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# Game of Defending a Target: A Linear Quadratic Differential Game Approach ⋆

**Dongxu Li** * **Jose B. Cruz, Jr.** *

* *Department of Electrical and Computer Engineering, The Ohio State University, 205 Dreese Lab, 2015 Neil Ave, Columbus, OH 43202, USA; e-mail: li.447@osu.edu, cruz@ece.osu.edu.*

**Abstract:**
Pursuit-evasion (PE) differential games have recently received much attention in military applications involving adversaries. We extend the PE game problem to a problem of defending target, where the roles of the players are changed. The evader is to attack some fixed target, whereas the pursuer is to defend the target by intercepting the evader. We propose a practical strategy design approach based on the linear quadratic game theory with a receding horizon implementation. We prove the existence of solutions for the Riccati equations associated with games with simple dynamics. Simulation results justify the method.

## 1. INTRODUCTION

The increasing use of autonomous assets in modern military operations has recently led to renewed interest in Pursuit-Evasion (PE) differential games (Hespanha et al. [2000], Schenato et al. [2005], Li and Cruz [2006], Li et al. [2008]). The PE problem is usually formulated as a zero-sum game, in which the pursuer(s) tries to minimize a prescribed cost functional while the evader(s) tries to maximize the same functional (Isaacs [1965], Başar and Olsder [1998]). Due to the development of Linear Quadratic (LQ) optimal control theory, a large portion of the literature focuses on PE differential games with a performance criterion in a quadratic form and linear dynamics, c.f. Ho et al. [1965], Turetsky and Shinar [2003].

The roles of the pursuer and the evader in a PE situation may vary with applications. For instance, in the UAV (unmanned autonomous vehicle) applications of surveillance and persistence area denial (Liu et al. [2007]), the evader usually acts as an intruder to strike some (stationary) target; while the pursuer tries to destroy the intruder to protect the target. Here, the roles of the players are no longer merely to chase and to escape. The evader tries to reach the vicinity of the target to attack, and the pursuer wants to intercept the evader before it reaches the target. This type of problem is called "game of Defending Targets" (DT) (Pachter [1987]). In contrast to a nominal (two-player) PE game, in which the game terminates when the distance between the pursuer and the evader is small enough, a game of DT also terminates once the evader is sufficiently close to the target. Although dynamic programming techniques are still applicable, an analytical solution to a DT problem becomes extremely difficult to obtain due to the additional constraint imposed by the terminal set.

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In this paper, we focus on a general DT game involving a single pursuer and a single evader. The challenges in problem definition and solution techniques are discussed. To circumvent these difficulties, we use penalty terms as *soft constraints* to replace the inherent *hard constraints* in DT problems from a practical point of view. With additional assumptions on the linear dynamics of the players, LQ differential game theory can be applied. We further propose a receding horizon implementation scheme for the resulting LQ strategies. We demonstrate by simulation that this method performs well in a DT game with simple motion and we compare it to an optimal strategy that can be derived using a geometric approach. Moreover, we prove the existence of solutions for the Riccati equation associated with games with simple linear dynamics, where the conditions on definiteness of the matrices are absent, which are usually required in LQ differential game theory.

The paper is organized as follows. In Section 2, we introduce the problem of DT and discuss technical difficulties. In Section 3, we formulate a DT game with linear dynamics and a quadratic objective based on soft constraints. A implementation scheme is then introduced for the players' LQ strategies. In Section 4, we verify by simulation the proposed strategy in a DT game with simple dynamics. The concluding remarks are provided in Section 5.

## 2. THE PROBLEM OF DEFENDING A TARGET

Consider an evader, a pursuer and a target in a $n_S$-dimensional space $S \subseteq \mathbb{R}^{n_S}$, $n_S \in \mathbb{N}$. Let $x_e \in \mathbb{R}^{n_e}$ and $x_p \in \mathbb{R}^{n_p}$ be the state variables of the evader and the pursuer respectively, with $n_p, n_e \geq n_S$. The dynamics of the players are described by

$$\dot{x}_p(t) = f_p(x_p(t), u_p(t)); \quad \dot{x}_e(t) = f_e(x_e(t), u_e(t)) \\ \text{with } x_p(0) = x_{p_0} \text{ and } x_e(0) = x_{e0}. \tag{1}$$

Here, $u_p(t) \in U_p \subseteq \mathbb{R}^{m_p}$ and $u_e(t) \in U_e \subseteq \mathbb{R}^{m_e}$ are the control inputs. Suppose that the first $n_S$ elements of $x_p$

($x_e$) stand for the physical position of the pursuer (evader) in $S$. Define a projection $P : \mathbb{R}^{n_p} \mapsto S$ for the pursuer as

$$P(x_p) = \left[x_{p_1}, \cdots, x_{p_{n_S}}\right]^{\mathrm{T}}. \qquad (2)$$

A similar projection can also be defined for the evader, and we use the same notation $P$. For simplicity, we assume that the target is at the origin. Given some $\varepsilon > 0$, define the set $\Lambda_1$ and $\Lambda_2$ as

$$\Lambda_1 = \{(x_p, x_e) \in \mathbb{R}^{n_p} \times \mathbb{R}^{n_e} \mid \|P(x_e)\| \le \varepsilon\};$$
$$\Lambda_2 = \left\{(x_p, x_e) \in \Lambda_1^c \mid \|P(x_p) - P(x_e)\| \le \varepsilon\right\}, \qquad (3)$$

where $\Lambda_1^c$ is the complementary set of $\Lambda_1$; $\|\cdot\|$ is the standard Euclidean norm. Let $\Lambda = \Lambda_1 \cup \Lambda_2$, and here, the set $\Lambda$ defines the terminal of a DT game, i.e., the game terminates at $T \ge 0$ with $T = \min\left\{t > 0 \mid \left(x_p(t), x_e(t)\right) \in \Lambda\right\}$. With the notations introduced above, a DT game problem can be described as follows:

*Given the initial states $x_{p0}$, $x_{e0}$, the pursuer needs to find a proper control input $u_p(t)$ (as a function of time) based on its information, such that the game ends in $\Lambda_2$, i.e., $\left(x_p(T), x_e(T)\right) \in \Lambda_2$ ; while the evader tries to drive the state trajectory into $\Lambda_1$ by choosing a proper input $u_e(t)$.*

Generally speaking, the game described above is a differential game of *kind* [1] (Başar and Olsder [1998]), for which an analytical solution is usually derived by introducing the following auxiliary cost functional

$$J = \begin{cases} 0 \text{ if } (x_p(T), x_e(T)) \in \Lambda_2, \\ 1 \text{ if } (x_p(T), x_e(T)) \in \Lambda_1. \end{cases}$$

In this way, the game is converted into a differential game of *degree*. The value function $V$ (if it exists) can only take two values: $V(x_p, x_e) = 1$ and $V(x_p, x_e) = 0$, which is not differentiable. Please note that the description above is not a rigorous definition of a DT game because an information structure has not been specified for the players, which will have a great impact on existence of a Value.

In what follows, we illustrate the tremendous difficulties in solving a DT problem above as a traditional PE game. First of all, optimal strategies of the players can possibly be derived only when it is known where the terminal state is given the initial states $x_p, x_e$. It requires the existence of a value at $x_p, x_e$, i.e., $V(x_p, x_e) = 1$ or $V(x_p, x_e) = 0$. However, there is no theoretical result regarding standard formulation and existence of a value $V$ for a DT game in the literature, and establishment of such existence is far from trivial, even for a traditional two-player PE game (Evans and Souganidis [1984]). It usually requires specification of players' information structures as well as the continuity of $V$ at the terminal set, which is no longer the case here. Now, let us assume the existence of a value $V$ and define the set

$$\mathcal{S}_j = \{(x_p, x_e) \in \mathbb{R}^{n_p} \times \mathbb{R}^{n_e} \mid V(x_p, x_e) = j\}$$

with $j \in \{0, 1\}$. Here, the set $\mathcal{S}_0$ (or $\mathcal{S}_1$) contains all the states, from which the pursuer (or evader) can force the state trajectory to end in $\Lambda_2$ (or $\Lambda_1$) against any control input of the other player. Only with the knowledge of $\mathcal{S}_i$, may an optimal strategy of the pursuer (or evader) be solved based on a proper formulation of a game of degree within $\mathcal{S}_i$. However, a formal analytical derivation of the set $\mathcal{S}_i$ through its boundary $\partial\mathcal{S}_i$ is a formidable task due to the dimension of the state space and the involvement of an additional target, considering the complication involved in a simple two-player PE game example treated in Başar and Olsder [1998], pp.436. Furthermore, the numerical method for calculating reachable sets based on the level set method (c.f. Mitchell et al. [2005]) can be problematic here because the time horizon can be arbitrarily large, and the method suffers from the exponential growth in the number of states. In general, a DT game is a very difficult problem.

[1] When we speak of differential game of *kind*, we mean one with finitely many, usually two, outcomes; the counterpart is called differential game of *degree*, which has a continuum of possible payoffs. The latter is the concept mostly used in the field of differential games.

## 3. A LQ DIFFERENTIAL GAME APPROACH

### *3.1 Linear Quadratic Formulation with Soft Constraints*

In the previous section, both the theoretical and the practical difficulties are discussed for a DT game. These difficulties mainly result from the hard constraints imposed on the terminal of the game as in (3). For optimal control and differential game problems, hard constraints are often approximated by soft constraints with weighting parameters (Ho et al. [1965]). With this approximation, problems with fixed horizons are formulated. The state of the art in this approach is the choice of the optimization horizon and the weighting parameters, which have a large impact on the relevance of the results. To this end, we reformulate a DT problem with a fixed horizon and soft constraints.

We focus on linear dynamics of the players:

$$\dot{x}_p(t) = A_p x_p(t) + B_p' u_p(t) \quad \text{with} \quad x_p(t_0) = x_{p0}, \quad (4a)$$
$$\dot{x}_e(t) = A_e x_e(t) + B_e' u_e(t) \quad \text{with} \quad x_e(t_0) = x_{e0}. \quad (4b)$$

Here, $x_p(t) \in \mathbb{R}^{n_p}$, $x_e(t) \in \mathbb{R}^{n_e}$ for $n_p, n_e \ge n_S, t \ge t_0$; $u_p(t) \in U_p \subseteq \mathbb{R}^{m_p}$ and $u_e(t) \in U_e \subseteq \mathbb{R}^{m_e}$ are the control inputs; $A_p, A_e, B_p', B_e'$ are real matrices with proper dimensions. We write an aggregate dynamic equation as

$$\dot{x}(t) = Ax(t) + B_e u_e(t) + B_p u_p(t), \qquad (5)$$

where

$$x = \begin{bmatrix} x_e \\ x_p \end{bmatrix}, A = \begin{bmatrix} A_e & 0 \\ 0 & A_p \end{bmatrix}, B_e = \begin{bmatrix} B_e' \\ 0 \end{bmatrix} \text{ and } B_p = \begin{bmatrix} 0 \\ B_p' \end{bmatrix}$$

with $x \in \mathbb{R}^n$ and $n = n_p + n_e$. Regarding the players' information structure, we consider the feedback strategies $\gamma_p : \mathbb{R}^n \times \mathbb{R} \mapsto U_p$ and $\gamma_e : \mathbb{R}^n \times \mathbb{R} \mapsto U_e$. Namely, given $x \in \mathbb{R}_n$ and time $0 \le t < T$, $\gamma_p(x, t) \in U_p$ and $\gamma_e(x, t) \in U_e$. Denote by $\Gamma_p$ and $\Gamma_e$ the set of admissible feedback strategies for the pursuer and the evader respectively.

We consider the objective functional as

$$J_1(\gamma_p, \gamma_e; x_0) = \int_0^T \left(u_e^{\mathrm{T}}(\tau) u_e(\tau) - u_p^{\mathrm{T}}(\tau) u_p(\tau)\right) \mathrm{d}\tau$$
$$+ w_e \|P(x_e(T))\|^2 - w_p \|P(x_p(T)) - P(x_e(T))\|^2. \quad (6)$$

Here, $\gamma_p, \gamma_e$ are the feedback strategies, and $u_p, u_e$ are the control inputs associated with the strategies; $w_p > 0$ and $w_e > 0$ are some weighting parameters. In (6), soft constraints (penalty terms) are considered as the weighted squared distance between the evader and the target as well as the pursuer with a fixed time duration $T$. Here,

$w_e$, $w_p$ and $T$ are the design parameters. For comparison purposes, we also consider the following objective

$$\begin{aligned} J_2(\gamma_p, \gamma_e; x_0) = \int_0^T &\Big( u_e(\tau)^{\mathrm{T}} u_e(\tau) - u_p^{\mathrm{T}}(\tau) u_p(\tau) \\ &+ w_e^+ \|P(x_e(\tau))\|^2 - w_p^+ \|P(x_p(\tau)) - P(x_e(\tau))\|^2 \Big) \mathrm{d}\tau \\ &+ w_e \|P(x_e(T))\|^2 - w_p \|P(x_p(T)) - P(x_e(T))\|^2 \end{aligned} \quad (7)$$

with $w_e^+, w_p^+ > 0$. Compared to (6), here, the penalty terms also appear inside the integral.

The objective $J_1$ or $J_2$ can be put into a quadratic form with respect to $x$, $u_p$ and $u_e$. Note that the terms $\|P(x_e)\|^2$ and $\|P(x_p) - P(x_e)\|^2$ in (6) or (7) satisfy $\|P(x_e)\|^2 = x^{\mathrm{T}} Q^e x$ and $\|P(x_p) - P(x_e)\|^2 = x^{\mathrm{T}} Q^p x$ respectively, with

$$Q^e = \begin{bmatrix} I_{n_S} & 0_{n_S \times (n - n_S)} \\ 0_{(n-n_S) \times n_S} & 0_{(n-n_S)} \end{bmatrix} \text{ and}$$

$$Q^p = \left[\begin{array}{cc|cc} I_{n_S} & 0_{\square} & -I_{n_S} & 0_{\square} \\ 0_{\square} & 0_{n_e - n_S} & 0_{\square} & 0_{(n_e-n_S)\times(n_p-n_S)} \\ \hline -I_{n_S} & 0_{n_S \times (n_e - n_S)} & I_{n_S} & 0_{n_S \times (n_p - n_S)} \\ 0_{\square} & 0_{(n_p-n_S)\times(n_e-n_S)} & 0_{\square} & 0_{n_p - n_S} \end{array}\right], \quad (8)$$

where $0_{p\times q}$ is the zero matrix of dimension $p \times q$; $0_p$ ($I_p$) is the $p \times p$ zero (identity) matrix; $0_{\square}$ is a zero matrix of a proper dimension that can be easily determined in the context. Both $J_1$ and $J_2$ can be written in the following quadratic form with $i = 1, 2$.

$$\begin{aligned} J_i = \int_0^T &(u_e^{\mathrm{T}} u_e - u_p^{\mathrm{T}} u_p + x^{\mathrm{T}}(\tau) Q_i x(\tau)) \mathrm{d}\tau \\ &+ x^{\mathrm{T}}(T) Q_f(w_p, w_e) x(T), \end{aligned} \quad (9)$$

where $Q_f(w_p, w_e) = w_e Q^e - w_p Q^p$, $Q_1 = 0$ and $Q_2 = Q_f(w_p^+, w_e^+)$.

This is a zero-sum game, where the evader (pursuer) seeks a strategy $\gamma_e \in \Gamma_e$ ($\gamma_p \in \Gamma_p$) to minimize (maximize) the objective $J_1$ or $J_2$ subject to (5). The following LQ differential game theory specifies a saddle-point equilibrium solution.

*Theorem 1.* Suppose that $U_p = \mathbb{R}^{m_p}$ and $U_e = \mathbb{R}^{m_e}$. The game with objective $J_i$ ($i = 1, 2$) and players' dynamics (5) admits a feedback saddle-point solution given by $\bar{u}_e^i(t) = \gamma_e^*(x(t), t) = K_e^*(t) x(t)$ and $\bar{u}_p^i(t) = \gamma_p^*(x(t), t) = K_p^*(t) x(t)$ with $K_e^*(t) = -B_e^{\mathrm{T}} Z_i(t)$ and $K_p^*(t) = B_p^{\mathrm{T}} Z_i(t)$, where $Z_i(t)$ is bounded, symmetric and satisfies

$$\begin{aligned} \dot{Z}_i &= -A^{\mathrm{T}} Z_i - Z_i A - Q_i + Z_i (B_e B_e^{\mathrm{T}} - B_p B_p^{\mathrm{T}}) Z_i \\ &\text{with } Z(T) = Q_f(w_e, w_p). \end{aligned} \quad (10)$$

Readers can refer to Başar and Olsder [1998] and Engwerda [2005] for a detailed derivation of the linear feedback strategies ($\gamma_e^*$, $\gamma_p^*$) at equilibrium.

*Remark 1.* The game is almost the same as the PE game formulated in Ho et al. [1965] except for the additional penalty term due to the target. With this additional term, the matrices $Q_i$ and $Q_f$ are no longer (positive semi-) definite. Thus, the existence of solutions for the Riccati equation (10) may be an issue.

### *3.2 Implementation Issue*

The discussion under the framework of LQ game theory is to take advantage of the availability of analytical solutions for LQ games. However, its usefulness in solving DT games remains to be tested. The biggest gap between the LQ game formulation and a generic DT game is on the definiteness of the terminal time $T$. To ensure that the LQ formulation in the previous section can be useful in solving DT games, we propose a repetitive implementation scheme as follows.

We choose $\Delta t > 0$ as the sampling time interval. At each sampling time $t_k = t_0 + k\Delta t$ for $k \in \{0, 1, 2, \cdots\}$, a saddle-point equilibrium strategy pair $(\gamma_p^*, \gamma_e^*)$ are solved over the interval $[t_k, t_k + T_k]$, where $T_k > \Delta t$ is the planning horizon. We will discuss shortly how to choose $T_k$ such that the corresponding Riccati equation has a solution over the interval. Then, the strategy $\gamma_p^*$ (or $\gamma_e^*$) obtained over $[t_k, t_k + T_k]$ is implemented during the next $\Delta t$ interval, i.e., $[t_0 + k\Delta t, t_0 + (k+1)\Delta t)$. The same procedure is repeated at each sampling time $t_k$. We call it LQ Receding Horizon Algorithm (LQRHA). The detailed calculation at each time $t_0 + k\Delta t$ is given in Table 1.

Table 1. Calculation at Each $t_k$ in the LQRHA

| |
|---|
| 1. Input: state $(x_p, x_e)$ at time $t_k$ |
| 2. Determine the parameters $w_p, w_e$ $(w_p^+, w_e^+), T_k$ |
| 3. Solve the saddle equilibrium feedback strategies ($\gamma_e^*$, $\gamma_p^*$) over the time interval $[t_k, t_k + T_k]$ |
| 4. Output: $K_p^*(\cdot), K_e^*(\cdot)$ |

We now discuss how to choose a proper planning horizon $T_k$, such that the corresponding Riccati equation (10) admits a bounded solution on $[0, T_k]$, i.e., the interval $[0, T_k]$ contains no *escape time* (Başar and Bernhard [1995]). In fact, the finite escape time (if it exists) of a Riccati equation can be determined in such a way that is suggested by the following theorem.

*Theorem 2.* The Riccati Differential Equation (RDE) (10) has a bounded solution over $[0, T]$ if and only if the following matrix linear differential equation

$$\begin{bmatrix} \dot{X}(t) \\ \dot{Y}(t) \end{bmatrix} = \begin{bmatrix} A & -S \\ -Q_i & -A^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} X(t) \\ Y(t) \end{bmatrix}, \begin{bmatrix} X(T) \\ Y(T) \end{bmatrix} = \begin{bmatrix} I_n \\ Q_f \end{bmatrix} \quad (11)$$

has a solution on $[0, T]$ with $X(\cdot)$ nonsingular over $[0, T]$. In (11), $A, Q$ and $S = B_e B_e^{\mathrm{T}} - B_p B_p^{\mathrm{T}}$ are the corresponding matrices in (10). Moreover, if $X(\cdot)$ is invertible, $Z(t) = Y(t) X^{-1}(t)$ is a solution of (10).

Refer to Engwerda [2005], pp. 194 or Başar and Bernhard [1995], pp. 354 for a proof.

According to Theorem 2, the finite escape time $T_e > 0$ satisfies that $T - T_e$ is the largest time such that matrix $X(T - T_e)$ is singular. In practice, suppose that the optimization horizon $\hat{T}_k$ (without considering finite escape time) may be chosen heuristically depending on the applications, e.g., $\hat{T}_k = \mathcal{T}(x_k)$. Here, function $\mathcal{T}(\cdot)$ may be a function of state $x$ at time $t_k$. By solving linear differential equation (11), it can be checked whether $T_e \in [0, \hat{T}_k]$. If $T_e \notin [0, \hat{T}_k]$, then $T_k$ can be chosen as $\hat{T}_k$; otherwise, $T_k$ can be set as $T_k = T_e^{\delta}$ with $T_e^{\delta} = T_e - \delta$ for some $\delta > 0$. With $T_k$ chosen above, the Riccati equation (10) has a bounded solution over $[0, T_k]$. It should be noted that the sampling interval size $\Delta t$ should satisfy $T_k > \Delta t$. One way to ensure the inequality is to let $\Delta t < T_e$. Finally, $T_e$ only needs to be calculated once because the Riccati equation (10) does not depend on state $x$. Finally, in general, $w_p, w_e, w_p^+, w_e^+$ may also be properly selected at each time instant $t_k$ by the players, depending on the game situation.

## 4. A DT GAME: PLAYERS WITH SIMPLE MOTION

In this section, we show the usefulness of the LQ strategy in solving DT problems.

### *4.1 Players with Simple Motion*

Suppose that the players have the following simple dynamics, which are given (in a *x-y* coordinate) as

$$\begin{cases} \dot{x}_p = u_p v_p \cos(\theta_p) \\ \dot{y}_p = u_p v_p \sin(\theta_p) \end{cases}; \begin{cases} \dot{x}_e = u_e v_e \cos(\theta_e) \\ \dot{y}_e = u_e v_e \sin(\theta_e) \end{cases}. \tag{12}$$

with given initial states $x_p(0), x_e(0)$. In (12), $x_e, y_e$ ($x_p, y_p$) are the evader's (pursuer's) states (displacement) along the $x$ and $y$ axis; $v_e$ ($v_p$) is the speed; $u_e, \theta_e$ ($u_p, \theta_p$) are the control inputs, where $u_e(u_p) \in [0,1]$ is a scalar that controls the speed from 0 up to $v_e$ ($v_p$), and $\theta_e$ ($\theta_p$) is the moving orientation. Note that the dynamics in (12) is not linear, and in the following, we use the the player's dynamics that is equivalent to (12), given as

$$\begin{bmatrix} \dot{x}_e \\ \dot{y}_e \end{bmatrix} = \begin{bmatrix} v_e & 0 \\ 0 & v_e \end{bmatrix} \begin{bmatrix} u_{ex} \\ u_{ey} \end{bmatrix}, \begin{bmatrix} \dot{x}_p \\ \dot{y}_p \end{bmatrix} = \begin{bmatrix} v_p & 0 \\ 0 & v_p \end{bmatrix} \begin{bmatrix} u_{px} \\ u_{py} \end{bmatrix}. \tag{13}$$

In (13), $(u_{\varsigma x}, u_{\varsigma y})$ are the controls with $\sqrt{u_{\varsigma x}^2 + u_{\varsigma y}^2} \leq 1$, where $\varsigma \in \{e, p\}$ stands for evader or pursuer. Clearly, $(u_\varsigma, \theta_\varsigma)$ and $(u_{\varsigma x}, u_{\varsigma y})$ forms a one-to-one mapping by considering $\theta_\varsigma \in [0, 2\pi)$. Here, the dynamics in (13) is linear in the input $(u_{\varsigma x}, u_{\varsigma y})$ with a boundedness constraint. Although the LQ feedback strategies derived in the previous sections are not bounded, we still rely on them to design the feedback control law $\gamma_\varsigma$ here. We further use the following nonlinear function $\varphi(\cdot)$ to ensure the boundedness of players' control inputs in implementation.

$$\varphi(r) = \begin{cases} r & \text{if } \|r\| \leq 1 \\ r/\|r\| & \text{if } \|r\| > 1 \end{cases} \quad \text{for } r \in \mathbb{R}^m \text{ with } m \geq 1. \tag{14}$$

The actual control $u_\varsigma$ applied is $u_\varsigma = \varphi(\gamma_\varsigma(x,t))$.

### *4.2 Existence of a Bounded Solution for the RDE*

We first show the existence of solutions for the Riccati equation (10) associated with the dynamics in (13) without considering $\sqrt{u_{\varsigma x}^2 + u_{\varsigma y}^2} \leq 1$. Note that here, the matrix $Q_f$ and $Q_i$ lack the positive semidefiniteness, which is required in the existing optimal control or differential game literatures. We define $H_i \triangleq \{A, Q_i, B_p, B_e\}$ $(i = 1, 2)$, and denote by $\text{Ric}(H_i)$ the Right Hand Side (RHS) of the Riccati equation (10). Define the set $R_\square^{H_i} \triangleq \{W \in \mathbb{R}^{n\times n} | W = W^\text{T} \text{ and } \text{Ric}(H_i)\square 0\}$ with $\square \in \{\geq, \leq, =\}$. We denote by $W(\cdot, X_0)$ the solution of equation (10) with $W(t_0, X_0) = X_0$ for some arbitrary and fixed $t_0$.

*Lemma 3.* Suppose that there exist

$$W_1 \in R_\leq^H \text{ and } W_2 \in R_\geq^H \text{ with } W_1 \leq W_2,$$

then $W_1 \leq W_0 \leq W_2$ for some $W_0 \in \mathbb{R}^{n\times n}$ and $W_0 = W_0^\text{T}$ implies that $W(t, W_0)$ exists for $t \in (-\infty, t_0]$ with $W_1 \leq W(t, W_1) \leq W(t, W_0) \leq W(t, W_2) \leq W_2$ for $t \in (-\infty, t_0]$.

Refer to Theorem 3.1 in Freiling and Jank [1996].

*Theorem 4.* For a game with objective $J_1$ and $J_2$ and the simple linear dynamics in (13),
(i) if $v_e \leq v_p$, the corresponding Riccati equation (10) associated with $J_1$ in (6) has a bounded solution;
(ii) if $v_e < v_p$ , the corresponding Riccati equation (10) associated with $J_2$ in (7) has a bounded solution.

**Proof.** With the linear simple dynamics, $Q^e, Q^p$ in (8) become

$$Q^e = \begin{bmatrix} I_{n_S} & 0_{n_S} \\ 0_{n_S} & 0_{n_S} \end{bmatrix} \text{ and } Q^p = \begin{bmatrix} I_{n_S} & -I_{n_S} \\ -I_{n_S} & I_{n_S} \end{bmatrix},$$

and $Q_1, Q_2, Q_f$ in $J_1$ and $J_2$ in (6) and (7) can be determined accordingly. In the following, we use Lemma 3 to prove the theorem.
(i) We need to find matrices $W_1$ and $W_2$ satisfying Lemma 3. Define

$$S = B_e B_e^\text{T} - B_p B_p^\text{T} = \begin{bmatrix} v_e I_{n_S} & 0 \\ 0 & -v_p I_{n_S} \end{bmatrix}.$$

Choose $W_1$ as

$$W_1 = \begin{bmatrix} -\omega_1 I_{n_S} & \omega_1 I_{n_S} \\ \omega_1 I_{n_S} & -\omega_1 I_{n_S} \end{bmatrix}$$

for some $\omega_1 > 0$. Substitute $W_1$ into $\text{Ric}(H_1)$,

$$\text{Ric}(H_1) = W_1 S W_1 = \omega_1^2 (v_e - v_p) \begin{bmatrix} I_{n_S} & -I_{n_S} \\ -I_{n_S} & I_{n_S} \end{bmatrix} \leq 0. \tag{15}$$

Note that $v_e \leq v_p$, and thus, $W_1 \in R_\leq^{H_1}$ with $H_1 \triangleq \{A, Q_1, B_p, B_e\}$. For any $x = [x_e^\text{T}, x_p^\text{T}]^\text{T} \in \mathbb{R}^n$,

$$\begin{aligned} x^\text{T}(Q_f - W_1)x &= (w_e - w_p + \omega_1)\|x_e\|^2 + 2(w_p - \omega_1)x_e^\text{T} x_p \\ &\quad + (\omega_1 - w_p)\|x_p\|^2 \\ &= (\omega_1 - w_p)\|x_p - x_e\|^2 + w_e\|x_e\|^2. \end{aligned} \tag{16}$$

Clearly, if we select $\omega_1$ with $\omega_1 \geq w_p$, then (16)$\geq 0$, and thus $Q_f \geq W_1$. On the other hand, choose $W_2$ as

$$W_2 = \begin{bmatrix} \omega_2 I_{n_S} & 0_{n_S} \\ 0_{n_S} & 0_{n_S} \end{bmatrix} \tag{17}$$

for some $\omega_2 > 0$, and $W_2 \in R_\geq^{H_1}$. For any $x = [x_e^\text{T}, x_p^\text{T}]^\text{T}$,

$$\begin{aligned} x^\text{T}(W_2 - Q_f)x &= \omega_2\|x_e\|^2 - w_e\|x_e\|^2 + w_p\|x_p - x_e\|^2 \\ &= (\omega_2 - w_e + w_p)\|x_e\|^2 - 2w_p x_e^\text{T} x_p + w_p\|x_p\|^2. \end{aligned}$$

There exists some $\omega_2 > 0$ such that $x^\text{T}(W_2 - Q_f)x \geq 0$ for any $x \in \mathbb{R}^n$, i.e., $W_2 \geq Q_f$. Hence, $W_1 \leq Q_f \leq W_2$. By Lemma 3, $W(t, Q_f)$ exists for all $t \leq T$.

(ii) The proof is similar to part (i). With the same $W_1, W_2$ chosen in (i), we only need to check if $W_1 \in R_\leq^{H_2}$ and $W_2 \in R_\geq^{H_2}$ with $H_2 \triangleq \{A, Q_2, B_p, B_e\}$. Substitute $W_1$ into $\text{Ric}(H_2)$, and let $RW_1 \triangleq \text{Ric}(H_2) = -Q_2 + W_1 S W_1$. In (15), define $r \triangleq (v_e - v_p)\omega_1^2$. For any $x = [x_e^\text{T}, x_p^\text{T}]^\text{T}$,

$$x^{\mathrm{T}}RW_1x = (r - w_e^+ + w_p^+)\|x_e\|^2 + (r + w_p^+)\|x_p\|^2 \\ -2(r + w_p^+)x_e^{\mathrm{T}}x_p \\ = (r + w_p)\|x_p - x_e\|^2 - w_e\|x_e\|^2. \qquad (18)$$

Note that $v_e < v_p$, and hence, $r < 0$. For $\omega_1 > \sqrt{\frac{w_p}{v_p - v_e}}$, $x^{\mathrm{T}}RW_1x \leq 0$ for any $x \in \mathbb{R}^n$. Therefore, $W_1 \in R_{\leq}^{H_2}$. On the other hand, substitute $W_2$ into $\mathrm{Ric}(H_2)$, and let $RW_2 \triangleq \mathrm{Ric}(H_2) = -Q_2 + W_2SW_2$. Similar to $RW_1$, given any $x = [x_e^{\mathrm{T}}, x_p^{\mathrm{T}}]^{\mathrm{T}}$,

$$x^{\mathrm{T}}RW_2x = (\omega_2^2 - w_e + w_p)\|x_e\|^2 + w_p\|x_p\|^2 - 2w_px_e^{\mathrm{T}}x_p.$$

Clearly, there exists some $\omega_2 > 0$ such that $x^{\mathrm{T}}RW_2x \geq 0$ for any $x \in \mathbb{R}^n$, i.e., $RW_2 \geq 0$, and $W_2 \in R_{\geq}^{H_2}$. According to (i), $W_1 \leq Q_f \leq W_2$ and by Lemma 3,$W(t, Q_f)$ exists for $t \leq T$. This completes the proof.

### *4.3 Performance Verification*

In this section, we apply the LQRHA to a DT problem with the player's dynamics specified in (12). We assume that $v_p = v_e$, and let $\bar{x}_p = [x_p, y_p]^{\mathrm{T}}$, $\bar{x}_e = [x_e, y_e]^{\mathrm{T}}$ and $\bar{x} = \left[\bar{x}_p^{\mathrm{T}}, \bar{x}_e^{\mathrm{T}}\right]^{\mathrm{T}}$ be the aggregate state variables.

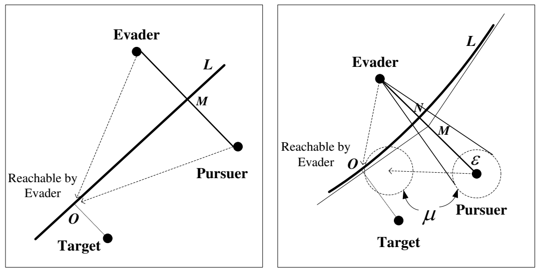
The optimal strategy of the players in this simple game can be derived using a geometric approach (Isaacs [1965], pp. 144), which is briefly summarized as follows. We first refer to a *reachable set* as the set of all the points in $\mathbb{R}^2$, to which the reachability of the evader (without being intercepted) is guaranteed by some strategy against any pursuer's control input. We denote by $\mathcal{R}(\bar{x}_p, \bar{x}_e)$ the reachable set given the initial position $\bar{x}_p, \bar{x}_e$. We first consider that the evader is captured at the coincidence of the pursuer and the evader, i.e, $\bar{x}_p = \bar{x}_e$. In this case, the set $\mathcal{R}(\bar{x}_p, \bar{x}_e)$ is the half plane on the evader's side that is divided by the straight line $L$ passing through the midpoint $M$ of the interval between the pursuer and the evader and perpendicular to the interval, which is illustrated in Fig. 1(a). Thus, the optimal strategy in a DT game drives the pursuer (or evader) towards point $O$ in Fig. 1(a), which is the point in $\mathcal{R}(\bar{x}_p, \bar{x}_e)$ that is the closest to the target. If the capture radius $\varepsilon$ of the pursuer is considered, the splitting line $L$ turns into a hyperbola passing through the midpoint $N$ of the interval between the evader and the intersection of the line formed by the pursuer and the evader and the capture circle $\mu$. The asymptotes pass through the midpoint $M$ and are perpendicular to the tangents from the evader to $\mu$. The reachable set is the half plane on the evader's side, as shown in Fig.1(b). The optimal strategy drives the pursuer (evader) to the point $O$.

For comparison, we consider a suboptimal feedback strategy: $u_p^d = -v_p\frac{x_e - x_p}{\|x_e - x_p\|}$ $(x_e \neq x_p)$. We refer to it as "direct strategy". Under $u_p^d$, the pursuer always proceeds directly towards the evader. The insight of the direct strategy is similar to the "proportional guidance law", where the pursuer (interceptor) always wants to align its orientation with the evader.

We design the LQ strategy based on the dynamics in (13) and implement it in LQRHA. Note that a bounded solution for the Riccati equation (10) associated with (13) exists. At each sampling time $t_k$, the planning horizon $T_k$ is chosen as $(\|x_e\| - \varepsilon)/v_e$, the minimum time it takes for the evader to reach the target without the pursuer. The actual controls that are applied are

$$\tilde{u}_p^i(t) = \varphi(\gamma_p^*(\bar{x}, t)), \text{ and } \tilde{u}_e^i(t) = \varphi(\gamma_e^*(\bar{x}, t))$$

with function $\varphi$ defined in (14).




(a) Capture at coincidence (b) Capture with Radius $\varepsilon$

Fig. 1. Researchable Set and Optimal Strategy

We assume that the evader exploits the optimal strategy specified in Fig.1(b). We first use the LQRHA to determine the pursuer's strategy in the DT problem. Suppose that $v_p = v_e = 1$ and the initial positions of the players are $x_{e0} = [3, 3]^{\mathrm{T}}$ and $x_{p0} = [3, -1]^{\mathrm{T}}$. We fix the parameters as $w_p = w_p^+ = w_e = w_e^+ = 10$. Let $\Delta t = 0.1$, $\varepsilon = 0.5$, and implement the LQRHA based on $J_1$ and $J_2$. Fig.2 depicts the pursuit trajectories when the pursuer uses the LQ strategies. For comparison, we simulate the DT game with both the direct strategy and the optimal strategy from the pursuer, and the resulting players' trajectories are illustrated in Fig.3. The terminal times of the DT game for each case are listed in Table 2.

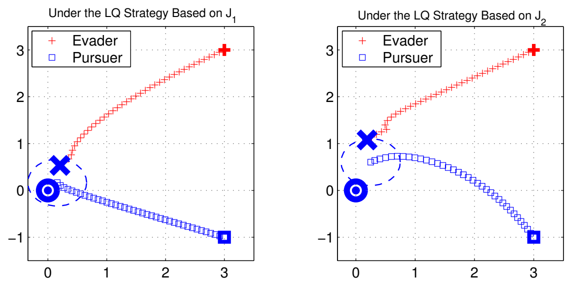



Fig. 2. Pursuit Trajectories under the LQ Strategy Based on $J_1$ and $J_2$ by the Pursuer

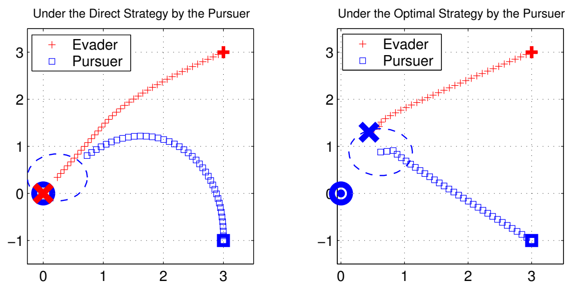



Fig. 3. Pursuit Trajectories under the Direct/Optimal Strategy by the Pursuer

Fig.2 shows that the pursuer can successfully intercept the evader by exploiting the LQ strategy (based on $J_1, J_2$)

Table 2. Performance Comparison

| | Optimal | LQ ($J_2$) | LQ ($J_1$) | Direct |
|---|---|---|---|---|
| Time(s) | 3.1 | 3.5 | 3.8 | 3.9 |

against the optimal strategy of the evader. From Fig.3, the pursuer intercepts the evader under its optimal strategy, but fails to catch the evader under the direct strategy. Note that the trajectories are straight lines only when both players exploit optimal strategies, as suggested by the optimal strategy. According to Table 2, the time of interception under the LQRHA based on $J_2$ is less than that based on $J_1$. With $J_2$, the pursuer appears to be more aggressive in intercepting the evader, which can be observed from the trajectories in Fig.2. This is reasonable by considering the penalties imposed on distances in the integral in (7). In LQRHA, $w_p, w_p^+, w_e, w_e^+$ are the design parameters, and based on a number of simulations, we have found that the actual results are almost the same with a large range of variation in those parameters.

Furthermore, we numerically calculate the set of the pursuer's initial positions in $\mathbb{R}^2$, from which it can successfully intercept the evader, under all 4 different strategies. We refer to the set as "interception region" and denote it by $\mathcal{I}$. Here, the evader exploits the optimal strategy. Fix the evader's initial position at $(3, 3)$. In Fig.4, the perimeters of $\mathcal{I}$ under different pursuer's strategies are indicated. The interception region $\mathcal{I}$ contains all the points inside the perimeter. According to Fig.4, $\mathcal{I}$ resulted from the LQRHA covers almost the entire set $\mathcal{I}$ associated with the optimal strategy. We can conclude that in this example, the LQHRA determines a fairly good interception guidance law for the pursuer.

Finally, we apply the LQRHA to determine the evader's strategies in the same DT problem against the pursuer's optimal strategy, and similar conclusions can be drawn from the evader's perspective. The results can also be extended into a general $n$-dimensional space.

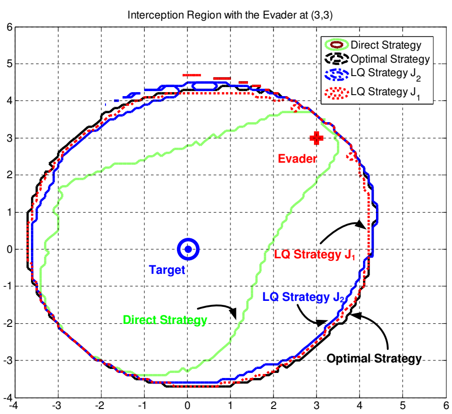



Fig. 4. Interception Regions $\mathcal{I}$ under the Different Strategies from the Pursuer

## 5. CONCLUSIONS

In this paper, we have studied the DT game under the framework of LQ differential game. The hard constraints associated with the original DT game are replaced by soft constraints with a fixed optimization horizon. A receding horizon scheme has been proposed for implementation of the LQ strategy designed. The existence of solutions for the Riccati equations associated with simple linear dynamics of the players has been proved without the definiteness of the matrices in the objective functional. We have demonstrated by simulation the performance of the player's LQ strategy with the LQHRA in a DT problem with simple dynamics, and have compared it to an optimal strategy. The LQRHA can determine good practical strategies for the players in DT games.

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