A(t)x(t) - w(t), \quad w(t)^T R_w w(t) \leq 1, \quad (10)$$

where the time varying matrices $A(t)$ are expressed as convex combinations of 2^m matrices A_j, $j = 1, \ldots, 2^m$. More precisely the expressions for $A(t)$ are

$$A(t) = \sum_{j=1}^{2^m} \sigma_j(t) A_j, \quad \sum_{j=1}^{2^m} \sigma_j(t) = 1. \quad (11)$$

The procedure to compute matrices A_j's is borrowed from Gomes da Silva, Jr. and Tarbouriech [2001] and recalled below. Let us rewrite the control policy as

$$u_i = sat(-kH_{i\bullet}x) = \theta_i(x)(-kH_{i\bullet}x),$$

where $\theta_i(x)$ are the "degree of saturation" of the control components defined as follows

$$\theta_i(x) = \begin{cases} \dfrac{u_i^-}{-kH_{i\bullet}x} & \text{if } -kH_{i\bullet}x < u_i^- \\ 1 & \text{if } u_i^- \leq -kH_{i\bullet}x \leq u_i^+ \\ \dfrac{u_i^+}{-kH_{i\bullet}x} & \text{if } -kH_{i\bullet}x > u^+ \end{cases}. \quad (12)$$

Let $\underline{\theta} = [\underline{\theta}_1, \ldots, \underline{\theta}_m]$ be a vector whose components $\underline{\theta}_i$ are such that $0 \leq \underline{\theta}_i \leq 1$ and represent lower bounds of θ_i. Lower bounds depend on $x(0)$ and can be computed as $\theta_i = \arg\min_{x\in\Sigma_0} \theta_i(x)$ where we remind the definition of $\Sigma_0 = \{x \in \mathbb{R}^n : x^T Px \leq x(0)^T Px(0)\}$. Also define the vector $\psi^\theta = [\psi_1^\theta, \ldots, \psi_m^\theta]$ with $\psi_i^\theta = \frac{1}{\underline{\theta}_i}$ and the associated portion of the state space

$$S(\psi^\theta) = \{x \in \mathbb{R}^n : -\psi^\theta \leq -kHx \leq \psi^\theta\}.$$

According to the above definition of the θ_is we derive that $S(\psi^\theta) \supseteq \Sigma_0$ and therefore if Σ_0 is invariant $S(\psi^\theta)$ is invariant as well. We will use this argument in the next theorem.

Consider now the 2^m vectors $\gamma_j \in \{1, \underline{\theta}_1\} \times \ldots \times \{1, \underline{\theta}_m\}$, with $j = 1, \ldots, 2^m$. In other words, γ_j is an m component vector with ith component γ_{ji} taking value 1 or $\underline{\theta}_i$. Then, each matrix A_j can be expressed as $A_j = -Bkdiag(\gamma_j)H$. Roughly speaking each vector γ_j stores the minimum and or maximum degree of saturation of all control components. Also, note that matrices A_js induce a partition of $S(\psi^\theta)$ into regions X_j, with $j = 1\ldots, 2^m$. Each region is defined as the set of state values such that the control components are saturated with degree of saturation equal to γ_{ji}, namely

$$X_j = \{x \in \mathbb{R}^n : \theta_i(x) = \gamma_{ji},\, i = 1, \ldots, m\}.$$

We remind here that γ_{ji} is the ith component of γ_j.

To complete the derivation of the polytopic form (10) it is left to be noted that given any $x(t) \in S(\psi^\theta)$ we can compute the associated degree of saturation from (12) and derive the weights $\sigma_j(t)$ of the convex combination (11). All the results in the rest of this section try to give an answer to Problem 1 with respect to the polytopic system (10). For each A_j, let us define a matrix

$$M_j = QA_j^T + A_jQ + \alpha Q + \frac{1}{\alpha}R_w^{-1}$$

for a given positive and arbitrarily chosen scalar α and let (λ_j^r, v_j^r) with $r \in \{1, \ldots, n\}$ be the negative eigenvalues and corresponding eigenvectors of M_j.

Theorem 1. Given system (1), the saturated linear state feedback control (8) drives the state $x(t)$ within the target set Π if

$$X_j \subseteq Span\{v_j^r\}, \quad \text{for all } j = 1, \ldots, 2^n. \tag{13}$$

Proof. First of all, note that if (13) holds true then Σ_0 is invariant and as $\Sigma_0 \subseteq S(\psi^\theta)$ also $S(\psi^\theta)$ is invariant (the state trajectory will never exit $S(\psi^\theta)$). Now, we must show that $\dot V(x) < 0$ for all x and w such that $x \notin \Pi$, $u \in \mathcal{U}$ and $w \in \mathcal{W}$. In formulas, we must have

$$\begin{aligned}\dot V(x) &= \dot x^T Px + x^T P\dot x = x^T A(t)^T Px + x^T PA(t)x \\ &- w^T Px - x^T Pw < 0\end{aligned} \tag{14}$$

for all x and w satisfying

$$1 - x^T Px \le 0 \tag{15}$$

$$w^T R_w w - 1 \le 0. \tag{16}$$

Using the $\mathcal{S}$-procedure, we can say that condition (14) is implied by conditions (15)-(16) if there exist $\alpha, \beta \ge 0$, such that for all x and w

$$\begin{bmatrix} x \\ w \end{bmatrix}^T \begin{bmatrix} A(t)^T P + PA(t)^T + \alpha P & -P \\ -P & -\beta R_w \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} - \alpha + \beta \le 0. \tag{17}$$

Trivially it must hold $\beta \le \alpha$. Assume without loss of generality $\beta = \alpha$. Remind that α and β can be chosen arbitrarily. After pre and post-multiplying by $Q = P^{-1}$, the above condition becomes

$$\begin{bmatrix} x \\ w \end{bmatrix}^T \begin{bmatrix} QA(t)^T + A(t)^T Q + \alpha Q & -I \\ -I & -\alpha R_w \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \le 0. \tag{18}$$

Now, as the state never leaves the region $S(\psi^\theta)$, i.e., $x(t) \in S(\psi^\theta)$, we can always express $A(t)$ as convex combination of the A_js as in (11).

By convexity, the above condition is true if it holds, for all $j = 1, \ldots, 2^n$,

$$\begin{bmatrix} x \\ w \end{bmatrix}^T \begin{bmatrix} QA_j^T + A_j^T Q + \alpha Q & -I \\ -I & -\alpha R_w \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \le 0. \tag{19}$$

Using the Shur complement the condition (19) is implied by (13). □

Stronger conditions are established in the following corollary which also highlights the dependence of M_j on the scalar α.

Corollary 2. Given system (1), the saturated linear state feedback control (8) drives the state $x(t)$ within the target set Π if there exists a scalar $\alpha \ge 0$ such that

$$M_j < 0, \quad \text{for all } j = 1, \ldots, 2^n. \tag{20}$$

Proof. Trivially, if we observe that (20) implies (13). □

As the above conditions are stronger sufficient conditions the target set obtained when such conditions are valid is (strictly) included within the current target set Π. In other words, when we refer to the above conditions we are approximating the target set Π with a smaller one contained in it. We conclude this section by observing that Corollary 2 yields a design condition to compute the target set given the feedback control law.

On this purpose, denote by Q_j the matrix of the smallest (in volume) ellipsoid satisfying $M_j < 0$, which is given by

$$Q_j := \arg \inf_Q \min_\alpha \{det(Q),\, M_j < 0\}. \tag{21}$$

Now, let matrix $\underline{A}$ be the matrix A_j with $j = 1, \ldots, 2^m$ when controls are unbounded. To be more precise, $\underline{A} = -BkH$ as all components of γ_j are equal to one. Also let us define $\underline{Q}$ the solution of (21) for $A_j = \underline{A}$. We derive that the minimum volume target set Π must inscribe the ellipsoid $\underline{\Pi}$ defined by $\underline{Q}$, i.e.,

$$\Pi \supseteq \underline{\Pi} := \{x \in \mathbb{R}^n : x^T \underline{Q}^{-1} x \le 1\}. \tag{22}$$

Similarly, let matrix $\overline{A}$ be the matrix A_j with $j = 1, \ldots, 2^m$ obtained when all controls are saturated at their lowest degree of saturation. To be more precise, $\overline{A} = -Bkdiag([\underline{\theta}_1, \ldots, \underline{\theta}_m])H$ as all components of γ_j are equal to $\underline{\theta}_i$ for $i = 1, \ldots, m$. If we also define $\overline{Q}$ the solution of (21) for $A_j = \overline{A}$, the target set Π must be inscribed in the ellipsoid $\overline{\Pi}$ defined by $\overline{Q}$, namely,

$$\Pi \subseteq \overline{\Pi} := \{x \in \mathbb{R}^n : x^T \overline{Q}^{-1} x \le 1\}. \tag{23}$$

Then, we can use (5) to compute

$$\begin{aligned}\epsilon_{min} &:= \max_x \{\|x\|_\infty : x \in \underline{\Pi}\} \\ \epsilon_{max} &:= \max_x \{\|x\|_\infty : x \in \overline{\Pi}\}\end{aligned} \tag{24}$$

and finally determine the interval $[\epsilon_{min}, \epsilon_{max}]$ where x is confined componentwise.

Example 2. Consider the graph depicted in Fig. 1, with one node and two arcs and incidence matrix $B = [1 \quad 1]$. Controls are subject to polytopic constraints (6). Take $H = [\frac{1}{2} \quad \frac{1}{2}]^T$ and $k = 1$. Then according to (12) we have (here x is a scalar)

$$\theta_1(x) = \begin{cases} \dfrac{4}{x} & \text{if } \frac{x}{2} > 2 \\ 1 & \text{if } \frac{x}{2} \in [-3, 2] \\ -\dfrac{6}{x} & \text{if } \frac{x}{2} < -3 \end{cases},$$

$$\theta_2(x) = \begin{cases} \frac{4}{x} & \text{if } \frac{x}{2} > 2 \\ 1 & \text{if } \frac{x}{2} \in [-1, 2] \\ -\frac{2}{x} & \text{if } \frac{x}{2} < -1 \end{cases} .$$

If we consider initial states $x(0)$ satisfying $-10 \leq x(0) \leq 10$, possible lower bounds for the θ's are $\underline{\theta}_1 = \frac{2}{5}$ and $\underline{\theta}_2 = \frac{1}{5}$. Note that $S(\psi^\theta) = \{x \in \mathbb{R}^n : -10 \leq x \leq 10\}$. Vectors γ's and matrices A's turn out to be

$$\begin{array}{llll} \gamma_1 = [1 \quad 1]^T & \gamma_2 = [0.4 \quad 1]^T & & \\ & \gamma_3 = [1 \quad 0.2]^T & \gamma_4 = [0.4 \quad 0.2]^T & \\ A_1 = -2 & A_2 = -1.4 & A_3 = -1.2 & A_4 = -0.6 \end{array} . \quad (25)$$

Dynamics (10) is then

$$\dot{x} = [-\sigma_1(t)2 - \sigma_2(t)1.4 - \sigma_3(t)1.2 - \sigma_4(t)0.6]x + w, \quad (26)$$

with $\sum_{j=1}^4 \sigma_j(t) = 1$. Furthermore, we have

$$\begin{array}{ll} M_1 = [-4 + \alpha]Q + \frac{1}{\alpha} & M_2 = [-2.8 + \alpha]Q + \frac{1}{\alpha} \\ M_3 = [-2.4 + \alpha]Q + \frac{1}{\alpha} & M_4 = [-1.2 + \alpha]Q + \frac{1}{\alpha} \end{array} .$$

To apply Theorem 1 and Corollary 2, note that there exists a Q great enough and $\alpha < 1.2$ such that $M_4 < 0$ and consequently $M_j < 0$ for all j.

4. CONSENSUS APPROACH

In this section we study the disturbed consensus as a particular example of Problem 1. Consider a network $G = \{\mathcal{G}, \mathcal{E}\}$ with $n = |\mathcal{G}|$ nodes and $m = |\mathcal{E}|$ arcs. Here we denote by $|X|$ the cardinality of set X. Node are associated to integrators of the form

$$\dot{\xi} = \mu + \omega, \quad (27)$$

where $\xi(t) \in \mathbb{R}^n$ is the state, $\mu(t) \in \mathbb{R}^n$ is the control, and $\omega(t) \in \mathbb{R}^n$ is the disturbance. Again, disturbances are bounded within ellipsoids, i.e.,

$$\omega(t) \in \Omega = \{\omega \in \mathbb{R}^n : \omega^T R_\omega \omega \leq 1\}, \quad (28)$$

and in addition we assume that it is zero-mean. It is well-known (see, e.g., Bauso et al. [2006-b]) that, if the graph is undirected or in case it is directed and balanced, if there are no disturbances, and if we apply the linear consensus protocol

$$\mu = -kL\xi,$$

where L is the Laplacian matrix describing $\mathcal{G}$ and k is a positive scalar, the system state converges to the consensus value $\xi_1 = \xi_2 = \ldots = \xi_n = \bar{\xi} = \frac{1}{n}\sum_{i=1}^n \xi_i(0)$. First observe that because of the presence of UBB disturbances convergence to $\bar{x}$ is, in general, not possible.

We also assume that sensors measuring the potential differences at the arcs are subject to saturations. To clarify this point, let us explain next how the linear consensus protocol modifies. Let $\nu = [\nu_{ij}]_{(i,j)\in\mathcal{E}} \in \mathbb{R}^m$ be a vector where component $\nu_{ij} = \xi_j - \xi_i$. Then it turns out that $L_i\xi = \sum_{j\in\mathcal{N}_i} \nu_{ij}$ where L_i indicates the ith row of L and $\mathcal{N}_i$ is the set of node i's neighbors. With this in mind, we can rewrite the linear protocol as

$$\mu = -kL\xi = \left[\sum_{j\in\mathcal{N}_1} -k\nu_{1j} \ldots \sum_{j\in\mathcal{N}_n} -k\nu_{nj}\right]^T \in \mathbb{R}^n.$$

Observe that each ith control averages the potential difference of all arcs incident to node i.

The linear consensus protocol under saturations becomes

$$\mu = \left[\sum_{j\in\mathcal{N}_1} sat_{[\alpha,\beta]}(-k\nu_{1j}) \ldots \sum_{j\in\mathcal{N}_n} sat_{[\alpha,\beta]}(-k\nu_{nj})\right]^T \in \mathbb{R}^n$$

and the resulting closed-loop is

$$\dot{\xi} = sat_{[\alpha,\beta]}(-kL\xi) + \omega, \quad \omega(t) \text{ as in (28)}. \quad (29)$$

The following definition of ϵ-consensus describes the cases where the system state is driven in finite time within a bounded tube of radius ϵ,

$$\mathcal{T} = \{\xi \in \mathbb{R}^n : |\xi_i - \xi_j| \leq 2\epsilon, \forall i, j \in \mathcal{G}\}. \quad (30)$$

Obviously, $\mathcal{T}$ will depend strongly on the amplitude of disturbances and saturations.

Problem 2. (ϵ-consensus) Given system (29), find conditions under which it is possible to drive the state $\xi(t)$ within the target set $\mathcal{T}$.

We explain next how to convert the above problem into Problem 1. For any given tree $T = \{\mathcal{G}, \mathcal{E}'\}$ obtained from G (we have $\mathcal{E}' \subset \mathcal{E}$, and $|\mathcal{E}'| = n-1$) consider the associated edge path incidence matrix $E \in \{0,1\}^{(n-1)\times m}$. In the edge path incidence matrix, the generic column associated to the edge $(i,j) \in \mathcal{E}$ is the incidence vector of the edges in $\mathcal{E}'$ on the unique path in T that joins node i with node j.

Then, we can derive a new state variable $\eta = [\eta_{ij}]_{(i,j)\in\mathcal{E}'} \in \mathbb{R}^{n-1}$ with generic component $\eta_{ij} = \nu_{ij} = \xi_j - \xi_i$. Dynamics in the new state variable is

$$\dot{\eta} = Esat_{[\alpha,\beta]}(\nu) + N^T\omega = Esat_{[\alpha,\beta]}(E^T\eta) + N^T\omega \quad (31)$$

where $N \in \{-1,0,1\}^{n\times(n-1)}$ is the incidence matrix of T and the second equation is obtained by noticing that $\nu = E^T\eta$. Note that dynamics (31) has the same form of (9). Also, denote by $Proj_\eta(\mathcal{T})$ the projection of the target $\mathcal{T}$ on the η space. Then, we can define a function $V(\eta) = \eta^T P\eta$ with $P \in \mathbb{R}^{(n-1)\times(n-1)}$ and positive definite such that $V(\eta) \subseteq Proj_\eta(\mathcal{T})$. Once we do this, the ϵ-consensus problem in the ξ space turns into an ϵ-stabilizability problem in the η space and we can apply the method explained in the first part of the work.

5. NUMERICAL EXAMPLE

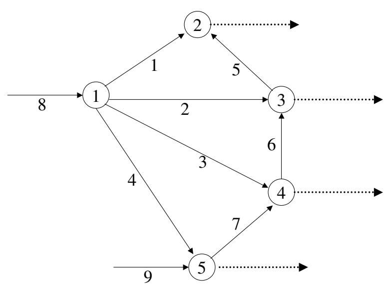

Fig. 2. Example of a system with 5 nodes and 9 arcs.

u^+	$\underline{\theta}$	ϵ_{min}	ϵ_{max}
1	$[0.25\,0.5\,0.5\,1\,1\,0.5\,0.25\,0.125\,0.125]^T$	1	31.62
2	$[0.5\,1\,1\,1\,1\,1\,0.5\,0.25\,0.25]^T$	1	8.27
3	$[0.75\,1\,1\,1\,1\,1\,0.75\,0.375\,0.375]^T$	1	4.16
4	$[1\,1\,1\,1\,1\,1\,1\,0.5\,0.5]^T$	1	2.88
5	$[1\,1\,1\,1\,1\,1\,1\,0.625\,0.625]^T$	1	1.97
6	$[1\,1\,1\,1\,1\,1\,1\,0.75\,0.75]^T$	1	1.49
7	$[1\,1\,1\,1\,1\,1\,1\,0.875\,0.875]^T$	1	1.19
8	$[1\,1\,1\,1\,1\,1\,1\,1\,1]^T$	1	1

Table 1. Values of $\underline{\theta}$, ϵ_{min} and ϵ_{max} for varying $u^+ = -u^-$ when $R_w = I$ and $k = 1$. Higher bounds u^+ yield higher $\underline{\theta}$ componentwise, and smaller ϵ_{max}.

Consider the constrained dynamics (1)-(3) for the flow network system with $n = 5$ nodes and $m = 9$ arcs depicted in Fig. 2. Consider the saturated linear control (8) where $k = 1$ and matrix $H \in \mathbb{R}^n$ is defined as

$$H = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 \\ -0.1 & 0 & 0.5 & 0 & 0 \\ -0.2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 \\ 0.1 & 0 & 0 & 1 & 0 \\ 0.7 & 1 & 1 & 0 & 0 \\ 0.3 & 0 & 0 & 1 & 1 \end{bmatrix}. \qquad (32)$$

First let us take without loss of generality $R_w = I$ and solve the semi-definite problem (22)-(23). According to our expectation, we always find $\underline{Q}$, if exists, equal to R_w while $\overline{Q}$ is a diagonal matrix

$$\underline{Q} = \begin{bmatrix} q_1 & 0 & \dots & 0 \\ 0 & q_2 & \dots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \dots & q_m \end{bmatrix}. \qquad (33)$$

with all equal eigenvalues $q_1 = \ldots = q_n$. The associated ellipsoid describes a sphere of radius $\sqrt{q_i}$. Table 1 displays the vector $\underline{\theta}$ and the values ϵ_{min} and ϵ_{max} for varying $u^+ = -u^- = 1, \ldots, 8$. Note that, for this specific case where $R_w = I$, we have $\epsilon_{min} = \sqrt{\lambda_{max}(\underline{Q})}$ and $\epsilon_{max} = \sqrt{\lambda_{max}(\overline{Q})}$ where we indicate by $\lambda_{max}(M)$ the maximum eigenvalue of a generic square matrix M.

Increasing bounds u^+ lead to higher $\underline{\theta}$ componentwise, and smaller ϵ_{max} while ϵ_{min} is always one. Smaller values of ϵ_{max} mean that the ellipsoid $\overline{\Pi}$ approximates better and better the minimum volume target set Π (remind the inclusions $\underline{\Pi} \subseteq \Pi \subseteq \overline{\Pi}$). For $u^+ = 8$ we have $\underline{\theta} = 1$ which means that no controls are saturated. In correspondence to this we also have $\epsilon_{max} = 1$ and the associated ellipsoid $\{x \in \mathbb{R}^n : x^T \overline{Q}^{-1} x \leq 1\}$ is the sphere of unitary radius. The latter represents exactly the minimum volume target set Π where we can drive $x(t)$.

Next, we choose a different matrix

k	$\underline{\theta}$	ϵ_{min}	ϵ_{max}
1/4	$[1\,1\,1\,1\,1\,1\,1\,0.5\,0.5]^T$	5.00(5.77)	16.93
1/3	$[0.75\,1\,1\,1\,1\,1\,0.75\,0.375\,0.375]^T$	3.75(4.33)	18.29
1/2	$[0.5\,1\,1\,1\,1\,1\,0.5\,0.25\,0.25]^T$	2.5(2.88)	19.79
1	$[0.25\,0.5\,0.5\,1\,1\,0.5\,0.25\,0.125\,0.125]^T$	1.22(1.41)	21.21

Table 2. Values of $\underline{\theta}$, ϵ_{min} and ϵ_{max} for varying k with R_w as in (34). In parenthesis the values when $\underline{Q}$ is imposed full diagonal as in (33).

$$R_w = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{4} & -\frac{1}{4} & 0 \\ 0 & 0 & -\frac{1}{4} & \frac{3}{4} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \qquad (34)$$

and again solve the semi-definite problem (22)-(23). Given $\underline{Q}$ and $\overline{Q}$, we compute $\underline{\Pi}$ and $\overline{\Pi}$ and using (24) also ϵ_{min} ϵ_{max}. Table 2 displays the vector $\underline{\theta}$ and ϵ_{min} and ϵ_{max} for varying $k = 1/4, 1/3, 1/2, 1$ (in this case we always find $\underline{Q} = k^2 R_w$). We can notice that by increasing k we have lower values of $\underline{\theta}$, which means that controls saturate more and more. Also we have lower values of ϵ_{min} which corresponds to smaller ellipsoids $\underline{\Pi}$. The values in parenthesis corresponds to the case where we additionally constrain $\underline{Q}$ to be full diagonal as in (33). The values of ϵ_{max} increase which means bigger ellipsoids $\overline{\Pi}$ and this is due to the small values of θ. The additional constrain of $\underline{Q}$ being full diagonal as in (33) makes the semi-definite problem (22)-(23) and therefore no values in parenthesis are displayed.

Now, we simulate the system with initial state $x(0) = [0\,4\,4\,4\,4]^T$ and demand $w(t)$ taking on one of the following values with uniform probability

$$\begin{array}{ll} w^{(1)} = [0\ \pm 1\ 0\ 0\ 0]^T & w^{(2)} = [0\ 0\ \pm[1\quad 1]\ 0]^T \\ w^{(3)} = [0\ 0\ \pm[-1\sqrt{2}\ 1\sqrt{2}]\ 0]^T & w^{(4)} = [0\ 0\ 0\ 0\ \pm 1]^T \\ w^{(5)} = [0\ 0\ \pm[1.22\ 0.3]\ 0]^T. & \end{array} \qquad (35)$$

With the above choice for $w(t)$ (it lays on the boundary of $\mathcal{W}$), we cause higher oscillations for $x(t)$. Fig. 3 displays the time plot of the state variable $x(t)$ when the saturated linear state feedback control (8) is applied with H as in (32) and for different values of k. Each component of the state x_i is comprised in the interval $[-\epsilon_{min}, \epsilon_{min}]$ (dashed lines) which also means that $\underline{\Pi} \equiv \Pi$. In Fig. 4 we show the time plot of the function $V(x(t)) - 1$ with $V(x(t)) = x^T(t)\underline{Q}^{-1}x(t)$. For about $t > 8$ the function $V(x(t)) - 1$ is negative which means $x \in \underline{\Pi}$. Finally, in Fig. 5 we display the projection onto the plane x_3-x_4 of the simulated state trajectory for $k = \frac{1}{2}$. Starting at point $[4\quad 4]^T$, the trajectory (dotted) is soon confined within the target set $\underline{\Pi}$ (solid ellipsoid).

6. CONCLUSIONS AND FUTURE WORKS

This work is a continuation of Bauso et al. [2006-a] and is in line with some recent applications of LMI techniques to inventory/manufacturing systems Boukas [2006]. In a future work, we will study the validity in probability of the LMI conditions derived in this paper. This is in accordance with some recent literature on *chance LMI constraints*

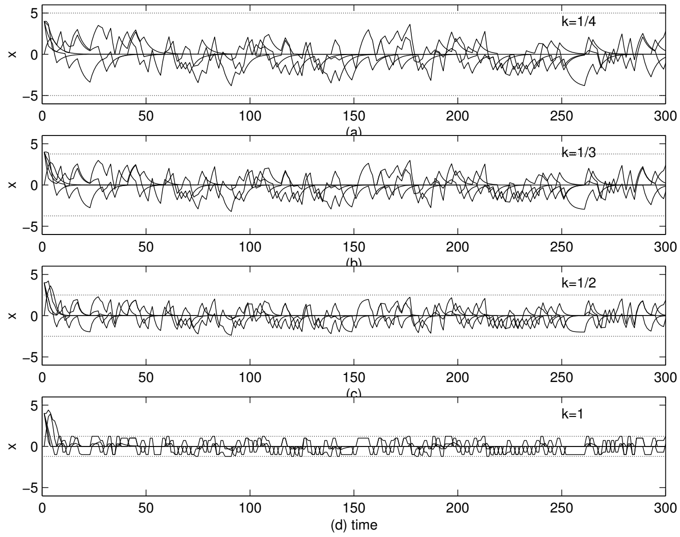

Fig. 3. Time plot of $x(t)$ when control (8) is applied with H as in (32) and for (a) $k = 1/4$, (b) $k = 1/3$, (c) $k = 1/2$ and (d) $k = 1$. Demand $w(t)$ takes on one of the values in (35) with uniform probability. We have x in the interval $[-\epsilon_{min}, \epsilon_{min}]$ (dashed lines).

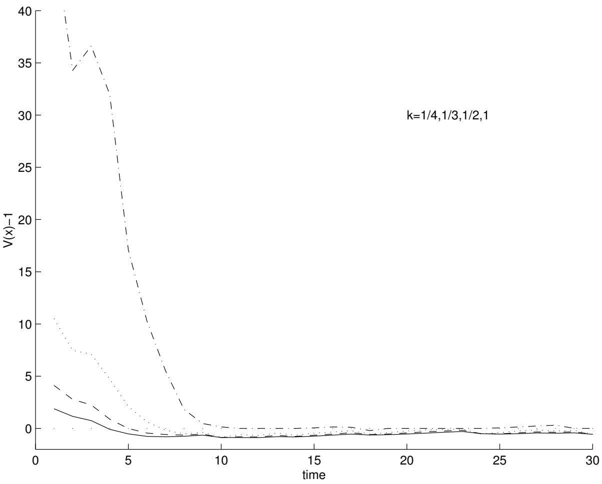

Fig. 4. Time plot of $V(x(t)) - 1$ with $V(x(t)) = x^T(t)\underline{Q}^{-1}x(t)$ when control (8) is applied with H as in (32) and for $k = 1/4$ (solid line), $k = 1/3$ (dashed line), $k = 1/2$ (dotted line) and $k = 1$ (dash-dot line). For about $t > 8$ the function $V(x(t)) - 1$ is negative which means $x \in \underline{\Pi}$

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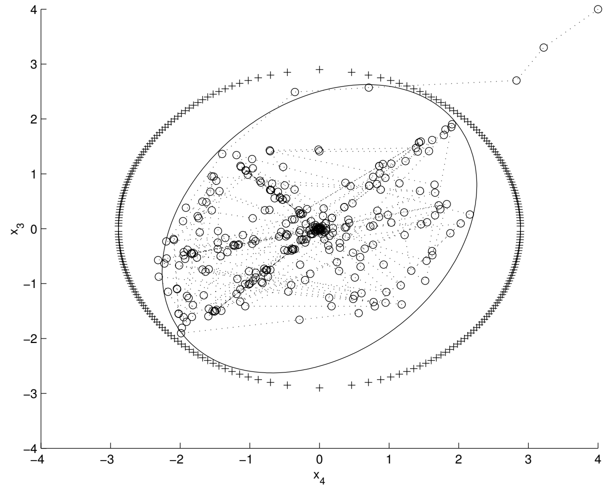

Fig. 5. Projection onto the plane x_3-x_4 of the simulated state trajectory for $k = \frac{1}{2}$. Starting at point $[4 \quad 4]^T$, the trajectory (dotted) is soon confined within the target set $\underline{\Pi}$. The external ellipsoids (crosses) describes $\underline{\Pi}$ when we additionally constrain $\underline{Q}$ to be full diagonal as in (33).

developed in the area of robust optimization Ben Tal and Nemirovsky [2002], Calafiore and Campi [2005].

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