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Unfalsified Virtual Reference Adaptive Switching Control of Plants with Persistent Disturbances

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Abstract: This paper addresses virtual reference adaptive switching control whereby a data-driven supervisor aims at stabilizing an unknown time-invariant dynamic system by switching at any time in feedback with system one element from a finite family of candidate controllers. Under the only assumption of problem feasibility, *viz.* the controller family contains a stabilizing controller, the resulting switched system is shown to be stable against arbitrary exogenous persistent bounded disturbances.

1. INTRODUCTION

In many practical control applications, despite only a partial knowledge of the plant, it is asked to design a feedback controller which can ensure stability to the controlled system. This paper considers the unfalsified *virtual reference adaptive switching control* (VRASC) approach, whereby a data-driven "high-level" unit, called the supervisor, aims at stabilizing an unknown time-invariant dynamic system by switching at any time in feedback with system to be controlled one element from a set of candidate controllers. A VRASC system is depicted in Fig.1. There, Π is the unknown plant to be controlled, and $K_{\sigma(\cdot)}$ is the time-varying controller whose selection is carried out by the supervisor $\mathcal{S}$ based on the plant input δu and output y.

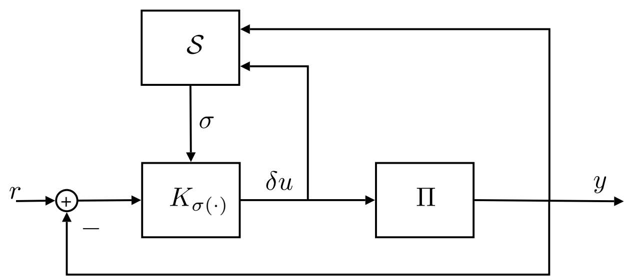

Fig. 1. Basic VRASC scheme.

The supervisor performs in real-time the scheduling task (when to switch) and the routing task (which controller select) (Morse (1995)), by monitoring a purely data-driven cost function. When the current performance is not satisfactory, *viz.* the switched-on controller is falsified by measured data (Safonov and Tsao (1997); Mosca and Agnoloni (2003)), another candidate controller is selected so as to replace the previous one. Thanks to the use of the virtual reference tool (Safonov and Tsao (1997)), such a selection is carried out without directly checking the performance of all candidate controllers via their effective use in the feedback loop. Using past plant input-output records, the supervisor infers the performance of the potential loop made up by a candidate controller and selects the one corresponding to the minimal value of the inferred performance.

In the noiseless case, stability properties of the adaptive switched system can be ensured. In fact it can be proved that the switching mechanism stops in a finite time (Safonov et al. (2007); Manuelli et al. (2007))). However, apart from few exceptions, little attention has been so far devoted on how to deal with persistent disturbances: the issue considered in this paper. For other contributions to the topic, see *e.g.* also Morse (1997), Jun and Safonov (1999), Zhivoglyadov et al. (2000), Hespanha et al. (2003), Freidovich and Khalil (2005) and Fekri et al. (2006).

Sect.2 recalls some preliminary concepts and known stability results pertinent to the noiseless case. Sect.3 analyses, from the stability viewpoint, the problems which may arise in the presence of disturbances. Finally, Sect.4 shows how suitable choises of the cost functionals ensure that the switched system be stable in the face of arbitrary bounded disturbances.

2. PRELIMINARIES AND PROBLEM FORMULATION

The study is intentionally focused on the simplest non-trivial case of a strictly causal SISO plant consisting of a discrete-time time-invariant dynamic system Π with input-increment $\delta u(t) := u(t) - u(t-1)$ and output $y(t)$, $t \in \mathbf{Z}_+ = \{0, 1, \ldots\}$. The only assumption on Π is that it belongs to a plant uncertainty set $\mathcal{P}$, which need not be known. Along with the uncertain plant Π, a finite family $\mathcal{K}$ of one-degree-of-freedom causal and causally invertible (CCI) controllers K_i is available, $\mathcal{K} = \{K_i, i \in \overleftarrow{N}\}$, $\overleftarrow{N} := \{1, 2, \ldots, N\}$. At any time t, the plant Π is fed by the control-increment $\delta u(t)$ coinciding with the output of one, say $K_{\sigma(t)}$, $\sigma(t) \in \overleftarrow{N}$, among the N candidate controllers, *viz.*, $K_{\sigma(t)}$ is switched-on in feedback to the plant. Consequently,

$$\begin{cases} y(t) = \Pi(\delta u)(t) \\ \delta u(t) = -K_{\sigma(t)}(y-r)(t) \end{cases} \tag{1}$$

where r denotes the reference to be tracked by the plant output y. Here, $K_{\sigma(t)}(y-r)(t)$ has to be introduced as a shorthand notation for $K_i(y-r)|_{i=\sigma(t)}(t)$. The latter means that the N candidate controllers are fed at all times by the tracking error $\epsilon := r - y$, and the output of the $\sigma(t)$-th controller is used as the input-increment to the plant at time t. In (1) Π and all K_i's are allowed to have arbitrary states at time zero. The switching mechanism is implemented by possibly bumpless control transfer techniques (Goodwin et al. (2001)) or common state multi-controller schemes (Morse (1995)) so as to reduce as much as possible the transients caused by switching from one controller to a different one. Plant input increments are customarily adopted for control design so as to ensure constant disturbance rejection and zero-offset. As anticipated, the high-level device responsible for orchestrating $\sigma(t)$ is the switching supervisor, and (1), combined with such a device, is referred to as an ASC system denoted $(\Pi/K_{\sigma(\cdot)})$.

2.1 Unfalsified control

The problem of controlling the unknown plant Π is dealt within a framework commonly referred to as *unfalsified control* (Safonov and Tsao (1997)). To this end, we recall some useful definitions available in literature (Safonov et al. (2007)).

Definition 1. A signal v is said to belong to l_{2e} if it is square summable over any bounded interval of time $\{1, 2, \ldots, t\}$, $t \in \mathbf{Z}_+$.

Definition 2. Assuming $r \in l_{2e}$ as (temporarily) the only exogenous variable, $(\Pi/K_{\sigma(\cdot)})$ is said to be *l_{2e}-gain stable* relatively to r if there exist finite nonnegative constants α and, β such that:

$$\|D^t\| \le \beta \cdot \|r^t\| + \alpha \tag{2}$$

$\forall t \in \mathbf{Z}_+$, $\forall r^t$, where $v^t := \{v(k)\}_{k=0}^t$ is the time truncation of $v(\cdot)$ over the interval $\{0, 1, \ldots, t\}$, $D(k) := [\delta u(k) \quad y(k)]'$ the plant I/O vector at time k, the prime denotes transpose, $\|D^t\|^2 := \sum_{k=0}^t |D(k)|^2$, $|-|$ and $\|-\|$ denote Euclidean and, respectively, l_2-norm.

It should be emphasized that stability of the system (1) is unfalsified by an input-output pair (r, D), during a specific infinite-length experiment, if (2) holds. On the opposite, with "stability" we mean that (2) holds true for every possible input, *viz.* stability of (1) is unfalsified by any possible pair (r, D).

In a VRASC system, for each $i \in \overleftarrow{N}$, the variable

$$v_i(t) = y(t) + K_i^{-1}(\delta u)(t) \tag{3}$$

$t \in \mathbf{Z}_+$, is uniquely computed in real-time provided that K_i is causal and causally stably invertible (CCSI). Note that if K_i^{-1} and K_i have the same initial condition, then

$$-K_i(y - v_i)(t) = -K_i(y - y - K_i^{-1}(\delta u))(t) = \delta u(t)$$

In words, v_i equals the fictitious or *virtual reference* that, if injected into the feedback system (Π/K_i), would reproduce $D(t)$, $t \in \mathbf{Z}_+$, the I/O pairs of the uncertain plant Π in (1). In other terms, if $(\Pi/K_{\sigma(\cdot)})$ is intended as the (time-varying) transformation (1) mapping r into D, we find:

$$D = (\Pi/K_{\sigma(\cdot)})r = (\Pi/K_i)v_i$$

where (Π/K_i) is the (time-invariant) transformation consisting of the plant Π fed-back by the i-th candidate controller K_i.

The introduction of the v_i's makes it possible to causally compute from D (virtual) nonnegative causal *loss functionals* $V_i(\cdot)$, $i \in \overleftarrow{N}$. Given D^{t-1}, the supervisor compares the N performance indices $V(t) := \{V_i(t)\}_{i=1}^N$ and selects, at every $t \in \mathbf{Z}_+$, the controller index $\sigma(t)$ via the following *hysteresis switching logic*:

$$\sigma(t) = l(\sigma(t-1), V(t)) \qquad \sigma(-1) = i_{-1} \tag{4}$$

$$l(i, V) = \begin{cases} i & \text{if} \quad V_i < V_{\gamma(V)} + h \\ \gamma(V) & \text{otherwise} \end{cases} \tag{5}$$

where $\gamma(V)$ denotes the least integer $i \in \overleftarrow{N}$ for which $V_i \le V_j$, $\forall j \in \overleftarrow{N}$, and h, a (typically small) positive real, is the so-called *hysteresis constant*.

2.2 Results in the noiseless case

In VRASC system analysis, a fundamental role is played by the next Hysteresis Switching Lemma which establishes the limiting behavior of $(\Pi/K_{\sigma(\cdot)})$.

Let S denote the class of all possible switching functions $s : \mathbf{Z}_+ \to \overleftarrow{N}$ giving rise to the switched system $(\Pi/K_{s(\cdot)})$. Consider the assumptions:

- **A1.** For each $s \in$ S and $i \in \overleftarrow{N}$, the performance index $V_i(t)$ admits a limit (even infinite) as $t \to \infty$;
- **A2.** There is at least one integer $m \in \overleftarrow{N}$ such that $V_m(\cdot)$ is bounded on $\mathbf{Z}_+$ for each $s \in$ S.

Hysteresis Switching Logic (HSL) Lemma (Morse et al. (1992)): For any initial condition and reference r, let D denote the I/O plant data to the supervisor, and σ the switching function resulting from (1), (4) and (5). Then, if A1 and A2 hold, there is a finite time t_f beyond which σ is constant as no more switching occurs. Moreover, $V_{\sigma(t_f)}(\cdot)$ is bounded.

In order to simplify the notation, from now on, the final controller index $\sigma(t_f)$ will be denoted by f, *viz.*, $K_{\sigma(t_f)} = K_f$. A pre-requisite for a VRASC system to yield stable and well-behaved final feedback loops (Π/K_f), $\forall \Pi \in \mathcal{P}$ is that the set $\mathcal{K}$ of candidate controllers be adequately chosen relatively to $\mathcal{P}$. In this connection, the minimal requirement on $\mathcal{K}$ is the so called:

Problem Feasibility: For every $\Pi \in \mathcal{P}$, there are indices i, $i \in \overleftarrow{N}$, such that (Π/K_i) is l_{2_e}-gain stable.

Key properties which enable cost functions to reliably detect any instability exhibited by the adaptive system are as follows (Safonov et al. (2007))

Definition 3. The pair $(V, \mathcal{K})$ is said to be *cost-detectable* if for every $K_{\sigma(\cdot)} \in \mathcal{K}$ with finitely many switching times, the following statements are equivalent: 1) V_f is bounded as $t \to \infty$, 2) l_{2e}-gain stability of the system $(\Pi/K_{\sigma(\cdot)})$ is unfalsified by (r, D).

Definition 4. Given the pair $(V, \mathcal{K})$, V is said to be l_{2e}-*gain-related* if for each $D \in l_{2e}$ and K_i, $i \in \overleftarrow{N}$: 1) $V_i(t)$ is monotone in t, 2) the virtual reference signal v_i exists, and 3) for every K_i, $i \in \overleftarrow{N}$, and $D \in l_{2e}$, $V_i(t)$ is bounded as $t \to \infty$ if and only if l_{2e}-gain stability of the system (Π/K_i) is unfalsified by (v_i, D).

If each K_i, $i \in \overleftarrow{N}$, is CCSI, the virtual reference v_i in (3) is well-defined. In such a case, cost detectability is equivalent to l_{2e}-gain-relatedness (see Safonov et al. (2007)). Therefore, one can focus on the latter in order to study the stability properties of the VRASC system. With this respect the following lemma holds that summarizes some of the results of Safonov et al. (2007).

Lemma 1. Let all the candidate controllers be CCSI, and the hysteresis switching logic (4)-(5) used. Then, provided that

B1. V is l_{2e}-gain-related and
B2. problem feasibility holds,

the HSL holds and the resulting VRASC system $(\Pi/K_{\sigma(\cdot)})$, $\Pi \in \mathcal{P}$, is l_{2e}-gain stable relatively to r.

Remark 1. For every index s corresponding to an internally stabilizing controller, l_{2e}-gain stability of the system (Π/K_s) is always unfalsified by the I/O pair (v_s, D) regardless of the switching sequence $\sigma(t)$, $t \in \mathbf{Z}_+$. As a consequence, under problem feasibility, there always are indices i, $i \in \overrightarrow{N}$, for which l_{2e}-gain stability of the system (Π/K_i) is unfalsified by (v_i, D). This together with the third property in Def. 4 ensures that there always exist indices i for which the cost $V_i(t)$ remains bounded as t increases to infinity, thus allowing for the application of the HSL.

Existing related literature proves the existence of l_{2e}-gain-related performance indices V_i. In the light of Lemma 1, under problem feasibility, such performance indices satisfy the HSL and yield stability (2) relatively to r to the adaptive system $(\Pi/K_{\sigma(\cdot)})$ resulting from (1), (4) and (5). For instance, a simple form for l_{2e}-gain-related V_i's is as follows

$$V_i(t) = \max_{\tau \le t} \frac{\|\epsilon_i^\tau\|^2 + \rho\|\delta u^\tau\|^2}{m^2 + \|v_i^\tau\|^2} \tag{6}$$

where $t \in \mathbf{Z}_+$, $\rho > 0$, and $\epsilon_i(k) := v_i(k) - y(k)$ denotes the tracking error in the feedback loop (Π/K_i) driven by the virtual reference v_i. The positive scalar $m^2 > 0$ appears so as to prevent the denominator from assuming values too close to zero. The form of (6) is a natural one in that it is an estimate of the performance of (Π/K_i) expressed in terms of the $l_2 - l_2$ induced gain of the map embodied by (Π/K_i) from v_i to D. Further, its evaluation requires, once data D^t are collected, only computation of v_i^t. The adoption of performance indices V_i computed via the maximum operator is a way for ensuring A1.

The following stability result descends directly from Lemma 1 and from the fact that the V_i's defined in (6) are l_{2e}-gain-related.

Theorem 1. (Safonov et al. (2007)) Let all the candidate controllers be CCSI, and the hysteresis switching logic (4)-(5) used along with the V_i's as in (6). Under the only assumption of problem feasibility, for any initial condition and reference r, the HSL holds and the resulting VRASC system $(\Pi/K_{\sigma(\cdot)})$, $\Pi \in \mathcal{P}$, is l_{2e}-gain-stable relatively to r.

Remark 2. Theorem 1 indicates that the simple loss functional (6) enables one to adaptively select a final controller K_f yielding a feedback system (Π/K_f) stable relatively to r, provided that only problem feasibility hold. The extension of VRASC systems to cover the case where not all the CCI controllers K_i are stably invertible, can be achieved in many ways (*e.g.* Manuelli et al. (2007)). For instance, in linear case, defining $C_i(d) = S_i(d)/R_i(d)$ as the transfer function of K_i, one can compute $\hat{v}_i(t) := (S_i(d)/S_i(1))v_i(t)$, where $S_i(1)$ acts so as to obtain a steady-state offset-free response of $\hat{v}_i$ to a step input. In this scenario, a simple way to cover the general CCI controllers case is to use

$$V_i(t) = \max_{\tau \le t} \frac{\|\hat{\epsilon}_i^\tau\|^2 + \rho\|\delta u^\tau\|^2}{m^2 + \|\hat{v}_i^\tau\|^2} \tag{7}$$

with $\hat{\epsilon}_i(t) := \hat{v}_i(t) - y(t)$. No controller inversion is required here and only input output data collected from the acting loop are requested. It is no difficult to prove that loss functionals (7), along with hysteresis switching logic (4)-(5), yield the same results as in Theorem 1.

Hereafter, d will denote the unit backward shift operator. The streamlined notation $M(d)v(t) = N(d)u(t)$, $M(d) = \sum_{k=0}^{\partial M} m_k d^k$, $N(d) = \sum_{k=0}^{\partial N} n_k d^k$, will be adopted as a representation of the difference equation $m_0 v(t) + m_1 v(t-1) + \cdots + m_{\partial M} v(t - \partial M) = n_0 u(t) + n_1 u(t-1) + \cdots + n_{\partial N} u(t - \partial N)$.

3. EFFECTS OF PERSISTENT DISTURBANCES

For the sake of convenience, the subsequent developments are carried out by assuming that all candidate controllers in $\mathcal{K}$ are CCSI. In Sect.4, it is shown how simple variants can be adopted so as to cover the general CCI controllers case.

So far, only stability of $(\Pi/K_{\sigma(\cdot)})$ relatively to the output reference r has been considered. However, it is essential to check whether a finite-gain property similar to (2) holds true for every bounded exogenous sequence injected at any possible position in the feedback loop $(\Pi/K_{\sigma(\cdot)})$. Because of the presence of the supervisor, in a VRASC system the situation is more complex than in a classical time-invariant feedback loop. The disturbances have in fact a different effect relatively to the way they affect the data D to the supervisor. Specifically, the disturbances might downgrade the information available to the supervisor, thus deteriorating the performance of the loop to an unacceptable level. To see this, consider the case where the disturbances enter the system as follows

$$\begin{cases} y(t) = \Pi(\delta\nu + n_u)(t) + n_y(t) \\ \delta u(t) = -K_{\sigma(t)}(y - r)(t) \end{cases} \tag{8}$$

As shown in Fig.2, the supervisor has now a noisy information y on the output of Π. Similarly, the input to Π equals $\delta u + n_u$, while the supervisor has access only to δu.

This case looks realistic for applications wherein the VRASC controller is regarded as a single block $(K_{\sigma(\cdot)}, \mathcal{S})$ having input y and output δu. In this way, the adaptive system consists of two block, the plant Π and the VRASC controller, while the inputs n_u, n_y appear at the interface between such blocks, so as to describe the standard formulation of the problem of input-output stability. In this scenario, for the supervisor, n_u and n_y play the role of process disturbances acting "inside" Π.

The performance index (6) is not proper to reflect correctly the I/O configuration in the noisy case and, consequently, it may cause a lack of information to the supervisor. In particular, the switching mechanism need not stop. In order to focus on how this case can arise, let (D,σ)

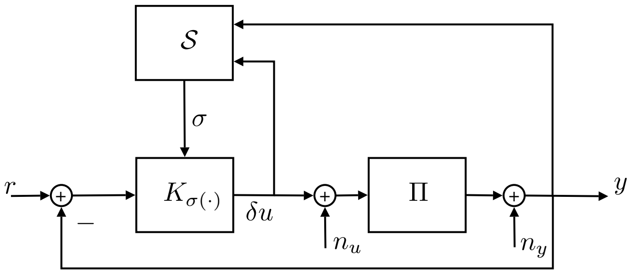

Fig. 2. Noisy scheme.

denote the unique solution of (8), subject to the hysteresis switching logic (4)-(5), for any initial state and bounded sequences n_u, n_y, where the performance indices $V_i(t)$, $i \in \overleftarrow{N}$, are given by (6). As a matter of fact, one cannot conclude that such functionals V_i's remain bounded even for the indices related to the stabilizing controllers. For example, this happens in case $r \in l_2$. Indeed, suppose that both the plant and controllers are linear. Further, suppose that at the generic time instant $\bar{t}$ a certain controller K_i is switched on in the loop. Let $P(d) = B(d)/A(d)$ and $C_i(d) = S_i(d)/R_i(d)$ be the transfer function of Π, respectively, K_i. As shown in Fig.3, according to (3), the data given by the i-th virtual loop are obtained as

$$\begin{aligned}\epsilon_i(t) &= \frac{A(d)R_i(d)}{\chi_{*/i}(d)}\,(v_i(t) - n_y(t)) - \frac{B(d)R_i(d)}{\chi_{*/i}(d)} n_u(t)\\ \delta u(t) &= C_i(d)\,\epsilon_i(t)\end{aligned}\tag{9}$$

with $\chi_{*/i}(d) := A(d)R_i(d) + B(d)S_i(d)$ the characteristic polynomial of (Π/K_i). Consequently, omitting the argument, one can write

$$\begin{aligned}\left\| \begin{bmatrix} \rho^{1/2}\delta u \\ \epsilon_i \end{bmatrix} \right\|^2 &\geq \min_{\omega\in[0,2\pi]} \left(\frac{|R_i|^2 + \rho |S_i|^2}{|\chi_{*/i}|} \right) \cdot l^2 \cdot \|p_i^t\|^2 = \\ &= N^2 \cdot \|p_i^t\|^2\end{aligned}\tag{10}$$

with $p_i(t) := A(d)(v_i(t) - n_y(t)) - B(d)n_u(t)$, $\forall i \in \overleftarrow{N}$; l^2, $0 < l \leq 1$, accounts for the truncation effects on the l_2 norm. Further, $N^2 > 0$ because R_i and S_i are coprime polynomials. Roughly, since ϵ_i and δu are obtained by filtering the disturbances, provided that such disturbances are persintently exciting sequences of sufficiently high order, the numerator of $V_i(t)$ grows at least linearly with $t - \bar{t}$. On the opposite, since for the switched-on controller the virtual reference converges exponentially to the true reference $r \in l_2$, the denominator of $V_i(t)$ turns out to be bounded as long as K_i is kept in the loop. Consequently, it cannot be argued that assumption A2 in the HSL lemma holds, and, hence, that the switching mechanism stops.

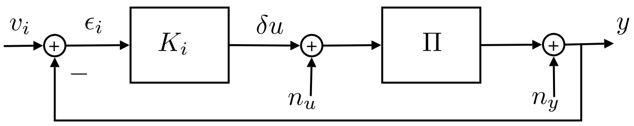

Fig. 3. The i-th virtual closed loop.

3.1 *A novel notion of stability*

This state of affairs can be understood by noting that the convergence of the virtual reference to the true one, for the switched-on controller, yields no account of the disturbances. In fact the supervisor cannot distinguish between disturbances and some process dynamics, as n_u and n_y do not affect the virtual reference of the switched-on controller. Consequently, even if the supervisor preserves the capability of detecting any trend to unboundedness due to instability, its capability of holding in the loop for ever a (final) stabilizing controller is lost.

For this reason, the minimal requirement on the functionals V_i's is their boundedness for the indices $i \equiv s$ related to the stabilizing controllers K_s, $s \in \overleftarrow{N}$,

$$\lim_{t\to\infty} V_s(t) < \infty \quad \text{for any possible } \sigma(\cdot) \tag{11}$$

It is important to remark that the performance indices (6) do not satisfy (11) because they were chosen to be l_{2e}-gain-related. However, the notion of l_{2e}-gain stability introduced in Def. 2 is inconsistent with the noisy system (8): in the presence of disturbances, even for indices s corresponding to internally stabilizing controllers, the l_{2e}-gain stability of the system (Π/K_s) can be falsified by the I/O pair (v_s, D). Consequently, for system (8) the considerations of Remark 1 do not hold.

Therefore, a different notion of stability has to be considered that takes into account the presence of bounded but possibly persistent disturbances affecting the plant.

Definition 5. System (8) is said to be *weakly l_{2e}-gain-stable* relatively to r if, for every reference $r \in l_{2_e}$, every initial condition, and every bounded disturbances n_u and n_y, there exist finite nonnegative constants α, β, and γ such that

$$\|D^t\| \leq \beta \cdot \|r^t\| + \gamma \cdot \sqrt{t+1} + \alpha \tag{12}$$

for any $t \in \mathbf{Z}_+$.

The parameter γ stands from the contribution of n_u and n_y to the data D, while α allows for consideration of systems with non-zero initial state.

4. CHOICE OF PERFORMANCE INDICES

On the basis of the novel stability notion introduced in Def. 5, one can consider novel key properties that the cost functions have to satisfy in order to reliably detect losses of stability of the adaptive system $(\Pi/K_{\sigma(\cdot)})$.

Definition 6. Given the pair $(V, \mathcal{K})$, V is said to be *weakly l_{2e}-gain-related* if for each $D \in l_{2e}$ and K_i, $i \in \overleftarrow{N}$: 1) $V_i(t)$ is monotone in t, 2) the virtual reference v_i exists, and 3) for every K_i, $i \in \overleftarrow{N}$, and $D \in l_{2e}$, $V_i(t)$ is bounded as $t \to \infty$ if and only if weak l_{2e}-gain stability of the system (Π/K_i) is unfalsified by (v_i, D).

Def. 6 and the theorem that follows indicate some guidelines on how to choose a performance index so as to achieve weak l_{2e}-gain stability.

4.1 Performance indices

Theorem 2. Let all the LTI candidate controllers be CCSI, and the hysteresis switching logic (4)-(5) used. Then, provided that

C1. V is weakly l_{2e}-gain-related and
C2. problem feasibility holds,

the HSL lemma holds and the resulting VRASC system $(\Pi/K_{\sigma(\cdot)})$, $\Pi \in \mathcal{P}$, is weakly l_{2e}-gain stable relatively to r.

Proof. Suppose that bounded disturbances n_u and n_y enter the system as in (8). Thus, for any index s corresponding to an internally stabilizing controller, weak l_{2e}-gain stability of the system (Π/K_s) is always unfalsified by the I/O pair (v_s, D) regardless of the switching sequence $\sigma(t)$, $t \in \mathbf{Z}_+$. As a consequence, by virtue of the third property in Def. 6, one may conclude that $V_s(\cdot)$ remains bounded as t increases to infinity for every $s \in \overrightarrow{N}$ such that (Π/K_s) is internally stable. Therefore, under problem feasibility, the HSL holds. Further, the cost function V_f related to the final switched-on controller K_f is bounded. This, along with the assumption C1, implies that the weak l_{2e}-gain stability of (Π/K_f) is unfalsified by (v_f, D), *i.e.* there exist finite nonnegative constants α, β and γ such that

$$\|D^t\| \leq \beta \cdot \|v_f^t\| + \gamma \cdot \sqrt{t+1} + \alpha \tag{13}$$

As the virtual reference v_f converges exponentially to the true reference r, there exists a finite nonnegative $\hat{\alpha}$ such that $\|v_f^t\| \leq \|r^t\| + \hat{\alpha}$, $\forall t \in \mathbf{Z}_+$. Consequently, one can conclude that

$$\|D^t\| \leq \beta \cdot \|r^t\| + \gamma \cdot \sqrt{t+1} + \tilde{\alpha} \tag{14}$$

where $\tilde{\alpha} := \beta \cdot \hat{\alpha} + \alpha$. As (14) holds for the data from every possible input, the adaptive system $(\Pi/K_{\sigma(\cdot)})$ is weakly l_{2e}-gain stable, relatively to r. $\square$

A possible choice, amongst many alternatives, for a weakly l_{2e}-gain-related performance index $V_i(t)$ consists of modifying the denominator of the V_i's in (6) so as to account for the effects of the additional noise inputs n_u, n_y. This may be achieved by considering

$$w_i(t) := \left(v_i^2(t) + m^2\right)^{1/2}, \quad i \in \overleftarrow{N} \tag{15}$$

where $m^2 > 0$ is a positive real, which might be chosen as an a priori estimate of the RMS value of the disturbances.

The following theorem holds.

Lemma 2. Let $\epsilon_i(t) := v_i(t) - y(t)$, $i \in \overleftarrow{N}$, with v_i as in (3). The performance index

$$V_i(t) := \max_{\tau \leq t} \frac{\|\epsilon_i^\tau\|^2 + \rho\|\delta u^\tau\|^2}{\|w_i^\tau\|^2}, \quad i \in \overleftarrow{N} \tag{16}$$

with $w_i(t)$ as in (15), is weakly l_{2e}-gain-related.

Proof. The first two properties in Def. 6 are trivially verified. With regard to the third property, notice that $V_i(t)$ bounded implies that, $\forall t \in \mathbf{Z}_+$, there exist indices i, $i \in \overleftarrow{N}$, such that $\|D_{\rho,i}^t\| \leq M \cdot \|w_i^t\|$, being $D_{\rho,i} := \begin{bmatrix} \rho^{1/2}\delta u \\ \epsilon_i \end{bmatrix}$, and M a nonnegative constant. By triangle inequality it follows that $\|D^t\| \leq \|v_i^t\| + \hat{M} \cdot \left(\|v_i^t\|^2 + m^2 \cdot (t+1)\right)^{1/2}$, for some bounded positive real $\hat{M}$. Hence, $\forall t \in \mathbf{Z}_+$

$$\|D^t\| \leq (1 + \hat{M}) \cdot \|v_i^t\| + m \cdot \hat{M} \cdot \sqrt{t+1} \tag{17}$$

On the opposite, suppose that the weak l_{2e}-gain stability of (Π/K_i) is unfalsified by (v_i, D), *i.e.* $\|D^t\| \leq \beta \cdot \|v_i^t\| + \alpha + \gamma \cdot \sqrt{t+1}$, for some nonnegative reals α, β and γ and $\forall t \in \mathbf{Z}_+$. Consequently, by triangular inequality, $\|D_{\rho,i}^t\|^2 \leq \|v_i^t\|^2 + \|y^t\|^2 + \rho\|\delta u^t\|^2 \leq \|v_i^t\|^2 + (1+\rho)\|D^t\|^2$. Finally,

$$\|D_{\rho,i}^t\| \leq \|v_i^t\| + \sqrt{1+\rho} \cdot \|D^t\| \tag{18}$$

Hence, boundedness of $V_i(t)$ follows directly from the definition of $w_i(t)$. In fact,

$$V_i(t) \leq \frac{\left((1 + \mu \cdot \beta) \cdot \|v_i^t\| + \mu \cdot \gamma \cdot \sqrt{t+1} + \mu \cdot \alpha\right)^2}{\|v_i^t\|^2 + m(t+1)} \tag{19}$$

where $\mu := \sqrt{1+\rho}$. $\square$

Remark 3. Similar results can be achieved in order to extend Def. 6 to the general CCI controller case. In particular, a simple weakly l_{2e}-gain-related performance index can be obtained re-defining $\hat{v}_i(t) := (S_i(d)/S_i(1))v_i(t)$ as the i-th virtual reference. A simple choice for the performance indices is given by

$$V_i(t) = \max_{\tau \leq t} \frac{\|\hat{\epsilon}_i^\tau\|^2 + \rho\|\delta u^\tau\|^2}{\|\hat{w}_i^\tau\|^2}, \quad i \in \overleftarrow{N} \tag{20}$$

with $\hat{w}_i(t) := \left(\hat{v}_i^2(t) + m^2\right)^{1/2}$, and $\hat{\epsilon}_i(t) := \hat{v}_i(t) - y(t)$.

4.2 Interpretation

In order to understand the meaning of (12) w.r.t. the adaptive system (1), it is interesting to give a detailed analysis of the switching mechanism generated by (4)-(5) along with (15)-(16).

Let problem feasibility be satisfied. Let K_s, $s \in \overleftarrow{N}$, be a stabilizing controller. For the index related to such a stabiling controller, one has

$$\|\epsilon_s^t\|^2 + \rho\|\delta u^t\|^2 \leq \tag{21}$$

$$M_1^2 + M_2^2(\|w_s^t\|^2 + \|n_y^t\|^2) + M_3^2\|n_u^t\|^2 \tag{22}$$

for some finite nonnegative constants M_i, $i = 1, 2, 3$, being $\|w_i^t\| \geq \|v_i^t\|$, $\forall t \in \mathbf{Z}_+$. Consequently,

$$V_s(t) \leq \max_{\tau \leq t} \left(\hat{M}^2 + \frac{M_2^2\|n_y^\tau\|^2 + M_3^2\|n_u^\tau\|^2}{\|w_s^\tau\|^2} \right) \tag{23}$$

for some finite nonnegative constant $\hat{M}$. From (15), as $\|w_s^t\|^2 \geq t \cdot m^2$, one gets

$$m^2 \cdot V_s(t) \leq m^2 \cdot \hat{M}^2 + \left(M_2^2\|n_y\|_\infty^2 + M_3^2\|n_u\|_\infty^2\right) \tag{24}$$

where $\|-\|_\infty$ denotes the l_∞-norm. Therefore, (24) ensures that the second assumption stated by the HSL Lemma is satisfied, *viz.* for any switching sequences $\sigma(\cdot) \in \mathcal{S}$, there is a finite time, t_f, beyond which σ is constant. Being V_f bounded, it follows by triangle inequality that

$$\begin{aligned} \|D^t\| &\leq \|v_f^t\| + L \cdot \|w_f^t\| \leq \\ &\leq \|r^t\| + \alpha + L \cdot \|w_f\|_\infty \cdot \sqrt{t+1} \end{aligned} \tag{25}$$

for some bounded positive real L. Notice that the second inequality in (25) takes into account the possible different initial condition between the controller K_f switched on in the loop and the controller K_f^{-1} used for the calculation of (3). Hence, as $v_f(t)$ converges exponentially to the reference $r(t)$, $\forall t \geq t_f$, there exists a finite nonnegative constant c such that $\|v_f\|_{\infty,t\geq t_f} = \|r\|_{\infty,t\geq t_f} + c$, where $\|-\|_{\infty,t\geq\tau}$ denotes the l_∞-norm over $t \geq \tau$. Therefore, one can conclude that $\|w_f\|_\infty = \left(\|v_f\|_\infty^2 + m^2\right)^{1/2}$ equals

$$\|w_f\|_\infty = \max\left\{k, \left(\left(\|r\|_{\infty,t\geq t_f} + c\right)^2 + m^2\right)^{1/2}\right\} \quad (26)$$

where $k := \max_{\tau\leq t_f}\{w_f(\tau)\}$. Clearly, k and c depend on the particular switching sequence $\sigma(\cdot) \in \mathcal{S}$, *i.e* they concern the initial state of the adaptive system and the process disturbances. Hence, we can now give an interpretation to

$$\|D^t\| \leq \|r^t\| + \gamma \cdot \sqrt{t+1} + \alpha \quad (27)$$

where $\gamma := L \cdot \|w_f\|_\infty$. In the concept of weak l_{2e}-gain stability given by (27), the term α accounts for the nonzero adaptive system state before the last switching time-instant, while the term γ takes into account the vector of the inputs to the adaptive system. In fact, $\|w_f\|_\infty$ correctly depends on the disturbances only for $t < t_f$. It also depends on the initial conditions of the adaptive system and the reference $r(t)$, $\forall t \in \mathbf{Z}_+$. On the opposite, by (24)-(25), L explicitly depends on the bounds $\|n_u\|_\infty$ and $\|n_y\|_\infty$ on the disturbances over $t \in \mathbf{Z}_+$.

4.3 Loss of l_{2e}-gain relatedness in the noiseless case

It is advisable to analyze the effects of the performance index (16) in the absence of disturbances. Using (16) instead of (6), we can no longer deduce the properties as in Th. 1, even if there are no disturbances. This is caused by the fact that the inferring mechanism is equipped with a cost function which turns out to be no longer l_{2e}-gain related. In fact, in this connection, the weak l_{2e}-gain stability of the adaptive system $(\Pi/K_{\sigma(\cdot)})$ becomes a sufficient condition for boundedness of the performance index $V(K_i, D, t)$. For instance, consider the case where the initial state of Π is $x_0 \neq 0$, while all exogenous inputs to the loop equals zero. Let K_j be the controller which is switched on in feedback to the plant at the initial time t_0, *i.e.* $\sigma(t_0) = j$. Consequently, $V_j(t) < h$, $\forall t \geq t_0$, implies $\sigma(t) \equiv j$, *i.e.* such a controller will never be switched off of the loop. More specifically, any initial controller K_j is kept in the loop, provided that it generates data $D_\rho := \begin{bmatrix} \rho^{1/2}\delta u \\ y \end{bmatrix}$ such that

$$\|D_\rho^t\|^2 \leq (t+1) \cdot m^2 \cdot h \quad (28)$$

Consequently, every controller which stabilizes the system in the weak sense is a potential final controller. Notice that such a selection is carried out by the supervisor based only on experimental data, *viz.* with no prior knowledge of a plant model. In other words, within the unfalsified control framework, with a specific single infinite-duration experiment one cannot distinguish the effect of internal noise on data from an unknown offset or bias in some internal parameter of the plant.

5. CONCLUSION

The unfalsified VRASC approach has been extended to handle the case of persistent plant disturbances (akin the noise configuration considered for internal stability of feedback-systems). It has been pointed out that the presence of persistent disturbances which corrupt the data to the switching supervisor can have the effect of making invalid the nice conclusions of convergence analysis of noiseless VRASC systems based on the HSL lemma. In order to recover the relevant convergence properties of VRASC systems in the persistent disturbance case, *viz.* finite switching stopping time and boundedness, a novel notion of stability has been introduced and constructively used so as to select new schemes which are shown to achieve the desired goals.

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