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Horizon-Switching Predictive Set-Point Tracking under mixed Control Saturations and Persistent Disturbances

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Abstract:
This paper extends of the horizon-switching predictive control approach, so far restricted to positional input saturation and the pure regulation problem, to the case of set-point tracking. It is proved a basic feasibility property which makes it possible to extend this approach so as to achieve offset-free asymptotic tracking under joint positional and incremental input saturations and constant disturbances. It is also addressed the same problem in the presence of time-varying disturbances. In such a case it is proved that, whenever the system is subject only to incremental input saturations, the goal is achieved with the property of bounded-noise bounded-state l_∞-stability.

1. INTRODUCTION

Control of input-saturated dynamic systems, though a fundamental issue in automatic control, has been given exhaustive and constructive systematic answers mainly only during the last fifteen years. From one side it was characterized the class of dynamical linear time-invariant (LTI) systems whose state can be asymptotically driven to zero with arbitrarily small controls (Sussmann et al. (1994)): the so called ANCBI systems. In discrete time, they coincide with all stabilizable LTI systems with eigenvalues in the closed unit disk. Hence, they encompass stable systems with integrator chains of arbitrary complexity, and are representative of a great deal of processes of practical interest. From another side, linear control structures were shown to only provide semi-global stabilization of input-saturated ANCBI systems (Lin (1995)). Non-linear state feedback schemes for input-saturated ANCBI plants were discussed in Sussmann and Yang (1991) and Sussmann et al. (1994). However, such schemes amount to low-gain control strategies which feature poor regulation performance. In an attempt to provide enhanced performance, gain-scheduling variants, akin to the approach adopted in Mosca (2005) and the present paper, were proposed in Alvarez-Ramírez and Suárez (1996) and Lin et al. (1996).

While positional input saturations have attracted a great deal of interest, fewer results apply to incremental input saturations or joint positional and incremental input saturations. Incremental input saturations are a serious challenge in many automatic control applications, *e.g.* flight control (Lenorovitz (1990); Dornheim (1992)). In particular, it is known that they can induce a considerable destabilizing effect due to phase-lag. Joint constraints on both input magnitude and increments were considered in Trygve et al. (1997) for the particular case of a plant consisting of a chain of cascade integrators. More generally, Lin et al. (1997) showed that the whole class of ANCBI systems is semi-globally stabilizable through linear feedback also in the presence of both constraints. For other contributions to the topic, see also Feng et al. (1992) and Lin et al. (1997). However, it is to be pointed out that all these contributions deal mainly with the stabilization issue, and put little attention on performance of the overall controlled system. A recent paper (Mosca (2005)) has reconsidered the problem from the viewpoint of both stability and performance of systems subject to only positional input saturations but also affected by persistent disturbances of arbitrary unknown intensity.

The present paper aims at extending the results of Mosca (2005) to the case of set-point tracking for LTI systems subject to persistent disturbances. The paper is organized as follows. Sect. 2 describes the so-called incremental model of the system to be controlled, the specific type of feedback-gain matrices that are adopted to realize possible control actions, and motivates the study. Sect. 3 proves that the algorithm proposed in Mosca (2005) can be used for the set-point tracking problem also in the presence of both incremental input saturation and persistent time-varying disturbances. Sect. 4 shows that, in the presence of only constant disturbances, with a simple modification in the switching logic, the approach can handle the case of joint positional and incremental input saturations.

2. PRELIMINARIES

Consider the following generic discrete-time LTI ANCBI system

$$\begin{cases} s(t+1) = \Phi_s s(t) + G_s u(t) + \xi(t) \\ q(t+1) = \Phi_q q(t) + G_q u(t) + c(t) \\ \quad\;\; y(t) = H_s s(t) + H_q q(t) + \zeta(t) \end{cases} \tag{1}$$

where $\Phi_s \in \mathbb{R}^{n_s}$ and $\Phi_q \in \mathbb{R}^{n_q}$; $x(t) := [s'(t)\ q'(t)]'$, the prime denotes transpose; $x(t) \in \mathbb{R}^{n_x}$, $n_x := n_s +$

n_q is the plant state; $|\lambda_i(\Phi_s)| < 1$, $i \in \overleftarrow{n_s}$, $\overleftarrow{n} := \{1, 2, ..., n\}$, $|\lambda_j(\Phi_q)| = 1$, $j \in \overleftarrow{n_q}$, with arbitrary geometric multiplicities; $u(t) \in \mathbb{R}^m$ is the control. Let

$$e(t) := y(t) - r(t) \tag{2}$$

denote the tracking error, $y(t) \in \mathbb{R}^m$ being the performance variable (output), and $r(t)$ the set-point to be tracked by the output. The vectors $\xi(t)$, $c(t)$ and $\zeta(t)$ represent arbitrary bounded disturbances. The following assumption is adopted:

$$(\Phi, G)\,\text{reachable} \tag{3}$$

where $\Phi := diag\{\Phi_s, \Phi_q\}$. The plant input $u(t)$ and its increments $\delta u(t) := u(t) - u(t-1), \forall t \in Z_+ := \{0, 1, \ldots\}$ are subject to the following saturation constraints

$$u(t) \in \mathcal{U} := \{u \in R^m : |u|_i < \overline{U}\}, \tag{4}$$

$$\delta u(t) \in \mathcal{D} := \{\delta u \in R^m : |\delta u|_i < \overline{\Delta}\} \tag{5}$$

where $i \in \overleftarrow{m}$, $\overline{U}$, $\overline{\Delta}$ positive extended reals, and $|u|_i$ and $|\delta u|_i$ denote the absolute value of the i-th component of u and, respectively, δu. It is known that an ANCBI system has the most general structure for which it makes sense to consider stability and boundeness for any arbitrary initial state of an input-saturated LTI system .

The aim is to find a subject to (4)-(5) stabilizing feedback control for (1) with possibly offset-free tracking. The design can be carried out by resorting to the so-called *incremental model* (IM$_\eta$) of (1)

$$\begin{cases} \chi(t+1) = \mathcal{A}\,\chi(t) + \mathcal{B}\,\delta u(t) + \delta v(t) \\ \quad\;\; \eta(t) = \mathcal{C}\,\chi(t) + \delta w(t) \end{cases} \tag{6}$$

where $\chi(t) := [\delta x'(t)\ \eta'(t-1)]'$, $\delta x(t) := x(t) - x(t-1)$; signals $\delta v(t) := [\delta\xi'(t)\,\delta c'(t)\,\delta w'(t)]'$, $\delta w(t) := \delta\xi(t) - \delta r(t)$, and

$$\mathcal{A} = \begin{bmatrix} \Phi & 0 \\ H & I_m \end{bmatrix} \quad \mathcal{B} = \begin{bmatrix} G \\ 0_m \end{bmatrix} \quad \mathcal{C} = \begin{bmatrix} H & I_m \end{bmatrix} \tag{7}$$

A necessary and sufficient condition for the existence of such a stabilizing linear state-feedback is as follows (Davison (1976))

$$\det \begin{bmatrix} \mathrm{I}_n - \Phi & G \\ H & 0_m \end{bmatrix} \neq 0 \tag{8}$$

In connection with the incremental model (6), let χ be its state at time 0, and $\Omega_h(\chi)$ the set of all control increments ω of length h, $\omega = [\delta u'(0), \ldots, \delta u'(h-1)]'$, which drive the system state to the zero-state 0_χ in h time-steps

$$\Omega_h(\chi) := \left\{\omega \in (\mathbb{R}^m)^h : \chi(h) = 0_\chi\right\} \tag{9}$$

where $\chi(h) = \mathcal{A}^h\chi + \sum_{k=0}^{h-1} \mathcal{A}^{h-1-k}\mathcal{B}\delta u(k)$. Note that $\Omega_h(\chi) \neq \emptyset, \forall\chi \in \mathbb{R}^n$, $n := n_x + m$, if $h \geq \nu$, with ν, $\nu \leq n$, the reachability index of $(\mathcal{A}, \mathcal{B})$. Let $\delta u_h(\chi)$ the element in $\Omega_h(\chi)$ of minimum energy

$$\sum_{k=0}^{h-1} \delta u'(k)\Psi_u\delta u(k) \;=\; \omega'\widehat{\Psi}_u\omega \tag{10}$$

where $\Psi_u = \Psi_u' > 0$ and $\widehat{\Psi}_u := \mathrm{Diag}\{\Psi_u, \ldots, \Psi_u\}$ -(h-times). For $h \geq \nu$, $\delta u_h(\chi)$ is as follows

$$\begin{aligned} \delta u_h(\chi) &:= [\delta u_h'(0|\chi), \ldots, \delta u_h'(h-1|\chi)]' \\ &= [\mathcal{F}_h'(0) \cdots \mathcal{F}_h'(h-1)]'\chi \\ &= \mathcal{F}_h\chi \end{aligned} \tag{11}$$

$$\mathcal{F}_h := -\widehat{\Psi}_u^{-1}\mathcal{R}_h'\mathcal{G}_h^{-1}\mathcal{A}^h \tag{12}$$

where $\mathcal{R}_h$ is the h-order reachability matrix

$$\mathcal{R}_h := \left[\mathcal{A}^{h-1}\mathcal{B}| \ldots |\mathcal{A}\mathcal{B}\,|\mathcal{B}\right] \tag{13}$$

and $\mathcal{G}_h$ the h-order reachability Gramian

$$\mathcal{G}_h := \mathcal{R}_h\mathcal{R}_h' \tag{14}$$

The integer h will be referred to as the *control horizon*.

In order to address the set-point tracking problem it is necessary to give some interpretation about the structure of (6). First, one has to note that the problem set-point tracking is subject to the some intrinsic limitation, similarly to the pure regulation problem. Indeed, time-varying disturbances $\delta c(t)$, entering the neutrally stable modes of $\chi(t)$ in (6), are generally not allowed to assume arbitrary values, *e.g.* $c(t)$ cannot assume arbitrary incremental values. Consequently, in the following it will be assumed that such a disturbance $c(t)$ in (1) be constant, *i.e.* $c(t) \equiv c$.

2.1 Motivation of the study

The motivation for studing possibile extensions of the approach proposed in Mosca (2005) hinges mainly upon two considerations. The first one is related to the possibility of decoupling the effects of the disturbances on the system. Consider the incremental model (6) and a similarity transformation $\mathcal{T} : \chi \to \chi_\epsilon := [\delta s'\,\delta q'\,\epsilon'^-]'$ (ϵ^- stands for $\epsilon(t-1)$)

$$\mathcal{T} := \begin{bmatrix} I_s & 0 & 0 \\ 0 & I_q & 0 \\ H_s(I_s - \Phi_s)^{-1} & 0 & I_m \end{bmatrix} \tag{15}$$

under which (6) diagonalizes *w.r.t.* δs. This transformation exists as $1 \notin sp(\Phi_s)$ and leads to a novel incremental model (IM$_\epsilon$) algebraically equivalent to (6)

$$\begin{cases} \delta s(t+1) = \Phi_s\delta s(t) + G_s\delta u(t) + \delta\xi(t) \\ \delta q(t+1) = \Phi_q\delta q(t) + G_q\delta u(t) \\ \epsilon(t) = \epsilon(t-1) + H_q\delta q(t) + W_s\delta u(t) + \delta n(t) \end{cases} \tag{16}$$

where $W_s := H_s(I_s - \Phi_s)^{-1}G_s$, $\delta n(t) := \hat{W}_s\delta\xi(t) + \delta\zeta(t) - \delta r(t)$, and $\hat{W}_s := H_s(I_s - \Phi_s)^{-1}$. Re-define

$$\mathcal{A} := \begin{bmatrix} \Phi & 0 \\ \hat{H} & \mathrm{I}_m \end{bmatrix} \mathcal{B} := \begin{bmatrix} G \\ W_s \end{bmatrix} \mathcal{C} := \begin{bmatrix} \hat{H} & \mathrm{I}_m \end{bmatrix} \tag{17}$$

with $\hat{H} := [\,0\ \ H_q\,]$. Because (11) is linear in χ_ϵ,

$$\delta u_h(\chi_\epsilon) = \delta u_h^s(\delta s) + \delta u_h^q(\delta q) + \delta u_h^\epsilon(\epsilon^-) \tag{18}$$

By (16) one has $\delta u_h^s(k|\delta s) := -\Psi_u^{-1}\mathcal{B}'(\mathcal{A}^{h-k-1})'\mathcal{G}_h^{-1}\delta\hat{s}$ where $\delta\hat{s} := [(\Phi_s^h\delta s)'\,0_q'\,0_\epsilon']'$, and similarly for $\delta u_h^q(\delta q)$. Moreover $\delta u_h^\epsilon(k|\epsilon^-) := -\Psi_u^{-1}\mathcal{B}'(\mathcal{A}^{h-k-1})'\mathcal{G}_h^{-1}\hat{\mathcal{A}}^h\chi_\epsilon$, where $\hat{\mathcal{A}}^h := \begin{bmatrix} 0_s & 0 & 0 \\ 0 & 0_q & 0 \\ 0 & H_q\sum_{i=0}^{h-1}\Phi_q^i & I_m \end{bmatrix}$. Hence, (18) allows one to consider separately the contribution in $\delta u_h(\chi_\epsilon)$ given by the disturbances $\delta\xi$ injected in the stable modes and the contribution caused by the disturbances δn affecting the critically unstable ones.

The second one focuses on the relationship between the positional system (1) and the incremental model (6). Seemingly, as $\delta w(t)$ enters the neutrally stable modes of $\chi(t)$, system (6) has not the structure of an input-saturated LTI system for which it makes sense to consider stability and boundedness under arbitrary l_∞-disturbances. Consequently, is not possible to ensure that the direct adoption to the present case of the approach of Mosca (2005) can achieve such goals. Indeed, even if $\delta w(t) \equiv 0$, $\forall t \in \mathbb{Z}_+$, the neutrally stable modes of $\chi(t)$ would be indirectly affected, via the stable ones, by $\delta\xi(t)$, which enters only the stable modes of $\chi(t)$. However, one has to take account that (6) is a representation for design of the real UPS (1). Hence, not only the positional and the incremental instantaneous values of the disturbances are bounded but also any partial sum of the incremental ones, viz. $\forall t, v \in [0, \infty)$

$$\begin{cases} |\xi(t)| \leq \overline{\Xi} \Rightarrow \left|\displaystyle\sum_{t=0}^{v} \delta\xi(t)\right| \leq 2\,\overline{\Xi} \\ |n(t)| \leq \overline{\mathcal{N}} \Rightarrow \left|\displaystyle\sum_{t=0}^{v} \delta n(t)\right| \leq 2\,\overline{\mathcal{N}} \end{cases} \tag{19}$$

These properties allow one to prove the intuition motivated conjecture that, for $h(t)$ sufficiently large, there exists an interval of L consecutive steps, such that the contribution to δu given by any sequence $\{\delta n(t)\}_{t=0}^{l}$, $l \in \overleftarrow{L}$, which enters the integrator modes, is of the same order of $\sum_{t=0}^{l} \delta n(t)$. Specifically, let $p := [\delta q' \ \epsilon'^{-}]'$ denote the neutrally stable substate of χ_ϵ. Assume that the control horizon $h(\cdot)$ grows unbounded. This implies that $\chi(\cdot)$ is unbounded. As Φ_s is a stability matrix, and $\delta u(t)$ and $\delta\xi(t)$ are bounded, there must be a time t large enough at which $\|\chi_\epsilon(t)\|^2 = \|\delta s(t)\|^2 + \|p(t)\|^2 \simeq \|p(t)\|^2$. As $\sum_{j=0}^{t} \delta n(j)$ is bounded, h is chosen after such a large t, according to the restricted system with state $p(t+l) = \hat{p}(t+l) + \tilde{p}(t+l) \cong \hat{p}(t+l)$, where $\hat{p}(t+l)$ is related to the noiseless system while $\tilde{p}(t+l)$ is the response to the bounded term $\sum_{j=t}^{t+l-1} \delta n(j)$. Under these circumstances, at times $t+l$ subsequent to such a large t, $h(t+l) = h(t) - l$ until $\|\hat{p}(t+l)\|$ decreases so as to make $\|\delta s(t+l)\|$ and/or $\|\tilde{p}(t+l)\|$ comparable with $\|\hat{p}(t+l)\|$ and, hence, significant again for the selection of the horizon. This means that a "*horizon resetting mechanism*" is inherently enforced.

3. INCREMENTAL INPUT SATURATION

Let

$$M_h(\chi) := \max\{\frac{|[\delta u_h\,(k|\chi)]_i|}{\overline{\Delta}};\ k+1 \in \overleftarrow{h};\ i \in \overleftarrow{m}\} \tag{20}$$

where $[\delta u]_i$ denotes the i-th component of the vector δu. Note that the whole sequence $\delta u_h(\chi)$ does not violate (5) if and only if $M_h(\chi) < 1$. As (6) is ANCBI, it is always possible to find a large enough horizon h so as to satisfy $M_h(\chi) < 1$. In fact, it can be shown (Mosca (2005)) that for an ANCBI system

$$M_h(\chi) \leq \overline{M}\, h^{-1} \|\chi\| \tag{21}$$

where $\overline{M}$ is a positive real depending on $(\mathcal{A}, \mathcal{B})$.

Let F_h be as follows

$$F_h = \begin{bmatrix} I_m & 0_{m\times m(h-1)} \end{bmatrix} \mathcal{F}_h = \mathcal{F}_h(0) \tag{22}$$

If only input-increment saturations are present, at a generic time t, $h(t)$ can be chosen, according to a suitable logic, such that $M_{h(t)}(\chi(t)) < 1$ and the input increment to (6) can be set as $\delta u(t) = F_{h(t)}\chi(t)$. Here, $F_{h(t)} = \mathcal{F}_{h(t)}(0)$ is recognized to be the feedback-gain matrix of the receding horizon regulation related to the zero-terminal state minimum energy control problem of horizon $h(t)$. Let $\delta u(t) = F_{h(t)}\chi(t)$ with F_h as in (22) and $h(t)$ chosen according to the following *hysteresis switching logic* ($\underline{h} \geq n$, $n = n_x + m$)

$$h(t) = \begin{cases} \tilde{h}(t), & \text{if } M_{\tilde{h}(t)}(\chi(t)) \leq 1 \\ \hat{h}(t), & \text{otherwise.} \end{cases} \tag{23}$$

$$\tilde{h}(t) := \max\{\underline{h},\ h(t-1) - 1\}$$

$$\hat{h}(t) := \min\{h \in \mathbb{Z}_+ : h \geq h(t-1);\ M_h(\chi(t)) \leq 1 - \mu\}$$

where $t \in \mathbb{Z}_+$; $\mu \in (0, 1)$ is the *hysteresis constant*; $h(0) = \hat{h}(0)$ with $h(-1) = \underline{h}$; and $M_h(\chi)$ as in (20).

3.1 Time-varying disturbances

The proof of Th. 1 hinges upon the following lemma, proved in Mosca (2005), which the reader is referred to.

Lemma 1. Consider the system (6). Then, there exist large enough integers h, N, $h - N > 0$, such that the two following inequalities jointly hold

$$M_h(\chi) \leq \gamma M_{h-N}(\chi) \tag{24}$$

$$\lambda^h \leq c/N \tag{25}$$

for γ, $c > 0$, $\lambda \in (0,1)$ and $\forall \chi \in \mathbb{R}^n$.

The first result, pertinent to the case of incremental input saturations, is summed up in the following theorem.

Theorem 1. Consider the reachable ANCBI system (1) under the incremental input saturations (5). Let the control increment be given by $\delta u(t) = F_{h(t)}\chi(t)$ with $F_{h(t)}$ as in (22) and $h(t)$ chosen according to the hysteresis switching logic (23). Then, the resulting closed-loop hysteresis switched system is bounded-noise bounded-state l_∞-stable irrespective of both the initial state $x(0) \in \mathbb{R}^{n_x}$ and the magnitude of $\xi(\cdot)$, $\zeta(\cdot)$ and $r(\cdot)$.

Proof. For the sake of convenience, one can refer *w.l.o.g.* to the system (16) with state χ_ϵ. Let

$$|\delta u_h(\chi_\epsilon)| \leq b$$

stand for $|\delta u_h\,(k|\chi_\epsilon)|_i \leq b$, $\forall i \in \overleftarrow{m}$ e $k+1 \in \overleftarrow{h}$, where $|\delta u|_i$ denotes the absolute value of the i-th component of δu. Let $\eta := \mu\overline{\Delta}$, with μ as in (23) and where $\overline{\Delta}$ follows from (5). Let

$$\delta\hat{\xi}(t) := [\delta\xi'(t)\ 0'_q\ 0'_m]', \quad \delta\hat{n}(t) := [0'_s\ 0'_q\ \delta n'(t)]' \tag{26}$$

Notice that the latter equation, along with (18), implies

$$\delta u_h(\delta\hat{\xi}(t) + \delta\hat{n}(t)) = \delta u_h^s(\delta\hat{\xi}(t)) + \delta u_h^\epsilon(\delta\hat{n}(t)) \tag{27}$$

According to Bellman's principle of optimality, if $\mathcal{A}_h = \mathcal{A} + \mathcal{B}\mathcal{F}_h(0)$ it follows that $\delta u_{h-1}(k|\mathcal{A}_h\chi_\epsilon) = \delta u_h(k+1|\chi_\epsilon)$. Consequently, one can write, $\forall i \in [0, \infty)$

$$\begin{aligned} &\delta u_{h(t+i)}(k|\chi_\epsilon(t+i)) = \\ &= \delta u_{h(t+i)+i}(k+i|\chi_\epsilon(t)) + \delta u_\xi^s(t+1) + \delta u_n^\epsilon(t+i) \end{aligned} \tag{28}$$

where

$$\delta u_{\xi}^{s}(t+i) := \delta u_{h(t+i)+i-1}^{s}(k+i-1|\delta\hat{\xi}(t)) + \\ + \ldots + \delta u_{h(t+i)}^{s}(k|\delta\hat{\xi}(t+i-1)) \quad (29)$$

denotes the contribution of the disturbance which enters the stable modes of χ_ϵ, while

$$\delta u_{n}^{\epsilon}(t+i) := \delta u_{h(t+i)+i-1}^{\epsilon}(k+i-1|\delta\hat{n}(t)) + \\ + \ldots + \delta u_{h(t+i)}^{\epsilon}(k|\delta\hat{n}(\tau+i-1)) \quad (30)$$

is the contribution related to the critically unstable ones. Let, by contradiction, $h(\cdot)$ be unbounded. Then, there exists a subsequence $\{t_j\}_{j=1}^{\infty}$ such that $\lim_{j\to\infty} h(t_j) = \infty$ and $h(t) \le h(t_j)$, $t \le t_j$. By virtue of Lemma 2, there exist large enough regulation horizons h and integers $L < h$, such that the following inequalities jointly hold

$$|\delta u_{h-l}^{s}(\delta\hat{\xi})| \le 2\hat{\Delta}\, \overline{\lambda}_{\Phi_s}^{h-L+1}\, \Xi \le \eta_1/L \quad (31)$$

$$|\delta u_{h-l}^{\epsilon}(\delta\hat{n})| \le 2\overline{\Delta}\, \overline{M}(h-L+1)^{-1}\overline{\mathcal{N}} \le \eta_2 \quad (32)$$

$\eta_1 + \eta_2 = \eta$, $\hat{\Delta} \in (0,\infty)$, $l \in \overleftarrow{L}$ and

$$M_h(\chi_\epsilon) \le \gamma M_{h-N}(\chi_\epsilon) \quad (33)$$

for $\gamma = 1 - 2\eta_2/\overline{\Delta} - [(L+1)/L](\eta_1/\overline{\Delta})$, $N \ge L+1$ and $\forall \chi_\epsilon \in \mathbb{R}^n$. Notice, in the definition of γ, that $2\eta_2$ is the L-counterpart of $\eta_1(L+1)/L$.

Choose an j so large that, with $\tau := t_j$, $h = h(\tau)$ satisfy (31), (32) and (33). Because of switching criterion (23),

$$|\delta u_{h(\tau)}(\chi_\epsilon(\tau))| \le \overline{\Delta} - \eta \quad (34)$$

as $h(\tau-1) \le h(\tau)$. So, according to Bellman's principle, if $\chi_\epsilon(\tau+l) = \mathcal{A}_{h(\tau)-l+1}\chi_\epsilon(\tau+l-1) + [\delta\xi'(\tau+l-1)\ 0'\ \delta n'(\tau+l-1)]'$, $l \in \overleftarrow{L}$, one has

$$|\delta u_{h(\tau)-l}(\chi_\epsilon(\tau+l))| \le \overline{\Delta} - [(L-l)/L]\eta_1 \quad (35)$$

provided that, with $h(\tau+l) = h(\tau) - l$, we can write for (30)

$$|\delta u_{n}^{\epsilon}(\tau+l)| \le \eta_2 \quad (36)$$

Put in other words this means that, for $l \in \overleftarrow{L}$, the control horizon h is allowed to decrease of one unit at each time-step, provided that (36) holds true. Let $N \ge L+1$ be the smallest integer at which $M_{h(\tau)-N+1}(\chi_\epsilon(\tau+N-1)) \le 1$ and $M_{h(\tau)-N}(\chi_\epsilon(\tau+N)) > 1$. By (33),$\forall k \in \overleftarrow{h(\tau)}$ and $\forall i \in \overleftarrow{m}$, one has

$$\begin{aligned}
&|\delta u_{h(\tau)}(k|\chi_\epsilon(\tau+N))|_i \le \\
&|\delta u_{h(\tau)}(k|\overline{\chi}_\epsilon)|_i + \eta_1/L + \eta_2 \le \\
&\overline{\Delta} M_{h(\tau)}(\overline{\chi}_\epsilon) + \eta_1/L + \eta_2 \le \\
&\gamma\overline{\Delta} M_{h(\tau)-N}(\overline{\chi}_\epsilon) + \eta_1/L + \eta_2 \le \\
&\gamma\overline{\Delta} + \eta_1/L + \eta_2 = \overline{\Delta} - \eta
\end{aligned} \quad (37)$$

with $\overline{\chi}_\epsilon := \mathcal{A}_{h(\tau)-N+1}\chi_\epsilon(\tau+N-1)$; the first inequality follows from (31) and (32), while the last one follows from $|\delta u_{h(\tau)-N}(k|\overline{\chi}_\epsilon)|_i = |\delta u_{h(\tau)-N+1}(k+1|\chi_\epsilon(\tau+N-1)|_i \le \overline{\Delta}$, $\forall k \in \overleftarrow{h(\tau)-N}$. Therefore $h(t_j+N) \le h(t_j)$.

Moreover, future regulation horizons will never exceed $h(t_j)$. Let $v := t_j + N$ and consider $|\delta u_{h(v)-1}(k|\chi_\epsilon(r+1))|_i = |\delta u_{h(v)-1}(k|\mathcal{A}_{h(v)}\chi_\epsilon(v)) + \delta u_{h(v)-1}^{s}(k|\delta\hat{\xi}(v)) + \delta u_{h(v)-1}^{\epsilon}(k|\delta\hat{n}(v))|_i$.

Recall $|\delta u_{h(v)-1}(k|\mathcal{A}_{h(v)}\chi_\epsilon(v))|_i = |\delta u_{h(v)}(k+1|\chi_\epsilon(v))|_i \le \overline{\Delta} - \eta$. Thus, $h(v+1) \ge h(v)$ implies $|\delta u_{h(v)-1}^{s}(k|\delta\hat{\xi}(v)) + \delta u_{h(v)-1}^{\epsilon}(k|\delta\hat{n}(v))|_i > \eta$, for some k and i. The latter, in turn, by (31) and (32), yields $h(v) \le h(t_j) - L$. Consequently,

$$\begin{aligned}
&|\delta u_{h(t_j)}(k|\chi_\epsilon(v+1))|_i = \\
&= |\delta u_{h(t_j)}(k|\mathcal{A}_{h(v)}\chi_\epsilon(v)) + \\
&+ \delta u_{h(t_j)}^{s}(k|\delta\hat{\xi}(v)) + \delta u_{h(t_j)}^{\epsilon}(k|\delta\hat{n}(v))|_i \\
&\le \gamma\overline{\Delta} M_{h(t_j)-L-1}(\mathcal{A}_{h(v)}\chi_\epsilon(v)) + \eta_1/L + \eta_2 \\
&\le \gamma\overline{\Delta} M_{h(v)-1}(\mathcal{A}_{h(v)}\chi_\epsilon(v)) + \eta_1/L + \eta_2 \\
&\le \gamma(\overline{\Delta} - \eta) + \eta_1/L + \eta_2 < \overline{\Delta} - \eta
\end{aligned}$$

Hence, $h(t_j+N+1) \le h(t_j)$. By arguing again in a similar way, one proves by mathematical induction that

$$h(t_j + k) \le h(t_j),\ \forall k \in \mathbb{Z}_+ \quad (38)$$

provided that (36) holds. To see this, rewrite (30) for $h(\tau+l) = h(\tau) - l$, $l \in \overleftarrow{L}$,

$$\delta u_{n}^{\epsilon}(\tau+l) := \delta u_{h(\tau)-1}^{\epsilon}(k+l-1|\delta\hat{n}(\tau)) + \\ + \ldots + \delta u_{h(\tau)-l}^{\epsilon}(k|\delta\hat{n}(\tau+l-1))$$

As $\mathcal{F}_h(k+1) = \mathcal{F}_{h-1}(k)\mathcal{A}_h$, $\mathcal{A}_h = \mathcal{A} + B\mathcal{F}_h(0)$, one can write $\forall j \ge 1$

$$\mathcal{F}_{h+j}(k+j)l = \mathcal{F}_{h+j-1}(k+j-1)\mathcal{A}_{h+j}l = \\ = \mathcal{F}_{h+j-1}(k+j-1)l + \mathcal{F}_{h+j-1}(k+j-1)\mathcal{B}\mathcal{F}_{h+j}(0))l$$

where the last equality holds because, as can be checked, $\mathcal{A}l = l$. Hence, the j-th component of $\delta u_{n}^{\epsilon}(\tau+l)$ is given by

$$\begin{aligned}
&\delta u_{h(\tau)-l+j}(k+j|\delta\hat{n}(\tau+l-j-1)) = \\
&= \mathcal{F}_{h(\tau)-l+j}(k+j)\delta\hat{n}(\tau+l-j-1) = \\
&= (\mathcal{S}_0 + \mathcal{S}_1 + \ldots \mathcal{S}_j)\delta\hat{n}(\tau+l-j-1)
\end{aligned}$$

where $\mathcal{S}_0 := \mathcal{F}_{h(\tau)-l}(k)$, $\mathcal{S}_i = \mathcal{F}_{h(\tau)-l+i-1}(k+i-1)\mathcal{B}\mathcal{F}_{h(\tau)-l+i}(0)$, $\forall i \in [1,j]$ e $\forall j \in [1,l)$. Consequently (30) becomes

$$\delta u_{n}^{\epsilon}(\tau+l) = \mathcal{S}_0\sum_{j=0}^{l-1}\delta\hat{n}(\tau+j) + \\ + \mathcal{S}_1\sum_{j=0}^{l-2}\delta\hat{n}(\tau+j) + \ldots + \mathcal{S}_{l-1}\delta\hat{n}(\tau) \quad (39)$$

Hence, (19) yields

$$|\delta u_{n}^{\epsilon}(\tau+l)| \le 2\,(|\mathcal{S}_0| + |\mathcal{S}_1| + \ldots + |\mathcal{S}_{l-1}|)\overline{\mathcal{N}}$$

Notice that $\forall i \in [1,j]$, $\exists\, \overline{\mathcal{S}} > 0$ such that

$$\begin{aligned}
|\mathcal{S}_0| &\le \overline{\mathcal{S}}\,|O_{h(\tau)-L}| \\
|\mathcal{S}_i| &\le \overline{\mathcal{S}}\,|O_{h(\tau)-L}|\,|O_{h(\tau)-L+1}|
\end{aligned} \quad (40)$$

where O_h stands for a quantity at least of the same order of h^{-1} as $h \to \infty$. Finally, because $l \in \overleftarrow{L}$, and $L \to \mu h$ as $h \to \infty$ (see Mosca (2005), proof of Lemma 3, for technical details), one finds

$$|\delta u_n^\epsilon(\tau+l)| \leq$$
$$\leq 2\,\overline{\mathcal{S}}\,\left(\left|O_{h(\tau)-L}\right| + (L-1)\ \left|O_{h(\tau)-L}\right|\left|O_{h(\tau)-L+1}\right|\right)\overline{\mathcal{N}} =$$
$$= 2\overline{\mathcal{S}}\left(\left|O_{(1-\mu)h(\tau)}\right| + \ \frac{\mu}{1-\mu}\ \left|O_{(1-\mu)h(\tau)}\right|\right)\overline{\mathcal{N}} =$$
$$= 2\,\overline{\mathcal{S}}\,\frac{1}{1-\mu}\ \left|O_{(1-\mu)h(\tau)}\right|\,\overline{\mathcal{N}} \qquad (41)$$

Hence (36) holds and this completes the proof.

4. JOINT POSITIONAL AND INCREMENTAL INPUT SATURATION

In this section attention will be devoted to the possible extension of the switching-horizon predictive control approach to handle joint positional and incremental input saturation. For the sake of simplicity, the discussion on such an issue will be restricted to the case constant disturbances, the general case still being under development. Consider system (6) with δv, $\delta w \equiv 0$. Let

$$\mathcal{M}_h(\chi) := \max\{\tilde{\alpha}\frac{|[\delta u_h(k|\chi)]_i|}{\overline{\Delta}}\ ,\ \alpha\frac{|[u_h(k|\chi)]_i|}{\overline{U}};\quad k+1 \in \overleftarrow{h}\,;\ i \in \overleftarrow{m}\} \qquad (42)$$

where ($\tilde{\alpha} = 1, \alpha = 0$) corresponds to only incremental saturations; ($\tilde{\alpha} = 0, \alpha = 1$) pertains to only positional saturations; ($\tilde{\alpha} = 1, \alpha = 1$) to joint incremental and positional saturations. E.g., if $\tilde{\alpha} = 1$ and $\alpha = 0$, the whole sequence $\delta u_h(\chi)$ does not violate (5) if and only if $\mathcal{M}_h(\chi) < 1$. The fundamental question for extending Mosca (2005) to the present case is whether, given an arbitrary χ, there exist h such that $\mathcal{M}_h(\chi) < 1$ for any of the possible pair $(\tilde{\alpha}, \alpha)$. If this is the case, one can always find a (virtual) input increment sequence (11) of large enough length h for which the saturation constraints (4) and (5) are jointly satisfied.

Remark 1. Consider the orthogonal decomposition

$$\mathbb{R}^N = \mathcal{R}((\mathcal{A}^N)') \oplus \mathcal{N}(\mathcal{A}^N) \qquad (43)$$

where $\mathcal{N}(\mathcal{A}^N) = \mathcal{N}(\mathcal{A}^h)$, $\forall h \geq N = dim(\mathcal{A})$, and $\mathcal{R}(\cdot)$ and $\mathcal{N}(\cdot)$ denote range-space and, respectively, null-space. Because, if $\chi^\perp \in \mathcal{N}(\mathcal{A}^N)$, $\delta u_h(k|\chi^\perp) = 0$, $\forall h \geq N$, we can restrict the study to states in $\mathcal{R}((\mathcal{A}^N)')$. This amounts to assuming w.l.o.g. $\mathcal{A}$ non singular.

Under such an assumption, the following properties hold.

Lemma *Consider the incremental model (6). Let it be reachable and ANCBI, and $\mathcal{F}_h$ as in (11). Then, the following properties hold,* $\forall\, k+1 \in \overleftarrow{h}$,

$$\mathcal{F}_{h+1}(k) = \mathcal{F}_h(k)\left[I + O(h^{-1})\right] \qquad (44)$$

If $\mathcal{A}_h := \mathcal{A} + \mathcal{B}F_h$,

$$\mathcal{F}_h(k+1) = \mathcal{F}_h(k)\mathcal{A}_h\left[I + O(h^{-1})\right] \qquad (45)$$

Finally, if $\ l = [0'\ \epsilon']' \in \mathbb{R}^N$,

$$\mathcal{F}_{h+1}(k+1)l = \mathcal{F}_h(k)\left[I + O(h^{-1})\right]l \qquad (46)$$

Proof. From (11) one finds

$$\begin{aligned}\mathcal{F}_{h+1}(k) &= -\Psi_u^{-1}\mathcal{B}'(\mathcal{A}^{h-k})'\mathcal{G}_{h+1}^{-1}\mathcal{A}^{h+1}\\ &= -\Psi_u^{-1}\mathcal{B}'(\mathcal{A}^{h-1-k})'(\mathcal{A}'\mathcal{G}_{h+1}^{-1}\mathcal{A})\mathcal{A}^h\end{aligned} \qquad (47)$$

Now

$$\begin{aligned}&(\mathcal{A}'\mathcal{G}_{h+1}^{-1}\mathcal{A})^{-1}\\ &= \mathcal{A}^{-1}\mathcal{G}_{h+1}\mathcal{A}^{-T} = \mathcal{G}_h + \mathcal{A}^{-1}\mathcal{B}\mathcal{B}'\mathcal{A}^{-T}\\ &= \qquad \mathcal{G}_h(I + \mathcal{G}_h^{-1}\mathcal{A}^{-1}\mathcal{B}\mathcal{B}'\mathcal{A}^{-T})\end{aligned} \qquad (48)$$

Therefore

$$\begin{aligned}\mathcal{A}'\mathcal{G}_{h+1}^{-1}\mathcal{A} &= (I + \mathcal{G}_h^{-1}\mathcal{A}^{-1}\mathcal{B}\mathcal{B}'\mathcal{A}^{-T})^{-1}\mathcal{G}_h^{-1}\\ &= \mathcal{G}_h^{-1}(I + O(h^{-1}))\end{aligned} \qquad (49)$$

where, as shown in Mosca (2005), $\mathcal{G}_h^{-1} = O(h^{-1})$. Hence, (44) follows. The first equality in the next equation was shown (Mosca (2005)) to hold for the feedback-gains in (11), provided that $\mathcal{A}_h := \mathcal{A} + \mathcal{B}F_h$,

$$\begin{aligned}\mathcal{F}_h(k+1) &= \mathcal{F}_{h-1}(k)\mathcal{A}_h\\ &= \mathcal{F}_h(k)\mathcal{A}_h\left[I + O(h^{-1})\right]\end{aligned} \qquad (50)$$

where the last equality follows from (44). By (45), the first equality in the following equation holds

$$\begin{aligned}\mathcal{F}_{h+1}(k)l &= \mathcal{F}_h(k)\mathcal{A}_h\left[I + O(h^{-1})\right]l\\ &= \mathcal{F}_h(k)\left[\mathcal{A} + \mathcal{B}F_h\right]\left[I + O(h^{-1})\right]l\\ &= \mathcal{F}_h(k)\left[I + O(h^{-1})\right]l\end{aligned} \qquad (51)$$

where the last equality holds because, $\mathcal{A}l = l$, and $F_h = O(h^{-1})$.

In order to compute $\delta u_h(k|\chi)$ for large h, let $\chi = \chi(0) := l + v$ where $l := [0' \quad \epsilon'(-1)]'$, $v := [\delta x'(0) \quad 0']'$. Then, by linearity of $\delta u_h(k|\cdot)$

$$\delta u_h(k|\chi) = \delta u_h(k|l) + \delta u_h(k|v) \qquad (52)$$

Further,

$$\begin{aligned}\delta u_h(k|l) &= \mathcal{F}_h(k)l \ = \ \mathcal{F}_h(k-1)\mathcal{A}_h\left[I + O(h^{-1})\right]l\\ &= \mathcal{F}_h(k-1)\left[I + O(h^{-1})\right]l\\ &= F_h\left[I + O(h^{-1})\right]l\end{aligned} \qquad (53)$$

where the second equality follows from (45), and the third from the fact that $\mathcal{A}_h l = l + O(h^{-1})$. Consequently,

$$\begin{aligned}u_h(k|\chi) &= u(-1) + \sum_{i=0}^{k}\delta u_h(i|\chi) =\\ &= u(-1) + (k+1)F_h l + u_h(k|v) + O(h^{-1})\end{aligned} \qquad (54)$$

where $O(h^{-1})$, the rightmost term in (54), arises by taking into account that $F_h = O(h^{-1})$, and consequently $\sum_{i=0}^{k} F_h O(h^{-1})l = O(h^{-1})$, $\ k+1 \in \overleftarrow{h}$.

Now, one must have

$$u_h(h-1|\chi) = u^\infty \qquad (55)$$

if u^∞ denotes the input vector to (1) which in steady-state yields the desired set-point r at the output of (1)

$$r = H(I-\Phi)^{-1}\left(Gu^\infty + \begin{bmatrix}\xi\\ c\end{bmatrix}\right) + \zeta \qquad (56)$$

Using (55) in (54), one finds

$$F_h l = \left[u^\infty - u(-1)\right]h^{-1} - u_h(h-1|v)h^{-1} + O(h^{-2}) \qquad (57)$$

Therefore, $k+1 \in \overleftarrow{h}$,

$$u_h(k|\chi) = u(-1) + \frac{k+1}{h}\left[u^\infty - u(-1)\right] + \\ + u_h(k|v) - \frac{k+1}{h} u_h(h-1|v) + O(h^{-1}) \quad (58)$$

We now turn to show that $\delta u_h(k|v) = O(h^{-1})$ and, similarly, $u_h(k|v) = O(h^{-1})$. In fact,

$$\begin{aligned}\delta u_h(k|v) &= \mathcal{F}_h(k)v = \\ &= \mathcal{F}_h(k-1)\mathcal{A}_h\left[I + O(h^{-1})\right]v = \\ &= F_h\mathcal{A}_h^k\left[I + O(h^{-1})\right]v \end{aligned} \quad (59)$$

where the second equality follows from (45). That $\delta u_h(k|v) = O(h^{-1})$ follows from the fact that $F_h = O(h^{-1})$ and that $\mathcal{A}_h$ is a stability matrix.
Using these two properties, it is easy to see that also $u_h(k|v) = O(h^{-1})$.
Summing up, $k+1 \in \overleftarrow{h}$,

$$\delta u_h(k|\chi) = F_h\mathcal{A}_h^k\chi + O(h^{-2}) \quad (60)$$

$$u_h(k|\chi) = u(-1) + \frac{k+1}{h}\left[u^\infty - u(-1)\right] + O(h^{-1}) \quad (61)$$

Eq. (60) and (61) show that for any initial state $\chi \in \mathbb{R}^N$ it is always possible to find a large enough control horizon h so as to make the virtual input increments $\delta u_h(\cdot|\chi)$ and virtual inputs $u_h(\cdot|\chi)$ compatible with constrains (4) and (5) provided that $u(-1), u^\infty \in \mathcal{U}$. Notice that the latter property amounts to assuming that ξ, ζ and r are jointly within the input control range. Properties (60) and (61) are summed up in the following

Feasibility Property. *Consider the reachable ANCBI system (1) subject to joint input positional and incremental saturation constrains (4) and (5). Then, for every $\chi \in R^n$ and in the presence of constant disturbances ξ, c, ζ and set-point r for which $u^\infty \in \mathcal{U}$, control horizons h can always be found so that $u_h(\chi) \in \mathcal{U}$ and $\delta u_h(\chi) \in \mathcal{D}$ provided that $u(-1) \in \mathcal{U}$.*

Thanks to the *Feasibility Property* and by exploiting (42), one can set up a switching logic for chosing h at each time t a natural extension of the one in (23), so as to obtain a closed-loop switched system enjoying offset-free asymptotic tracking under *joint* incremental and positional input saturations. A conjecture in this connection is that, similarly to Angeli et al. (1996), the use of a *reference governor* for moderating the time-variations of u^∞ can enable one to extend the convergence analysis of this paper to the general case of time-varying disturbances.

Remark - For aspects concerning memory and computational savings, the reader is referred to Mosca et al. (2008).

5. CONCLUSIONS

The paper provides a computationally affordable solution to the set-point tracking problem of discrete-time LTI systems subject to persistent disturbances and control saturations. The proposed solution enjoys the following features: It consists of a supervisory switching control logic whereby a feedback-gain, selected at any time from a family of pre-designed candidate feedback-gains, is switched-on in feedback to the plant according to the information on the current plant state. The controller selection is made in accordance with a predictive control philosophy, and each candidate feedback-gain is tuned on to a different horizon in a receding-horizon control sense. It is proved that the adopted switching logic ensures global asymptotic stability in the case of constant disturbances and joint positional and incremental input saturation, as well as bounded-noise bounded-state l_∞-stability in the presence of time-varying disturbances, whenever only incremental input saturations are present.

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