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On Singular Perturbations of Unstable Underactuated Mechanical Systems With Underactuation Degree ≥ 1 *

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Abstract: 10 years ago, K.J. Åström proposed that the essence of the complex control problem originated by the joint of the pilot–&–aircraft can be captured on labs, by means of unstable underactuated mechanical systems. Thus, the unactuated part describes the autonomous aircraft dynamics and the actuated the piloted one. In this constructive approach we propose a nonlinear controller based on classical feedback linearization and singular perturbation theory, which has a *compact and explicit expression*, providing the designer a handle to address transient performance and robustness issues to dominate undesirable friction and/or drag effects, even in the unactuated coordinates. Further, partial differential equations need not to be solved. A multivariable example and successful experiments on the Furuta's pendulum are reported. To the best of authors' knowledge it has the largest attraction basin experimentally tested so far.

1. INTRODUCTION

In applications where an operator is present the design of controllers is commonly carried out through the asymptotic stability of the reference velocities defined by the operator. Thus, operator's mission is to stabilize the system in a small neighborhood of the desired position. This occurs in piloted vehicles applications such as cars, ships and aircraft, where the pilot imposes the reference velocities. In aeronautics, stability and control augmentation systems are introduced to improve handling (or flying) qualities on the aircraft, defined in terms of those characteristics of the dynamic behavior of the aircraft that allow precise control with low pilot workload. In this way, control augmentation systems are sometimes additionally required to provide a particular type of response to pilot's set points, i.e., tracking of velocities and rates commands. The flight control system is mission critical because in some flight conditions the unstable mode is so fast that a pilot cannot stabilize the system. The flight control system should thus fulfill the dual task of stabilizing the aircraft without restricting the maneuverability unnecessarily. A typical example is the control of a VTOL aircraft where the roll angle and rate, and the horizontal and vertical velocities are the output feedback variables Bates and Postlethwaite [2002]. In this way, the proposed approach deals with the control of these piloted vehicles whose behavior is modeled by means of unstable underactuated mechanical systems motivated by Åström and S. [1997], Åkesson and Åström [2001] and Isidori [2003]. The applicability reveals a great benefit for understanding and testing the control of open–loop unstable aircraft. The maneuverability of these aircraft is higher in certain flight conditions. Unfortunately the unstable flight conditions include landing and take off. For the automatic control the presence of a pilot is a complication because the pilot may also drive the system unstable through manual control actions. The design of control strategies for such situations is a significant challenge as was pointed out in Stein [1989]. Thus, the complex dynamics induced by the joint pilot–&–aircraft makes the manufacturers encounter severe difficulties to the control problem. The usual solution adopted is to try to divide the available control authority between control and stabilization by means of hybrid control. In this approach a nonlinear controller is proposed whose nonlinear behavior decides the control authority autonomously.

We use the classical geometric framework of feedback linearization from Isidori [1995] to construct an output for the whole class. The main difference with the actual approaches is that the non–conservative forces are taken into account, like the friction and drag forces, which make the system be *non minimum–phase*. The usual approach/methodologies relies on solving a set of partial differential equations, like Controlled Lagrangians, IDA–PBC, Forwarding and first integrals, and in all of them the solution yields conservative quantities which in practise cannot be fulfilled. In the present approach the output redesign allows to dominate the non–conservative forces yielding a *minimum–phase system* and all based on singularly perturbed theory. Moreover the solution for the whole class is explicitly given in a compact form.

Research history and outline: in Acosta and López-Martínez [2007a] we showed an explicit solution of a class of underactuation mechanical systems of underactuation degree one. After that, in Acosta and López-Martínez [2007b] we generalize that result for the underactuation

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degree larger than one. Thus, in section 2, we briefly mention all necessary previous results of Acosta and López-Martínez [2007b], without proofs. Nevertheless, since most of results referenced here are an extension to the case of underactuated mechanical systems of underactuation degree one, we refer to readers to Acosta and López-Martínez [2007a]. Then, in Section 3 and 4, we extend the class solved in Acosta and López-Martínez [2007b] by means of the singularly perturbed theory, removing and relaxing some of the assumptions. Section 5 deals with the applications and examples and successful experimental results on the available Furuta pendulum are given. Finally, a conclusion section.

Notation: Throughout the paper all vectors are *column* vectors, even the gradient operator $\nabla_x = \frac{\partial}{\partial x}$, and the hessian $\nabla_x^2 = \frac{\partial^2}{\partial x^2}$. For vector functions $F : \mathbb{R}^n \mapsto \mathbb{R}^m$, we define the matrix $\nabla_x F(x) = [\nabla_x F_1(x), \ldots, \nabla_x F_m(x)]$. When clear from the context the subindex of the operator ∇ and the arguments of the functions will be omitted. For matrices $M \succ 0$, $M \in \mathbb{R}^{m\times n}$, means symmetric and positive definite. Acronyms used: r.h.s. means right hand side; i.e. means that is.

2. PRELIMINARY RESULTS

In order to present clearly the class of systems we address in this paper, we will star from a classical Lagrange's formulation for a general underactuated mechanical system. After that, we will impose the required assumptions to show the considered class. Thus, let $(q, \dot{q}) \in \mathbb{R}^n \times \mathbb{R}^n$ the generalized coordinates and velocities, respectively. We address the control problem of underactuated mechanical systems, i.e. there are fewer control inputs than degrees of freedom. The Lagrange's equations read

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + D(\dot{q}) + \nabla U(q) = [O\ I_m]^\top\ \tau \qquad (1)$$

where $M \in \mathbb{R}^{n\times n}$ is the symmetric and positive definite inertia matrix, $U \in \mathbb{R}$ is the potential function, $\tau \in \mathbb{R}^m$ the number of independent control inputs, the matrix $C \in \mathbb{R}^{n\times n}$ contains the Coriolis and centrifugal forces, and it can be calculated as

$$C(q,\dot{q})\dot{q} = \left[\nabla_q(M\dot{q}) - \frac{1}{2}\nabla_q(M\dot{q})^\top\right]\dot{q}, \qquad (2)$$

and the matrix $D(\dot{q}) \in \mathbb{R}^n$ is the vector of friction and/or drag forces. We now proceed to define the class of mechanical systems for which we can explicitly solve the control problem. In fact, the class considered refers to systems with underactuation degree $n - m$, $m \geq 1$. Then, we partition intuitively the set of generalized coordinates $q = (q_1, q_2) \in \mathbb{R}^{n-m} \times \mathbb{R}^m$, where the unactuated degree of freedom is represented by the q_1–coordinate and the actuated ones by the set of the generalized q_2–coordinates. After this partition the Lagrange's equations of motion can be written as

$$\begin{bmatrix} M_{11} & M_{12}^\top \\ M_{12} & M_{22}\end{bmatrix}\ddot{q} + \begin{bmatrix} F_1(q,\dot{q}) \\ F_2(q,\dot{q})\end{bmatrix} = \begin{bmatrix} 0 \\ I_m \end{bmatrix}\tau, \qquad (3)$$

where now $M_{11} \in \mathbb{R}^{(n-m)\times(n-m)}$, $M_{12} \in \mathbb{R}^{m\times(n-m)}$, $M_{22} \in \mathbb{R}^{m\times m}$ and we have introduced the scalar and vector functions $F_1(q,\dot{q}) \in \mathbb{R}^{n-m}$ and $F_2(q,\dot{q}) \in \mathbb{R}^m$, respectively. The first step of the approach presented here is to linearize partially the equations of motion (1), as done in Spong [1998] where it was called collocated partial feedback. Indeed, after some simple calculations following the ones given in Spong [1998], it is easy to see that the partially feedback–linearized system takes the affine in control form:

$$F(q,\dot{q}) \triangleq \begin{bmatrix} -M_{11}^{-1}F_1(q,\dot{q}) \\ O \end{bmatrix}, \quad G(q) \triangleq \begin{bmatrix} -M_{11}^{-1}M_{12}^\top \\ I_m \end{bmatrix}$$

$$\ddot{q} = F(q,\dot{q}) + G(q)u, \qquad (4)$$

and the vector function $F_1(q,\dot{q})$ was defined as

$$F_1(q,\dot{q}) \triangleq (I_{n-m}\ O)[C(q,\dot{q})\dot{q} + D(\dot{q}) + \nabla U(q)]. \qquad (5)$$

Remark 1. Notice that, from equations (1) to (4) some implicit assumptions were needed in Spong [1998], usually satisfied for simple mechanical systems, but we will put them in order anyway. On one hand, the matrix M_{11} need to be uniformly positive definite, and then invertible. On the other hand, the elements of the sub–matrix M_{12} are different from zero at least in a neighborhood at the origin. This fact was defined in Spong [1998] as **Strongly Inertially Coupled** rank condition and so we assume $\text{rank}(M_{12}) = n - m$.

The assumptions for the class were:

A.1 (Definition of the class)
- The elements of the *inertia matrix* M_{11} and M_{12} do not depend on the actuated coordinates and the matrix M_{22} either it is a constant matrix or a function of the actuated coordinates.
- The *potential function* is of the form
$$U(q) \triangleq V(q_1) + \vartheta(q_2). \qquad (6)$$
- The vector function defining *the nonlinear friction and/or drag forces* $D_1(\dot{q})\dot{q} \geq 0$ with $D_1(\dot{q}) \triangleq (I_{n-m}\ O)D(\dot{q})$, satisfies the linear growth bound only on the unactuated coordinates
$$\|D_1(\dot{q})\| \leq \gamma\|\dot{q}_1\|,\ \forall \dot{q}_1 \in \mathcal{D} \subseteq \mathbb{R}^{n-m} \qquad (7)$$
where $\gamma \geq 0$ is a positive constant.

A.2 (Rank condition) $m \geq (n - m)$.

A.3 (Underactuated coordinates) The unstable equilibria of the unactuated coordinates are isolated in Ω, i.e. $\nabla^2 V(q_{1\star}) < 0$ and $\nabla V(q_1) \neq 0$, $\forall q_1 \in \Omega \setminus \{q_{1\star}\}$.

A.4 (Integrability condition) Let $\nabla m_i = (\nabla m_i)^\top$ with $m_i(q_1)$, $i = 1...m$, are the rows of the matrix M_{12}.

The physical meaning of the above assumptions has been thoroughly described in Acosta and López-Martínez [2007b], and omitted here due to page limitations.

Under the above assumptions we proved among other results that the composite *smooth–static–feedback* controller given by

$$u = \Delta^{-1}(K_1(-M_{12}M_{11}^{-1}F_1 + \dot{M}_{12}\dot{q}_1) + K_2 M_{12}\dot{q}_1 - \nu) \qquad (8)$$

$$\nu = -K_3\left(M_{12}\dot{q}_1 + K_4\tilde{\eta}\right), \qquad (9)$$

with $\Delta \triangleq K_1 M_{12} M_{11}^{-1} M_{12}^\top - I_m$, and the constant and full–rank $(m \times m)$–matrices satisfying the conditions

C1. $K_1 \succ 0$, and such that the matrix $\bar{M}(q_1) \succ 0$ for all $q_1 \in \Omega \subseteq \mathbb{R}^{n-m}$, where $\bar{M} \triangleq M_{12}^\top K_1 M_{12} - M_{11}$;

C2. $K_2 \succ 0$ and $k_2 > \gamma$, with k_2 the minimal eigenvalue of the full–rank matrix $M_{12}^\top K_2 M_{12}$, $\forall q_1 \in \Omega$;

C3. $K_3 \succ 0$, and diagonal; and

C4. $K_4 \succ 0$,

ensure that, for all $(\mathcal{Z}, \eta) \in \Omega \times \mathcal{D} \times \mathbb{R}^m$, the closed–loop system is partially state feedback input–output linearizable through the output [1]

$$\tilde{\eta} \triangleq [K_1 M_{12} \; I_m]\dot{q} + K_2 \int_{q_{1\star}}^{q_1} M_{12}(\mu) \cdot d\mu - \eta_\star \tag{10}$$

In fact, the Assumptions A.1 and A.2 played a critical role in the stabilization problem and we called to this class of systems the "constructive class". The requirements imposed by these assumptions allowed us to find out an energy–like Lyapunov function for the zero dynamics. Thus, these conditions does not play a key role for the (local) stabilization and then we use this fact to extend the class. We will show that the proposed controller renders a (local) asymptotical and exponential stability of the desired equilibrium of the extended class, i.e. the singularly–perturbed one. In some sense, we can say that the controller designed for the "constructive" class is robust to the nonlinearity introduced by this "new" class.

Remark 2. Let us denote throughout the paper, for compactness and also to help the reader to identify quickly the zero dynamics, the state space of these zero dynamics as $\mathcal{Z} = (q_1, \dot{q}_1)^\top$ and its equilibrium as $\mathcal{Z}_\star$.

The control problem. Recalling that the main control objective is to maneuver manually the set point in aircraft flying, i.e. a pilot manual operation, then the state space will be, from now on, $(\mathcal{Z}, \eta)^\top$ and, in this way the control problem to be addressed, at first, will be to stabilize the equilibrium $(\mathcal{Z}_\star, \eta_\star)^\top$. Other control problems will be posed further.

3. THE SINGULARLY PERTURBED CLASS

In this section we prove that through the same output (10) the equilibrium of the extended class, i.e. the so–called *singularly–perturbed class* can be also stabilized. Then, this singularly–perturbed class relax the assumptions A.1 and A.2 in the following sense:

A.1' on one hand, in A.1 the matrix M_{22} can be function of the unactuated coordinates; and

A.2' on the other hand, the A.2 is completely removed.

Only an additional assumption is needed, to generalize this result:

Assumption A.5 $\left[\frac{\partial M_{22}(q_1)}{\partial q_1}\right]_{q_1 = q_{1\star}} = 0.$

The above Assumption is necessary to assure that the desired equilibrium of the zero dynamics $\mathcal{Z}_\star$ is not modified at least locally, for an arbitrary set point $\eta_\star$. It is also noticeable that in the particular case of regulation at $\eta_\star = O$ *this assumption is not needed at all*, but we will state the general case.

Proposition 3. Consider the underactuated mechanical system (4) under the Assumptions A.1–A.5 with the relaxations made in A.1' and A.2', and with the matrices K_i, $i = 1, ..., 4$ satisfying the conditions C1–C4. Then, the following assertions hold:

(i) the closed–loop system is partially state feedback input–output linearizable in the set $(\mathcal{Z}, \eta) \in \Omega \times \mathcal{D} \times \mathbb{R}^m$ through the output (10). The smooth–static linearizing controller is given by (8).

(ii) the external controller (9) assures that there exits a real number $k_3^\star > 0$ such that, for $k_3 > k_3^\star$ the systems is singularly perturbed and so, all trajectories starting in a compact ball remain bounded, where k_3 stands for the maximal eigenvalue of the matrix K_3. Moreover, the desired equilibrium $(\mathcal{Z}_\star, \eta_\star)$ is (locally) asymptotically (exponentially) stable.

Proof. The first claim follows directly following the usual procedure of calculating the derivative of (10), and becomes

$$\begin{aligned}\dot{\tilde{\eta}} &= [K_1 M_{12} \; I_m]\ddot{q} + K_1 \dot{M}_{12}\dot{q}_1 + K_2 M_{12}\dot{q}_1 \\ &= -K_1 M_{12} M_{11}^{-1} F_1 - \Delta u + K_1 \dot{M}_{12}\dot{q}_1 + K_2 M_{12}\dot{q}_1 = \nu,\end{aligned}$$

and isolating the controller u we get (8).

To prove the second claim, we first need to prove that the zero dynamics associated to the proposed output is locally exponentially stable. The zero dynamics can be obtained after expanding the first $(n - m)$ equations of (4) using the controller derived in (8) with $\nu = 0$. Thus, tedious but straightforward calculations show that under the assumptions made the linearized zero dynamics round the equilibrium becomes

$$\bar{M}_\star \delta\ddot{q}_1 + (M_{12}^\top K_2 M_{12} - \nabla D_1 + \Gamma)_\star \delta\dot{q}_1 - \nabla^2 V_\star \delta q_1 = O,$$

where we defined the matrix $\Gamma \triangleq \nabla(M_{12}^\top \dot{q}_2) - \nabla(M_{12}^\top \dot{q}_2)^\top$ and the variables for the linearization of the zero dynamics as $\delta q_1 \triangleq q_1 - q_{1\star}$, and we used the notation $(\cdot)_\star$ for the matrices evaluated at the equilibrium $\mathcal{Z}_\star$. Since by C2 we fix $M_{12}^\top K_2 M_{12} > \gamma$ and by assumption A.1 $\|\nabla D_1\| < \gamma$ then the the damping matrix, i.e. $M_{12}^\top K_2 M_{12} - \nabla D_1 \succ 0$. Additionally, $\bar{M}_\star \succ 0$ and by A.3 $(-\nabla^2 V)_\star \succ 0$. Thus, since Γ is skew–symmetric and all the remaining matrices are positive definite then, from the results of linear theory Roseau [1987], Merkin [1996] we know that for these second order and linear dynamics the equilibrium $\mathcal{Z}_\star$ of the zero dynamics is locally asymptotically and exponentially stable [2].

Now, we are in position to prove the last part regarding to the singular perturbations theory, using the external controller given by (9), just invoking the result given in [Khalil, 2002, Th. 11.4] [3]. For, we rewrite the closed–loop dynamics qualitatively in the following way

$$\dot{\mathcal{Z}} = \psi(\mathcal{Z}, \tilde{\eta}) \tag{11}$$

$$\epsilon\dot{\tilde{\eta}} = -(O \; M_{12})\mathcal{Z} - K_4\tilde{\eta}, \tag{12}$$

where $\psi(\cdot)$ is a smooth function, the small parameter $\epsilon \triangleq 1/k_3$ was defined as the minimal eigenvalue of the matrix K_3^{-1}, which is the corresponding maximal eigenvalue of the matrix K_3. When $\epsilon = 0$ the roots of (12) are $\tilde{\eta} = h(\mathcal{Z}) = -K_4^{-1} M_{12}\dot{q}_1$ with $h(O) = O$. Thus, the boundary–layer system can be obtained defining the variable $y \triangleq \tilde{\eta} - h(\mathcal{Z})$, and then the equations (11)–(12) in the new coordinates become

[1] Notice that, since q_1 is a vector we have introduced some abuse of notation, only for compactness. See the TCP example further to clarify the computation in the case of a MIMO system.

[2] Locally Exponentially Stable (LES).

[3] This result is an extension to infinite interval of time of the well–known Tikhonov's Theorem.

$$\frac{d\mathcal{Z}}{dt} = \psi(\mathcal{Z}, y - h(\mathcal{Z})) \tag{13}$$
$$\frac{dy}{d\tau} = -K_4 y + \mathcal{O}(\epsilon), \tag{14}$$

with $\tau = (t - t_0)/\epsilon$ the new time scale. First notice that, on the one hand the reduced system ($\epsilon = 0$), in this case, is exactly the zero dynamics which we actually know that its equilibrium is LES, and, on the other hand the so–called boundary–layer system is described by (14) and its equilibrium is also LES uniformly in $\mathcal{Z}$. Therefore all the conditions from [Khalil, 2002, Th. 11.4] are satisfied, and then there exists an $\epsilon^* > 0$ such that for all $\epsilon < \epsilon^*$ the equilibrium of the entire system is LES on the compact ball where all the conditions are satisfied.

Remark 4. Notice that, in the proposition 3, we could not invoke the well–known results from Isidori [1995] because we cannot use the crucial fact that the dynamics on the $\tilde{\eta}$–coordinate is linear, and so neither in the ν–coordinate.

4. ASYMPTOTIC OUTPUT TRACKING

The propositions 3 assure that, at least, the equilibrium of the whole system is LES. This result is very promising because that means certain robustness to small disturbances. In this way, with a slight modification of the external controller (9) we can also assure a bounded asymptotic output tracking to a prescribed reference function $\eta_\star(t)$. We state this result with only a sketch of the proof because is based on the one given in [Isidori, 1995, Prop. 4.5.1].

Proposition 5. Consider the underactuated mechanical system (4) under the Assumptions A.1–A.5 with the relaxations made in A.1' and A.2', and with the matrices K_i, $i = 1, ..., 4$ satisfying the conditions C1–C4. Suppose further that $\eta_\star(t)$ and $\dot{\eta}_\star(t)$ are defined for all $t \geq 0$ and bounded. Then, the smooth–static–feedback controller (8) and the external controller

$$\nu = \text{r.h.s. } (9) - K_3 K_4 K_3 \int_{q_{1\star}}^{q_1} M_{12}(\mu) \cdot d\mu + \dot{\eta}_\star(t) \tag{15}$$

assure a bounded asymptotic output tracking Isidori [1995].

Proof. [Sketch] We define a new output $\chi \triangleq \tilde{\eta} + K_3 \int_{q_{1\star}}^{q_1} M_{12}(\mu) d\mu$, which is a (local) diffeomorphism to $\tilde{\eta}$, and then the external $\tilde{\eta}$–dynamics becomes $\dot{\chi} = -K_3 K_4 \chi$, remaining the zero dynamics unchanged. Since the closed–loop system is in the form given in [Isidori, 1995, Prop. 4.5.1] and all the conditions stated there are satisfied then same arguments hold here.

5. EXAMPLES AND EXPERIMENTS

We test the approach via simulations in one example and via experiments in the available laboratory equipment. Since, the applicability of the approach include also multi input/output systems we include a two–coupled pendula as an example, while the laboratory equipment that has been used for the experiments is the Furuta pendulum. It should be notice that the two–coupled pendula is of underactuation degree two in contrast with the last application we propose, the rotary pendulum or so–called the Furuta pendulum, which is of underactuation degree one.

In addition, when designing controllers for model–based underactuated systems usually the proposed controllers are based on models neglecting the friction forces, even in the unactuated coordinates. However, in Gómez-Estern et al. [2004] the importance of the friction to stabilize this system was shown. So, in the approach proposed here, the friction forces can be taken into account easily in the dynamical model. Notice that we only are interested to stabilize the upper position of the pendulum, not to swing-up from any position, whose solution was reported in Acosta et al. [2001] and Gordillo et al. [2003]. To the best of the authors knowledge the solution proposed by singular perturbations enlarges the largest region of attraction obtained so far, even with friction. In theory, this solution could stabilize the upper position from any point over the upper half plane. The excellent performance and the large region of attraction has been tested in the actual laboratory pendulum and reported here. Table 1 presents a summary of the relevant physical parameters of the systems related to the potencial function and the inertia matrix. The outputs η, internal control laws u and external controllers ν are obtained by substituting in the equations (10), (8) and (9), respectively. Otherwise, the change of coordinates given by $(\mathcal{Z}, \eta) \rightsquigarrow (\mathcal{Z}, \dot{X})$ is a (global) diffeomorphism we present the examples in the latter intuitive physical coordinates.

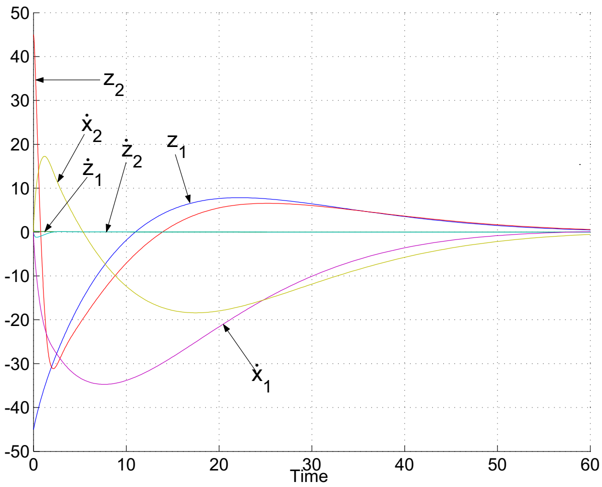

Fig. 1. Simulations of Two–Coupled Pendula.

5.1 *The Two–Coupled Pendula (TCP)*

This system consists of a platform than can be moved on the plane x-y, and two pendula which are mounted on the platform. The pendula are positioned with an angle shift of 90^o respect to z-axis and orientated respect to the x-axis with an angle of 30^o. In table 1, the following parameters have been defined $a = \cos 30^o$, $b = \sin 30^o$, the masses of the pendula are $m = 1\ kg$, the lengths are $l_1 = 10\ m$, $l_2 = 1\ m$, the viscous friction constants of the unactuated degrees of freedom are $F = 1\ N{\cdot}m/(rad/s)$ and $g = 10\ m/s^2$. The system has 4 degrees of freedom, the x and y coordinates and the angles of each pendulum respect to the z-axis, φ_1 and φ_2. This platform can be actuated with two forces orientated in x-axis and y-axis respectively. In this way, the system has two unactuated degrees of freedom φ_1 and φ_2, which will be denoted as z_1 and z_2 respectively. The control gains have been

Dynamical System	DOF n	Potential $U(q)$: $V(q_1)$	Potential $U(q)$: $\vartheta(q_2)$	Inertia Matrix $M(q)$: $M_{11}(q_1)$	Inertia Matrix $M(q)$: $M_{12}^{\top}(q_1)$	Inertia Matrix $M(q)$: $M_{22}(q)$
TCP	4	$mg(l_1cz_1 + l_2cz_2)$	-	$m\begin{bmatrix} l_1^2 & 0 \\ 0 & l_2^2 \end{bmatrix}$	$m\begin{bmatrix} l_1cz_1a & l_1cz_1b \\ l_2cz_2b & l_2cz_2a \end{bmatrix}$	$(M+2m)\ I_2$
FP	2	$mglcz$	-	J_p	$mrlcz$	$J_a + mr^2 + J_ps^2z$

Table 1. Physical Parameters of the Systems

selected as follows: $K_1 = 30I_2$, $K_2 = 100I_2$, $K_3 = I_2$ and $K_4 = I_2$. The redesign of the output has been calculated by introducing the following line integral in (10) as

$$\int_{(z_1,z_2)_\star}^{(z_1,z_2)} M_{12}d\mu = [a\sin(z_1) + b\sin(z_2),\, b\sin(z_1) + a\sin(z_2)]^\top$$

5.2 Experimental results on the Furuta pendulum (FP)

The model of the rotary pendulum, or so–called Furuta pendulum, used in this article is thoroughly described in Acosta et al. [2001] and Gordillo et al. [2003]. The values of the physical parameters–summarized in table 1–used in the experimental framework are: mass of the pendulum = 0.0679 (Kg), length of the pendulum = 0.28 (m), radius of the arm = 0.235 (m), mass of the arm = 0.2869 (Kg), constant torque = 7.4, moment of inertia of the motor = 0.0012(Kg· m^2). The full control system is shown in Fig. 2. The laboratory electro-mechanical system is composed by: a DC motor (15 Nm / 2000 rpm) with tachometer that measures the speed of the arm; a power supply (50 VA); a PWM servo-amplifier; a pendulum; an encoder that measures the angle of the pendulum and a slip ring that drives the signal to the base. The control system is composed by: a monitor PC with a target (DS1102) for control based on DSP (TMS320C31)and a software (DSPACE) for control, monitoring and supervisor. In this practical case, since in this approach the friction in the actuated coordinates can be compensated then, we include a non–linear compensator based on the LuGre model de Wit et al. [1995] to dominate the friction forces of the arm of the pendulum, in the linearizing controller u. The controller gains for the experiments were $K_1 = 100$, $K_2 = 500$, $K_3 = 10$ ($\epsilon = 0.1$, from proposition 3) and $K_4 = 25$. In figure 4 the initial conditions for the angular position of the pendulum was $z(0) = 1.45$ rad. To the best of our knowledge this is the largest region of attraction achieved in experimental results to stabilize this kind of pendulum. An approximation to the maximum theoretical value of $z(0)$ can be obtained through the condition $\det(\Delta(z)) = 0$, i.e. $K_1(mrl/J_p)\cos^2 z = 1$, whose value for the K_1 given is 1.5 rad. In fact, we also would like to compare with the maximum theoretical values obtained by passivity methods. In Bloch et al. [1999] the maximum theoretical z is given by the equation $|z| \leq \arcsin(\sqrt{r^2/(r^2+l^2)})$. In Viola et al. [2007] the condition which gives the largest admissible angle $|z| \leq \arccos(1 + (mrl/J_p)^2)^{-\frac{1}{2}}$. In both cases the formulas are not tunable, since depends only on the physical parameters, and gives rise in our pendulum to a maximum $z(0) = 0.7$ rad and $z(0) = 0.9$ rad respectively, which are approximately the half of the value presented above. In Fig. 3 we show a schematic of the experimental results obtained with the approach given $z(0) = 1.45$ rad; with the passive approach $z(0) = 0.82$ rad; and with an Linear Quadratic Regulator (LQR) where $z(0) = 0.75$ rad. The similar results obtained with the LQR and with the passive method are due to the friction forces which make the close–loop system non passive as commented before (see Gómez-Estern et al. [2004], Woolsey et al. [2004]). Figure 4 shows an experiment where the initial conditions for velocities are not near to zero. From the theoretical point of view, the region of attraction could tend to the horizontal position of the pendulum, by increasing K_1. Unfortunately, the system saturates and it was not possible enlarge more this practical region of attraction. The saturation of the input control is shown for both experiments in the figures. Notice that even with saturation the objective was achieved. We also report and experiment tracking an sinusoidal reference in Fig. 5 with the controller given in the proposition 5.

Fig. 2. Laboratory pendulum.

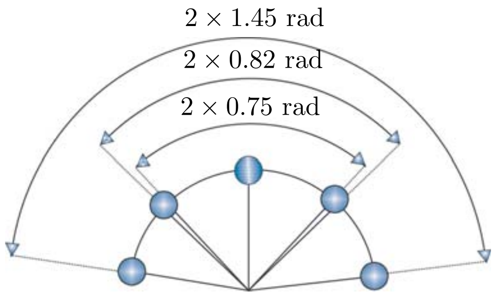

Fig. 3. Experimental basin of attraction.

6. CONCLUSIONS

The complex dynamics induced by the joint of the pilot–&–aircraft can be described by means of unstable underactuated mechanical systems, where the unactuated part describes the autonomous aircraft dynamics and the actuated one the piloted. This paper presents an easy

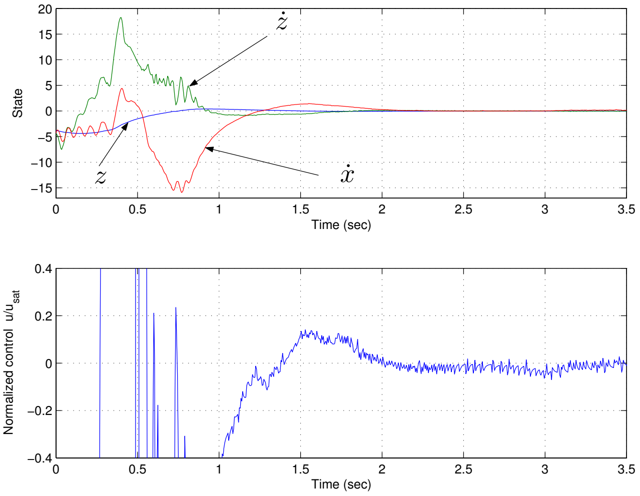

Fig. 4. Furuta Pendulum: regulation experiment.

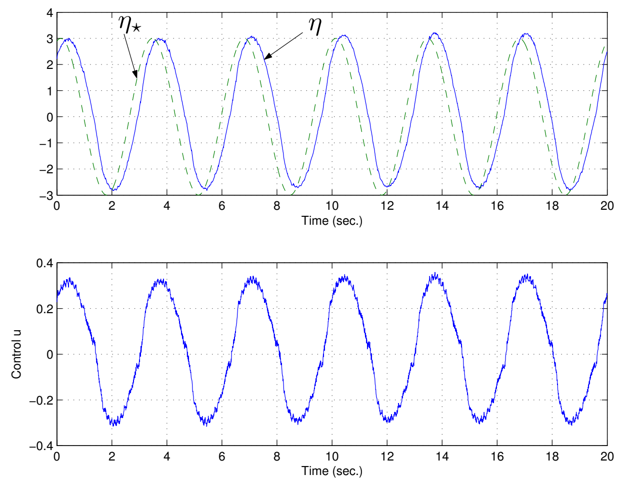

Fig. 5. Furuta Pendulum: tracking experiment.

constructive output to control unstable underactuated mechanical systems with underactuation degree larger than one, yielding a great benefit for understanding and testing strategies to control the unstable aircraft. The approach is supported by singular perturbations theory and yields an explicit controller for a whole class of systems. A multi input/output example with underactuation degree larger than one and successful experimental results on the Furuta pendulum are given. The latter presents, to the best of authors' knowledge, the largest attraction basin experimentally tested so far, which has been compared, via experiments, with another linear and non linear controllers.

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