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A probabilistic analysis of the average consensus algorithm with quantized communication

Ruggero Carli * **Fabio Fagnani** ** **Paolo Frasca** **
Sandro Zampieri *

* *Department of Information Engineering, Università di Padova, Via Gradenigo 6/a, 35131 Padova, Italy* {carlirug,zampi}@dei.unipd.it
** *Dipartimento di Matematica, Politecnico di Torino, Corso Duca degli Abruzzi, 24, 10129 Torino, Italy*
{paolo.frasca,fabio.fagnani}@polito.it

Abstract: In the average consensus problem the states of a set of agents, linked according to a directed graph, have to be driven to their average. When the communication between neighbors is uniformly quantized, such a problem can not be exactly solved by a linear time-invariant algorithm. In this work, we propose a probabilistic estimate of the error from the agreement, in terms of the eigenvalues of the evolution matrix describing the algorithm.

1. INTRODUCTION

In the last years, an increasing interest in the control theory community has been devoted to the study of the so-called consensus problems.

In the average consensus problem a set of linear systems has to be driven to the same final state, equal to the average of their initial states. This mathematical problem can be seen as the simplest example of coordination task. In fact it can be used to model either the control of multiple autonomous vehicles which all have to be driven to the centroid of the initial positions Cortés et al. [2006], or the decentralized estimation of a quantity from multiple measures coming from distributed sensors Xiao et al. [2005], or the load balancing procedure between processors Cybenko [1989], Elsässer et al. [2006]. Thanks to the work of Tsitsiklis [1984], Jadbabaie et al. [2003], Olfati-Saber and Murray [2004], Moreau [2005], Olfati-Saber et al. [2007], we are now able to solve the average consensus problem with mild conditions on the graph and on the evolution matrix.

Our contribution presents a consensus strategy in which the systems can exchange information through a time invariant strongly connected digital communication network. Beside the decentralized computational aspects induced by the choice of the communication network, here we also have to face the quantization effects due to the digital links. Similar ideas are already known in the control community, thanks to many works on quantized control and to the branch of the consensus literature, which studies systems whose states are themselves quantized Kashyap et al. [2007], Elsässer et al. [2006]. Compared to these works, anyway, our approach is different because we suppose that the agents' states are real numbers, but they can only exchange quantized information: communicated information will be quantized by a uniform quantizer. With this goal, we assume the links to allow the communication of integer numbers and we study a simple adaptation of the classical diffusion algorithm which is able, with such a constraint, to preserve the average of states and to drive the system near to the consensus. Its performance is thus defined in terms of the (asymptotical) distance of the states from the average of the initial conditions. A special attention is devoted to the scalability in N of its features. This algorithm, which is detailed in Section 2, has been first introduced in Carli et al. [2007a]: in this note we develop a probabilistic analysis of it, based on modelling the quantization error as an additive random noise. It comes out that the expected behavior depends only on the assumed distribution of the quantization errors and on the spectral properties of the evolution matrix.

1.1 Preliminaries and notations

In this section we collect some definitions and notations which are used through the paper: the reader can refer to Godsil and Royle [2001], Gantmacher [1959] for further readings.

The communications between agents are modelled by a *directed graph* $\mathcal{G} = (V, E)$. $V = \{1, \ldots, ...N\}$ is the set of vertices and E is the set of (directed) edges, i.e. a subset of $V \times V$. We say that the vertices i and j are communicant, or connected, if $(j, i) \in E$. This means that j can transmit information about its state to i. In this case we also say that j is a neighbor of i. The *adjacency matrix* A of $\mathcal{G}$ is a $\{0, 1\}$-valued square matrix indexed by the elements in V defined by letting $A_{ij} = 1$ if and only if $(j, i) \in E$ and $j \neq i$. Define the *in-degree* of a vertex i as $\sum_j A_{ij}$ and the *out-degree* of a vertex j as $\sum_i A_{ij}$. A graph is called *in-regular* (*out-regular*) of degree k if each vertex has in-degree (out-degree) equal to k. A graph is said a *undirected* (or *symmetric*) graph if $(i, j) \in E$ implies that $(j, i) \in E$. A graph is *strongly connected* if for any given pair of vertices (i, j) there exists a path (i.e. an ordered list of edges) which connects i to j. It is said to be *fully connected* or *complete* if for any couple of vertices there exists an edge joining them.

A matrix $M \in \mathbb{R}^{N\times N}$ is said *compatible* or *supported* by the graph $\mathcal{G}$ if $M_{ij} > 0$ implies $(j,i) \in E$. Given the matrix M, we can define an induced graph $\mathcal{G}_M$ by taking N nodes and putting an edge (j,i) in E if $M_{ij} > 0$. A matrix is said to be *nonnegative* if $M_{ij} \geq 0$ for all i and j, and is said *doubly stochastic* if it is nonnegative and the sums along each row and column are equal to 1. We remember that a matrix M is *normal* if $M^TM = MM^T$, or equivalently if it can be represented by a diagonal matrix with respect to a properly chosen orthonormal basis of $\mathbb{C}^N$.

Now we give some notational conventions. Given a matrix $M \in \mathbb{R}^{N\times N}$, $\mathrm{diag}(M)$ means a diagonal matrix with the same diagonal elements of the matrix M and $\mathrm{out}(M) = M - \mathrm{diag}(M)$. We denote by $\rho(M)$ the spectral radius of M: $\rho(M) = \max\{|\lambda| : \lambda \in \sigma(M)\}$, where $\sigma(M)$ is the set of the eigenvalues of M (its spectrum). When the matrix is doubly stochastic, it is also worth to define the *essential* spectral radius as $\rho_{ess}(M) = \max\{|\lambda| : \lambda \in \sigma(M) \setminus \{1\}\}$.

2. PROBLEM STATEMENT

In the standard consensus algorithm we have that the agent i updates its state according to the formula

$$x_i(t+1) = \sum_{j=1}^{N} P_{ij}x_j(t),$$

More compactly we can write

$$x(t+1) = Px(t), \tag{1}$$

where $x(t)$ is the column vector with entries $x_i(t)$ and P is the matrix with entries P_{ij}. The matrix P, called *evolution matrix*, or *diffusion matrix*, has to be adapted to the graph describing the communication network. From now on we will assume that P satisfies the following condition.

Hypothesis 1. P is a doubly stochastic matrix with positive diagonal and with $\mathcal{G}_P$ strongly connected.

It is well known in the literature Olfati-Saber and Murray [2004], Carli et al. [2006], that, if Hypothesis 1 holds, then the algorithm (1) solves the average consensus problem, namely

$$\lim_{t\to+\infty} x(t) = \frac{1}{N}\left(\sum_{i=1}^{N} x_i(0)\right)\mathbf{1}, \tag{2}$$

where $\mathbf{1}$ is the (column) vector of all ones. This can be shown because the previous conditions imply that

(A) 1 is the only eigenvalue of P on the unit circle centered in 0;
(B) the eigenvalue 1 has algebraic multiplicity one and $\mathbf{1}$ is its eigenvector;
(C) all the other eigenvalues are strictly inside the unit disk centered in 0.

Differently from most of the literature, in this work we assume that the communication network is constituted only of digital links. This implies that perfect exchange of information between the agents is not allowed. In fact, through a digital channel, the j-th agent can only send to the i-th agent symbolic data. In general, since there is a lack of communication, no algorithm is expected to converge in the usual sense (2): we will provide a suitable definition of convergence later in this paragraph.

To deal with the uniform quantization constraint, up to rescaling, we can suppose that the agents exchange integer numbers: these numbers are the integer approximation of the agents' states. Then the quantizer is $q : \mathbb{R}^d \to \mathbb{Z}^d$ and map each component of x into the nearest integer.

We propose the evolution scheme

$$x_i(t+1) = x_i(t) - q(x_i(t)) + \sum_{j=1}^{N} P_{ij}\, q(x_j(t)) \tag{3}$$

or, more concisely,

$$x(t+1) = x(t) + (P - I)\, q(x(t)). \tag{4}$$

Proposition 2. Let $x_a(t) = \frac{1}{N}\left(\sum_{i=1}^{N} x_i(0)\right)$. Then

$$x_a(t) = x_a(0).$$

Proof. Since P is doubly stochastic we have that

$$\begin{aligned}&x_a(t+1) = N^{-1}\mathbf{1}^T x(t+1) =\\ &N^{-1}\mathbf{1}^T x(t) + N^{-1}\mathbf{1}^T(P-I)\, q(x(t)) =\\ &N^{-1}\mathbf{1}^T x(t) = x_a(t),\end{aligned}$$

for all t. □

That is, the algorithm (4) preserves the average of the initial conditions.

We have proved that if the algorithm (4) converges, it converges to average of the initial states, but we still have to study its convergence properties. In general, since the agents do not have an exact knowledge of their neighbors' states, the system is not expected to converge in the sense (2). Instead, the states will reach a neighborhood of the hoped consensus value. Then, our definition of performance lies on the (asymptotical) distance from the average agreement, normalized by the number of agents. We define the performance index to be the limit in time of a scaled norm of the disagreement vector $\Delta(t) = Y\,x(t)$, with $Y = I - N^{-1}\mathbf{1}\mathbf{1}^T$. Notice that $\Delta(t) = x(t) - x_a(t)\mathbf{1} = x(t) - x_a(0)\mathbf{1}$. We then define

$$d(P, x(0)) = \limsup_{t\to\infty} \frac{1}{\sqrt{N}}||\Delta(t)||_2.$$

If we suppose to know the statistics of $x(0)$, then we can take the expectation with respect to the initial conditions,

$$d(P) = \mathbb{E}[d(P, x(0))].$$

In any case, it's possible to consider

$$d_\infty(P) = \sup_{x(0)} d(P, x(0)).$$

It's trivial to remark that $d(P) \leq d_\infty(P)$.

We start with a remark about the best achievable performance. It is clear that, when all states lie in the same quantization interval, that is $q(x_i(t)) = Q$ for all i, differences are not perceivable and states do not evolve. Therefore the best it can happen is that the algorithm reaches such an equilibrium in which $q(x_i(t)) = Q$ for all i and for all $t \geq T$. In this case we can only argue that $|\Delta_i(t)| \leq 1$ for all $t \geq T$.

Then the algorithm will at best assure the convergence into an interval of unitary size: $\forall\, \epsilon < 1$ initial conditions can be found such that $\|\Delta(t)\|_\infty > \epsilon \quad \forall t > 0$. Unfortunately, in several cases the error from the agreement can be much bigger.

We shall see in the sequel that it's possible to find an upper bound on the performance index, which is independent of the initial conditions, but eventually dependent on the number of agents.

3. WORST CASE BOUNDS

Studying the exact evolution of (4) is very difficult, because of nonlinearities induced by $q(\cdot)$, but it is possible to obtain some upper bound on $d_\infty(P)$ using a *worst case approach*. The results of this section are stated without proof: they will appear in a forthcoming paper Frasca et al. [2008].

Proposition 3. Define $e(t) = x(t) - q(x(t))$. Then the time evolution of $\Delta(t)$ is given by the recursion

$$\Delta(t+1) = P\Delta(t) + (I-P)e(t),$$

which gives

$$\Delta(t) = P^t\Delta(0) + \sum_{s=0}^{t-1} P^s(I-P)e(t-s-1). \tag{5}$$

Remark that by definition $||e(t)||_\infty \leq \frac{1}{2}$,and then $||e(t)||_2 \leq \frac{1}{2}\sqrt{N}$. Then, forgetting about the origin of $e(t)$, we can just consider (5) as an asymptotically stable linear dynamical system with an unknown, but bounded, forcing term.

The following results, based on this approach, state that the asymptotic error of the algorithm is bounded, and the bound does not depend on the initial conditions, but depends, in general, on P and, namely, on N.

Proposition 4. Consider the evolution equation (4). Then

$$d_\infty(P) \leq \frac{\|I-P\|_2}{2} \sum_{t=0}^{\infty} \|PY\|_2^t. \tag{6}$$

where the series converges if the Hypothesis 1 holds.

Theorem 5. If P is a normal matrix, then

$$d_\infty(P) \leq \frac{1}{2}\sum_{s=0}^{+\infty} \rho\left(P^s(I-P)\right) \leq \frac{1}{1-\rho_{ess}(P)}. \tag{7}$$

Since we are interested in studying the dependence of the performance on N, we shall consider a sequence of matrices P_N of dimension N: in general their essential spectral radius is a function of N, which we denote ρ_N. Consequently, the above theorem gives a bound which depends on N.

For some sequences, it is known that there exists $0 < B < 1$ such that $\rho_N \leq B < 1$ uniformly in N. Then, (7) gives a bound uniform in N,

$$\sup_{N\in\mathbb{N}} d_\infty(P_N) \leq \frac{1}{1-B}.$$

Instead, there are other interesting sequences, like those of Cayley graphs with fixed degree k studied in], for which $\rho_N \to 1$ for $N \to \infty$ at polynomial speed, and the bound (7) is polynomial in N. In such cases, a deeper analysis is needed to improve this bound.

Lemma 6. Let P be a symmetric matrix such that

$$\sigma(P) \setminus \{1\} \subset [L, U] \quad -1 < L \leq 0 \leq U < 1.$$

Then

$$\sum_{s=0}^{+\infty} \rho\left(P^s(I-P)\right) \leq \frac{4}{1+L} + \frac{1}{2}\log\left(\frac{1}{1-U}\right). \tag{8}$$

We shall see in Section 5 how this lemma allows to improve, asymptotically in N, the estimate given by (7), in many examples.

4. PROBABILISTIC RESULTS

In this section, instead of considering the worst case with respect to the quantization errors, we suppose that we have statistical information on $e(t)$. We consider it to be a random variable with bounded support and we perform a mean square analysis similar to that in Xiao et al. [2007] and Carli et al. [2007a]. To do that, we define a new performance index, which is likely to show some features of the original one.

Let $n(t)$ be a stochastic process such that $n_i(t)$ are i.i.d. random variables of zero mean and known variance σ^2 and have their supports inside $[-1/2, 1/2]$. Let $\tilde{P} = PY$ and

$$\tilde{\Delta}(0) = \Delta(0)$$
$$\tilde{\Delta}(t+1) = \tilde{P}\tilde{\Delta}(t) + (I-P)n(t).$$

The performance is given by

$$d^r(P) = \limsup_{t\to\infty} \sqrt{\frac{1}{N}\mathbb{E}[\|\tilde{\Delta}(t)\|_2^2]}.$$

Remark that $d^r(P)$ should in principle depend on $x(0)$, too. We did not write $d^r(P, x(0))$ because the following theorem proves that there is no such dependence.

Theorem 7. Let P be a matrix satisfying Hypothesis 1. Then

$$[d^r_\infty(P)]^2 = \frac{\sigma^2}{N}\operatorname{tr}\left[(I-P)(I-P)^*\left(I-\tilde{P}^*\tilde{P}\right)^{-1}\right].$$

where $\tilde{P} = PY$. In particular, if P is normal, and $\sigma(P) = \{1, \lambda_1, \ldots, \lambda_{N-1}\}$ denote the spectrum of P, we have that

$$[d^r(P)]^2 = \frac{\sigma^2}{N}\sum_{i=1}^{N-1}\frac{|1-\lambda_i|^2}{1-|\lambda_i|^2}. \tag{9}$$

Proof. Define $Q(t) = \mathbb{E}[\tilde{\Delta}(t)\tilde{\Delta}(t)^*]$, and remark that $\frac{1}{N}\mathbb{E}[||\tilde{\Delta}(t)||^2] = \frac{1}{N}\operatorname{tr} Q(t)$. Using the facts that $Y^k = Y$ for all positive integer k and $Y(P-I) = P-I$ it is easy to see that $\tilde{\Delta}$ satisfies the following recursive equation

$$\tilde{\Delta}(t+1) = \tilde{P}\tilde{\Delta}(t) + (P-I)n(t).$$

Now, thanks to the hypotheses on $n_i(t)$,

$$\begin{aligned}Q(t+1) &= \mathbb{E}[\tilde{\Delta}(t+1)\tilde{\Delta}(t+1)^*] \\ &= \mathbb{E}[\tilde{P}\tilde{\Delta}(t)\tilde{\Delta}(t)^*\tilde{P}^*] + (I-P)\mathbb{E}[n(t)n(t)^*](I-P)^* \\ &= \tilde{P}\,Q(t)\,\tilde{P}^* + \sigma^2(I-P)(I-P)^*,\end{aligned}$$

and by a simple recursion

$$Q(t) = \tilde{P}^t\,Q(0)\,(\tilde{P}^*)^t + \sigma^2\sum_{s=0}^{t-1}\tilde{P}^s(I-P)(I-P)^*(\tilde{P}^*)^s.$$

Recall now that, since P satisfies Hypothesis 1, then $\rho_{ess}(P^*P) < 1$. Moreover we have that $\rho(\tilde{P}) = \rho_{ess}(P) <$

1 and $\rho(\tilde{P}^*\tilde{P}) = \rho_{ess}(P^*P) < 1$. Using the linearity and cyclic properties of the trace,

$$\operatorname{tr} Q(t) = \operatorname{tr}\left[\tilde{P}^t\, Q(0)\, (\tilde{P}^*)^t\right] + $$
$$+ \operatorname{tr}\left[\sigma^2 \sum_{s=0}^{t-1} (PP^* - (P+P^*) + I)(\tilde{P}^*\tilde{P})^s\right]$$
$$= \operatorname{tr}\left[\tilde{P}^t\, Q(0)\, (\tilde{P}^*)^t\right] +$$
$$+ \sigma^2 \operatorname{tr}\left[(PP^* - (P+P^*) + I)(1-(\tilde{P}^*\tilde{P})^t)(1-\tilde{P}^*\tilde{P})^{-1}\right].$$

and hence

$$\lim_{t\to\infty} \operatorname{tr} Q(t) = \sigma^2 \operatorname{tr}\left[(PP^* - (P+P^*) + I)(1-\tilde{P}^*\tilde{P})^{-1}\right].$$

If moreover P is normal, we can find a unitary matrix O of eigenvectors and a diagonal matrix of eigenvalues Λ, such that $P = O\Lambda O^*$. This implies

$$\operatorname{tr}\left[(PP^* - (P+P^*) + I)(1-\tilde{P}^*\tilde{P})^{-1}\right] = \sum_{i=1}^{N-1} \frac{|1-\lambda_i|^2}{1-|\lambda_i|^2}.$$

□

Theorem 7 says that, for every matrix P which assures the convergence in the ideal communication case, the mean squared error tends in time to be equal to the variance of the communication noise times a simple functional of the matrix P,

$$\Phi(P) = \frac{1}{N}\sum_{i=1}^{N-1} \frac{|1-\lambda_i|^2}{1-|\lambda_i|^2}.$$

Then we study this functional [1], which is finite for every matrix P, but can depend on N. From now on, we state our results in terms of $\Phi(P)$.

Remark 8. If P is symmetric,

$$\Phi(P) = \frac{1}{N}\sum_{h=1}^{N-1} \frac{1-\lambda_h}{1+\lambda_h}.$$

Remark 9. Since under Hypothesis 1 all $|\lambda_i|$ with $i \geq 1$ are different from 1, then $\Phi(P)$ is finite for any matrix P.

Let us consider a sequence of matrices $(P_N)_{N\in\mathbb{N}}$. If the essential spectral radius is uniformly bounded away from one, then $\Phi(P_N)$ is clearly bounded. Instead, if $\rho(P_N) \to 1$ for $N \to \infty$, the bound could eventually diverge in N. Anyway, we are able to prove that this does not happen under mild conditions.

Lemma 10. Let $B_{c,r} \subset \mathbb{C}$ denote the closed ball of complex numbers with center c and radius r. Let $R = \inf\{r : B_{1-r,r} \supseteq \sigma(P)\}$.

Then $0 < R < 1$, and $\Phi(P)$ is bounded by

$$\Phi(P) \leq \frac{R}{1-R}. \tag{10}$$

Proof. $0 < R < 1$ is clear from $(A), (B), (C)$, and means that the spectrum is contained in a disc of radius R internally tangent in 1 to the unit disc of the complex plane. See Figure 1.

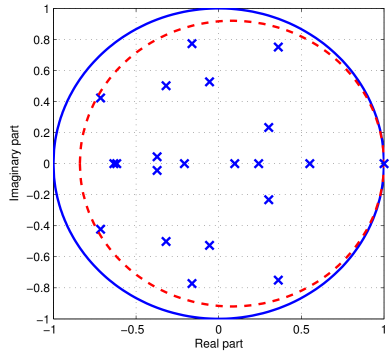

Fig. 1. The eigenvalues of a matrix P (x's) lie inside the ball $B_{1-R,R}$ (dashed circle), contained in the unit disk (solid) of the complex plane.

Then, we need to prove (10). For all i, the eigenvalue $\lambda_i \in B_{1-R,R}$, so $\lambda_i = (1-r) + r\,e^{i\theta}$ with $\theta \in [0, 2\pi[$ and $0 \leq r \leq R$. Moreover, if $i \geq 1$, then $\theta > 0$. Hence,

$$\frac{|1-\lambda_i|^2}{1-|\lambda_i|^2} = \frac{|r - re^{i\theta}|^2}{1-|1-r+r\,e^{i\theta}|^2} =$$
$$= \frac{r^2|1-e^{i\theta}|^2}{1-(1-r)^2 - 2r(1-r)\cos\theta - r^2} =$$
$$= \frac{r^2 2(1-\cos\theta)}{2r(1-r)(1-\cos\theta)} = \frac{r}{1-r} \leq \frac{R}{1-R} \quad \forall i.$$

Then $\frac{1}{N}\sum_{i=1}^{N-1} \frac{|1-\lambda_i|^2}{1-|\lambda_i|^2} \leq \frac{R}{1-R}\frac{N-1}{N}$ and we have completed the proof. □

We have the following useful corollary.

Theorem 11. Let $p = \min_i P_{ii}$ and R as above. Then,

$$R \leq 1-p$$

and

$$\Phi(P) \leq \frac{1-p}{p}. \tag{11}$$

Proof. By Gershgorin theorem,

$$\sigma(P) \subseteq \bigcup_i B_{P_{ii},1-P_{ii}} \subseteq B_{p,1-p}.$$

□

If in a family of matrices p is lower bounded uniformly in N, or just independent of N, then (11) gives a finite bound, uniform in N, on the asymptotic error. This is a useful hint to construct sequences of matrices whose performance scales well with N.

5. EXAMPLES AND SIMULATIONS

Examples are intended to investigate the dependence on N of d and d_∞, to compare simulative results with the theoretical ones and to validate our models.

Our plots are obtained running the algorithm from an adequate number of random initial conditions, for increasing N. To have a wide range of significant examples, we consider four different topologies, showed in Figure 2.

The *undirected circuit*, or *ring* (Figure 4) is the undirected graph in which each node communicates with two neighbors, and we take $P = (A+I)/3$, where A is the adjacency matrix. Then, the eigenvalues are

[1] It is worth to remark that the functional $\Phi(P)$ also arises, with a rather different meaning, in Delvenne et al. [2007], as a cost functional describing the transient of the diffusion methods for average consensus over Cayley graphs with ideal communication.

$$\lambda_h = \frac{1}{3} + \frac{2}{3}\cos\left(\frac{2\pi}{N}h\right) \quad h = 0, \ldots, N-1.$$

Namely, recalling Theorem 6, we remark that $L \geq -\frac{1}{3}$ and $U = 1 - \frac{4\pi^2}{3}\frac{1}{N^2} + o(\frac{1}{N^3})$ for $N \to +\infty$ and hence we get $\sum_{s=0}^{\infty} \rho(P^s(I-P)) \leq 9/2 + \frac{1}{2}\log\left(\frac{3}{4\pi^2}N^2\right)$. Then

$$d_\infty(P) = O(\log N) \text{ for } N \to +\infty.$$

Instead, the probabilistic analysis gives $\Phi(P) \leq 2$, by (11).

The *hypercube* graph (Figure 3) is the graph on $N = 2^n$ nodes obtained drawing the edges of a n-dimensional hypercube.[2] As P we simply choose $P = \frac{1}{n+1}(A+I)$. Then its eigenvalues are $\lambda_k = 1 - \frac{2k}{n+1}$ $\quad k = 0 \ldots n$, each with multiplicity $\binom{n}{k}$. Hence, by Theorem 6,

$$d_\infty(P) \leq \frac{1}{2}\left(\frac{3(n+1)}{2} + \frac{1}{2}\log\frac{n-1}{2}\right) = O(\log N).$$

In this case, $\Phi(P)$ can be exactly computed,

$$\Phi(P) = \frac{1}{2^n}\sum_{k=1}^{n} \frac{\left(\frac{2k}{n+1}\right)^2}{1-\left(\frac{n+1-2k}{n+1}\right)^2}\binom{n}{k} =$$

$$= \frac{1}{2^n}\sum_{k=1}^{n}\frac{k}{n+1-k}\binom{n}{k} = \frac{1}{2^n}\sum_{k=1}^{n}\binom{n}{k-1} = \frac{N-1}{N}.$$

Remark that in both examples, the worst case performance possibly worsen with N, while the probabilistic one does not.

Then we consider two examples of random graphs: in these cases our plots come from several realizations of both the graphs[3] and the initial conditions. The *random d-regular* graph (Figure 5) is a randomly chosen d-regular graph on N nodes, and we take $P = \frac{1}{d+1}(A+I)$.

The *random geometric* graph (Figure 5), is build by randomly placing N nodes in the unit square, which are neighbors whenever their distance is below a threshold[4] $R = \sqrt{\log N/(\pi N)}$. P is constructed so that it is doubly stochastic, exploiting the symmetry of the graph by the method explained in Carli et al. [2006]. In both these cases, the quantities (7) and (9) are numerically evaluated from the matrices P.

Even though the theoretical results suggest that the performance could worsen with increasing N, simulations show a nicer behavior. In all the considered cases, the errors $d(P)$ and $d_\infty(P)$ are small, compared to the quantization step, and (almost) not increasing with N. Moreover, the prediction of the probabilistic[5] model fits very well the simulated $d(P)$. Hence, we argue that the worst case analysis is too pessimistic, taking into account potential combinations of quantization errors which do not appear in practice. Instead, the probabilistic approach finds an *a posteriori* justification in its better agreement with simulations.

[2] Such a graph is widely used in networks of processors Elsässer et al. [2006].

[3] Disconnected realizations have been discarded.

[4] With this choice of R we are in the so-called *connectivity regime* Penrose [2003].

[5] The plots assume $\sigma^2 = 1/12$, as the distribution of quantization error was uniform.

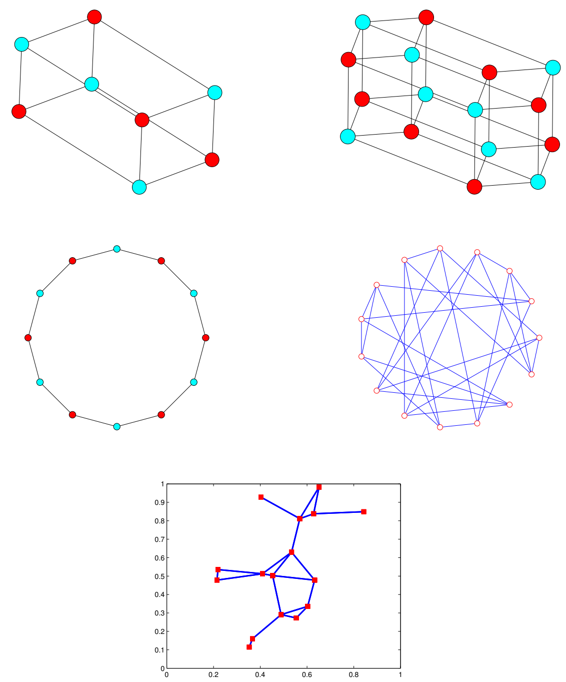

Fig. 2. Example graphs. Above, the hypercube graphs of dimension $n = 3$ and $n = 4$. In the middle, the undirected circuit with $N = 12$ and a 4-regular graph on 15 nodes. Below, a random geometric graph on 16 nodes.

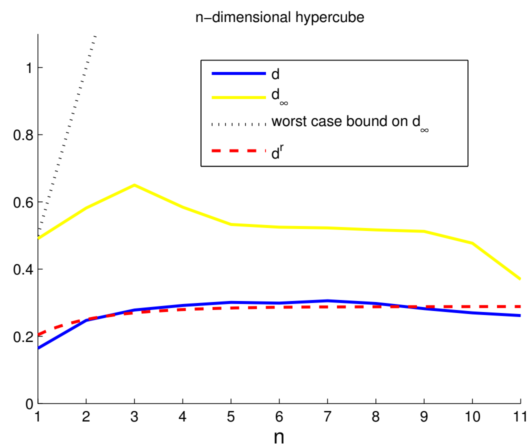

Fig. 3. Performance of the n-dimensional hypercube graph (of order $N = 2^n$).

6. CONCLUSION

In this paper we studied by a probabilistic model the quantized consensus algorithm (4) proposed in Carli et al. [2007a], arguing that its performance in driving the system near to consensus depends on the spectral properties of the evolution matrix P. Such a probabilistic model seems to fit simulative evidence better than a worst case approach. From the design point of view, we obtain with Theorem 11 a simple and distributed criterion to control the average asymptotic error of the method, uniformly in N. It would be enough to prescribe the agents a minimum weight to assign to their own values while averaging, that is a lower bound on the diagonal of the matrices.

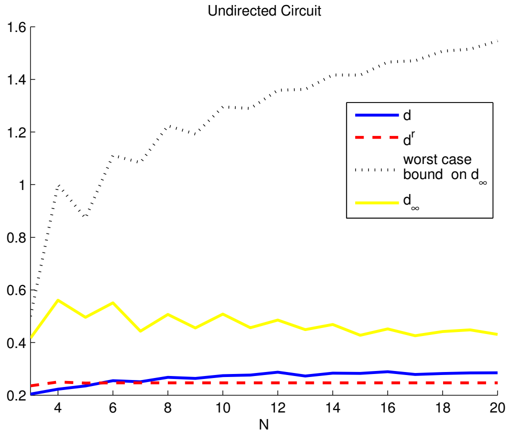

Fig. 4. Performance of the undirected circuit.

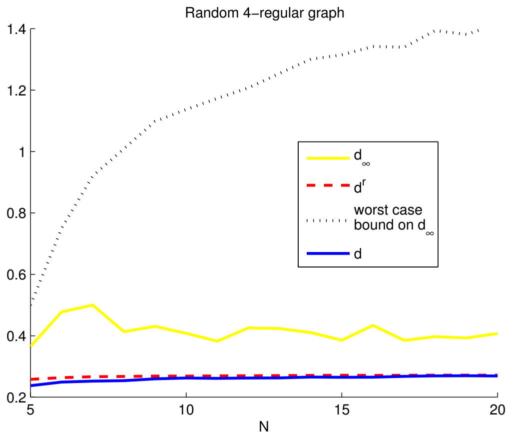

Fig. 5. Performance of the random 4-regular graph.

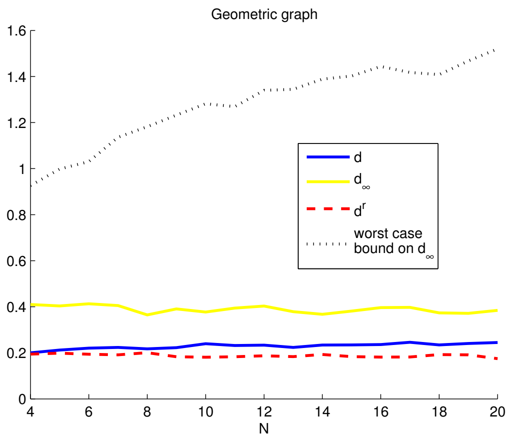

Fig. 6. Performance of the random geometric graph.

Further developments of this work will include the refinement of the above analysis and the interplays between worst case and probabilistic approach. Moreover, it will be interesting to construct methods with time varying quantization steps, following the spirit of Brockett and Liberzon [2000]: a first attempt in this direction is Carli et al. [2007b].

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