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Robust Switching of Switched Linear Systems ⋆

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Abstract: For switched dynamical systems, switching signals usually undergo perturbations and disturbances due to various reasons. A well-behaved switched system might not work properly when its switching signal is perturbed. In this work, we investigate the switching robustness for a class of switched linear systems. For this, we first define the distance between two switching signals by means of their switching matrices chains. Then, we prove that, if a periodic switching path steers the switched system exponentially stable, then any slightly perturbed switching signal also makes the system stable.

1. INTRODUCTION

The study of switched and hybrid dynamical systems has attracted much attention since 1990s. A switched dynamical system is a hybrid system which consists of several continuous subsystems and a switching signal that orchestrates the switching among them. The importance of the study on switched dynamical systems stems from the facts that the system framework represents a wide class of practical systems, and the two-level system structure provides an effective multiple-controller based switching control approach.

As a primary issue, the stability of switched systems has been extensively investigated in the literature. For the guaranteed stability, it was proved that the stability under arbitrary switching is equivalent to the existence of a common Lyapunov function for the subsystems Lin et al. [1996], Dayawansa & Martin [1999]. Accordingly, the Lyapunov-based tools, such as the multiple Lyapunov-like function approach and the linear matrix inequality approach, were proposed for stability analysis Branicky [1998], Hespanha [2004]. For the stabilizing switching design, no general constructive approach is available so far, but quite much development has been made for special classes of switched systems Wicks et al. [1994], Feron [1996], Bacciotti [2004]. The reader is referred to Decarlo et al. [2000], Savkin & Evans [2002], Liberzon [2003], Sun & Ge [2005] for surveys of recent developments.

It is well recognized that the interactions between the subsystem dynamics and the switching signal are quite involved. On one hand, different switching strategies may produce totally different global system behaviors. A well-known example is the switched server system composed by rather simple local models that could produce chaos Chase et al. [1993], multiple limit cycles Savkin & Matveev [2001], and other complex global behaviors. On the other hand, under the same switching strategy, a nominal system and its (slightly) perturbed system may generate different switching signals. For example, even the nominal system is well-behaved, the perturbed system might be ill-behaved due to the chattering of the switching signal Sun [2004a]. This means that a small perturbation in subsystem dynamics leads to a remarkable switching derivation.

For a switched system, a system perturbation can be imposed on the subsystem dynamics, or the switching mechanism, or both. For a switched system whose subsystems undergo structural perturbations, the stability issue has been addressed in Sun [2004b]. It was proved that any asymptotically stable system is also structurally stable. For structural stability in terms of the switching mechanism, the issue has not been discussed in the literature. By structural stability in terms of the switching mechanism, we mean a 'perturbation' of a normal stabilizing switching signal and its influence in system stability. Intuitively, for a robustly stabilizing switching signal, a small perturbed switching signal should also make the switched system stable. A problem thus arises naturally: How to characterize the distance between two switching signals? As switching signals are defined over infinite time horizon, it is not a trivial task to formulate a reasonable measure Sun & Ge [2006].

The study of the stability with respect to switching perturbation is well motivated and practiced for the reasons below. First, in practice we cannot implement a switching signal precisely. For example, time delay is unavoidable in may practical situations. Second, the switching device may mismanipulate in certain cases. For instance, the system should activate the ith subsystem, but it activates the jth instead. Third, component (subsystem) failures lead to displacement of switching signals. Finally, from the design viewpoint, we prefer to choose a switching signal which still works (for the stability purpose) under small perturbations.

2. PRELIMINARIES

Let $\mathbf{R}^n$ be the nth-dimensional Euclidian real space, $\mathbf{R}_+$ the set of nonnegative real numbers. Let $\|\cdot\|$ denote the

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standard 2-norm for a vector in $\mathbf{R}^n$ and the induced norm for a matrix in $\mathbf{R}^{n\times n}$. For $\Theta \subset \mathbf{R}$, $\mathrm{meas}\,\Theta$ denotes its Lebesgue measure. # denotes the cardinality of a set. Let $M = \{1, 2, \cdots, m\}$ be a finite index set.

Consider a switched linear system given by

$$\dot{x}(t) = A_{\sigma(t)}x(t), \tag{1}$$

where $x \in \mathbf{R}^n$ is the system state, $\sigma(t) \in M$ is the switching signal, and $A_k \in \mathbf{R}^{n\times n}$, $k \in M$ are known real constant matrices.

Given a time interval $[t_0, t_f)$ (other types of intervals can be considered in a similar manner), a switching path over the interval is a piecewise constant function p that maps $[t_0, t_f)$ to M (denoted $p_{[t_0,t_f)}$ in short). Suppose that its discontinuous (jump) time instants are $t_1 < t_2 < \cdots < t_s < \cdots < t_f$, we refer to the sequence

$$DS_p = \{(p(t_0), t_1 - t_0), (p(t_1), t_2 - t_1), \cdots, \\ (p(t_{s-1}), t_s - t_{s-1}), \cdots\}$$

as its duration sequence. In particular, the sequence $t_0, t_1, t_2, \cdots$ is referred to as switching time sequence. It is clear that p and DS_p are equivalent in the sense that one can determine the others and vice-versa. The switching path $p_{[t_0,t_f)}$ is said to be well-defined if the number of switches is finite in any subinterval of a finite length. Given a subinterval $[t_1, t_2)$ of $[t_0, t_f)$, the sub-path of $p_{[t_0,t_f)}$ over $[t_1, t_2)$ is denoted by $p_{[t_1,t_2)}$.

The switching signal σ is usually a function of the time and the state. It is said to be well-defined with respect to the switched system, if for each initial state x, it generates a well-defined switching path, denoted σ^x, that defined over $[0, \infty)$. In this way, a well-defined switching signal can be equivalently expressed by a set of switching paths $\{\sigma^x_{[0,\infty)} : x \in \mathbf{R}^n\}$.

Let $\phi(t; t_0, x_0, \sigma)$ denote the motion of system (1) at time t starting from x_0 at t_0 along switching signal σ.

3. SWITCHING DISTANCE

In this section, we are to characterize the distance between two switching signals. For this, we first consider the simplest case, that is, define the distance between two switching paths.

3.1 Distance Between Switching Paths

As the state transition matrix is multiple multiplication of matrix function of the form e^{At}, the system stability depends heavily on the asymptotic properties of the infinite chain of matrix multiplications

$$e^{A_{i_0}(t_1-t_0)} \mid e^{A_{i_1}(t_2-t_1)}e^{A_{i_0}(t_1-t_0)} \mid \cdots \mid \\ e^{A_{i_k}(t-t_k)}\cdots e^{A_{i_1}(t_2-t_1)}e^{A_{i_0}(t_1-t_0)} \mid \cdots \tag{2}$$

which is referred to as transition (matrices) chain. Indeed, suppose that the above matrix chain is convergent (to the zero matrix), the corresponding switching path makes the state convergent for all initial states, and vice-versa. On the other hand, a variation of a switching path means a variation of the duration sequence, which can also be seen as a variation of the transition chain. This simple observation provides a basic insight into the way for characterizing the distance between two switching paths.

Let Δ be the set of intervals of the form $[a, b)$ with $0 \le a < b$, and any union of such intervals. Given $\pi \in \Delta$, we can define a diffeomorphism $\psi_\pi : \mathbf{R}_+ \setminus \pi \mapsto \mathbf{R}_+$ by

$$\psi_\pi(t) = t - \mathrm{meas}\{s \le t : s \in \pi\}, \quad t \in \mathbf{R}_+ \setminus \pi.$$

Given two switching paths $p_{1[t_1,t_2)}$ and $p_{2[t_3,t_4)}$, p_1 is said to be a child-path of p_2, or p_2 is a parent-path of p_1, denoted by $p_1 \preceq p_2$, if there is a $\pi \in \Delta$ and a time transition $\delta \in \mathbf{R}$, such that

$$[t_1, t_2) = \cup_{t\in[t_3,t_4)\setminus\pi}\{\psi_\pi(t) - \delta\}, \\ p_2(t) = p_1(\psi_\pi(t) - \delta), \quad \forall\, t \in [t_3, t_4) \setminus \pi. \tag{3}$$

Correspondingly, let $\Delta^{p_2}_{p_1}$ be the set of π that satisfies (3). It should be stressed that, set $\Delta^{p_2}_{p_1}$ may contain more than one element.

For two switching paths $p_1 \preceq p_2$, define the distance to be

$$|p_2 - p_1| = \inf_{\pi\in\Delta^{p_2}_{p_1}} \mathrm{meas}\,\pi. \tag{4}$$

It is clear that $|p_2 - p_1| = 0$ iff p_1 is a pure time transition of p_2, that is, $t_4 - t_3 = t_2 - t_1$ and $p_1(t_1 + s) = p_2(t_3 + s)$ for all $s \in [0, t_2 - t_1)$. In this case, we denote $p_2 = p_1^{\mapsto t_3-t_1}$.

Given switching paths p_1, p_2, and p_3, p_3 is said to be a common parent-path of p_1 and p_2, denoted $p_3 \in \mathcal{CP}(p_1, p_2)$, if $p_1 \preceq p_3$ and $p_1 \preceq p_3$. It can be seen that, for any two switching paths, there must exist a common parent-path. Indeed, suppose that

$$DS_{p_j} = \{(i^j_0, h^j_0), (i^j_1, h^j_1), (i^j_2, h^j_2), \cdots\}, \quad j = 1, 2$$

then, the switching path

$$DS_{p_3} = \{(i^1_0, h^1_0), (i^2_0, h^2_0), (i^1_1, h^1_1), (i^2_1, h^2_1), \\ (i^1_2, h^1_2), (i^2_2, h^2_2), \cdots\}$$

is a common parent-path of p_1 and p_2.

Definition 1. For any switching paths p_1 and p_2, the (absolute) distance bewteen them is defined as

$$d(p_1, p_2) = \inf_{p_3\in\mathcal{CP}(p_1,p_2)} |p_3 - p_1| + |p_3 - p_2|. \tag{5}$$

Proposition 1. The distance between switching paths possesses the following properties:

1) (positive definiteness) $d(p_1, p_2) \ge 0$, and $d(p_1, p_2) = 0$ iff $p_1 = p_2^{\mapsto s}$ for some $s \in \mathbf{R}$;
2) (symmetricalness) $d(p_1, p_2) = d(p_2, p_1)$; and
3) (triangular inequality) $d(p_1, p_2) \le d(p_1, p_3)+d(p_2, p_3)$.

Proof. Properties 1) and 2) straightforwardly follow from the definition. To prove property 3), let ε be an arbitrarily small positive real number. By definition, there is a common parent-path p_4 of paths p_1 and p_3, such that

$$d(p_1, p_3) \ge |p_4 - p_1| + |p_4 - p_3| - \varepsilon.$$

Similarly, there is a common parent-path p_5 of paths p_2 and p_3, such that

$$d(p_2, p_3) \geq |p_5 - p_2| + |p_5 - p_3| - \varepsilon.$$

On the other hand, we can find a common parent-path p_6 of paths p_4 and p_5, such that

$$|p_6 - p_4| \leq |p_5 - p_3|, \quad \text{and} \quad |p_6 - p_5| \leq |p_4 - p_3|.$$

As p_6 is also a common parent-path of p_1 and p_2, we have

$$\begin{aligned} d(p_1, p_2) &\leq |p_6 - p_1| + |p_6 - p_2| \\ &\leq |p_6 - p_4| + |p_4 - p_1| + |p_6 - p_5| + |p_5 - p_2| \\ &\leq |p_5 - p_3| + |p_4 - p_1| + |p_4 - p_3| + |p_5 - p_2| \\ &\leq d(p_1, p_3) + d(p_2, p_3) + 2\varepsilon. \end{aligned}$$

Due to the arbitrariness of ε, the triangular inequality holds. ◇

It follows from the proposition that, the set of switching paths forms a metric space with the distance defined here.

3.2 Distance Between Two Switching Signals

Suppose that σ_1 and σ_2 are two well-defined switching signals for the switched system. Accordingly, for each initial state x, they generate the switching paths σ_1^x and σ_2^x, respectively.

Definition 2. (1) for a time τ and initial state x, the τ-distance at x between switching signals σ_1 and σ_2 is

$$D_x^\tau(\sigma_1, \sigma_2) = d({\sigma_1}^x_{[0,\tau]}, {\sigma_2}^x_{[0,\tau]}), ; \qquad (6)$$

(2) the relative distance at x between switching signals σ_1 and σ_2 is

$$RD_x(\sigma_1, \sigma_1) = \limsup_{\tau \to \infty} \frac{1}{\tau} D_x^\tau(\sigma_1, \sigma_2); \qquad (7)$$

(3) the supremal relative distance between switching signals σ_1 and σ_2 is

$$SRD(\sigma_1, \sigma_2) = \limsup_{x \in \mathbf{R}^n} RD_x(\sigma_1, \sigma_2). \qquad (8)$$

Remark 1. The τ-distance measures the absolute distance between the the switching signals with respect to an initial state over interval $[0, \tau]$. The relative distance measures the average distance in time over an infinite horizon, and the supremal relative distance measure the largest possible relative distance for all initial states. It should be stressed that the relative distance is more subtle than the (absolute) τ-distance in characterizing the distant between two switching signals. In fact, for any two switching paths defined on an infinite horizon, the absolute ∞-distance must be infinite if the relative distance is positive, but the reverse is not necessarily true.

Theorem 1. The distance between switching signals satisfies:

1) (positive definiteness) $SRD(\sigma_1, \sigma_2) \geq 0$ with $SRD(\sigma_1, \sigma_1) = 0$;
2) (symmetricalness) $SRD(\sigma_1, \sigma_2) = SRD(\sigma_2, \sigma_1)$; and
3) (triangular inequality) $SRD(\sigma_1, \sigma_2) \leq SRD(\sigma_1, \sigma_3) + SRD(\sigma_2, \sigma_3)$.

Proof. Properties 1) and 2) are straightforward. To prove 3), note that

$$\begin{aligned} &SRD(\sigma_1, \sigma_2) \\ &= \limsup_{x \in \mathbf{R}^n} RD_x(\sigma_1, \sigma_2) \\ &= \limsup_{x \in \mathbf{R}^n} \limsup_{\tau \to \infty} \frac{1}{\tau} D_x(\sigma_1^x, \sigma_2^x) \\ &\leq \limsup_{x \in \mathbf{R}^n} \limsup_{\tau \to \infty} \frac{1}{\tau} \left(D_x(\sigma_1^x, \sigma_3^x) + D_x(\sigma_2^x, \sigma_3^x) \right) \\ &\leq \limsup_{x \in \mathbf{R}^n} \limsup_{\tau \to \infty} \frac{1}{\tau} D_x(\sigma_1^x, \sigma_3^x) \\ &\quad + \limsup_{x \in \mathbf{R}^n} \limsup_{\tau \to \infty} \frac{1}{\tau} D_x(\sigma_2^x, \sigma_3^x) \\ &= SRD(\sigma_1, \sigma_3) + SRD(\sigma_2, \sigma_3). \end{aligned}$$

4. ROBUST SWITCHING

For robust switching, we mean that a stabilizing switching signal still works when it is slightly perturbed. For technical reasons, we focus on a special class of switched linear systems.

A switched linear system is consistently exponentially stabilizable, if there exists a switching path p such that

$$\|\phi(t; t_0, x_0, p)\| \leq \beta \exp(-\alpha(t - t_0)) \|x_0\|, \quad \forall\ t \geq t_0,\ x_0 \in \mathbf{R}^n$$

for some positive real numbers α and β. It was proved that consistent exponential stabilizability is equivalence to the existence of a stabilizing periodic switching path Sun [2004b], which is the start point of the following result.

Theorem 2. Suppose that periodic switching path p exponentially stabilizes the switched system. Then, there is a positive real number γ, such that any switching signal σ with $SRD(p, \sigma) \leq \gamma$ also exponentially stabilizes the switched system.

Proof. Suppose that T is the period of p. Then, it can be seen that the state transition matrix $\Phi(T, 0, p)$ is Schur stable. As a result, for any fixed real number $\delta \in (0, 1)$, there is a natural number k, such that

$$\|\Phi(kT, 0, p)\| = \|\Phi(T, 0, p)^k\| \leq \delta.$$

Fix a real number $\rho \in (\delta, 1)$. By the continuity of the state transition matrix with respect to the switching path, there is a positive real number μ, such that

$$\|\Phi(kT, 0, \bar{p})\| \leq \rho, \quad \forall\ |\bar{p} - p| \leq \mu. \qquad (9)$$

On the other hand, let x be any given but fixed state, ϵ be any given positive real number, and σ^x is the switching path generated by switching signal σ at x. Recall that $SRD(p, \sigma) < \gamma$ means that $RD_x(p, \sigma^x) < \gamma$, which further means that the τ-distance between the perturbed switching and the nominal switching is upper bounded by $\tau(\gamma + \epsilon)$ for sufficiently large τ. That is, there is a τ_0, such that

$$D_x^\tau(p, \sigma^x) \leq \tau(\gamma + \epsilon), \quad \forall\ \tau \geq \tau_0.$$

Now choose $\gamma = \epsilon = \mu\varpi$ where ϖ is a positive real number to be determined later. Fix an arbitrarily given natural number N such that $NkT \geq \tau_0$. By the definition of the distance between switching paths, we can partition time interval $[0, NkT)$ into N (possibly empty) subintervals $[0, NkT) = \cup_{i=1}^{N}[t_{i-1}, t_i)$, such that

$$D_x^{NkT}(p, \sigma^x) = \sum_{i=1}^{N} d\left(p_{[(i-1)kT, ikT)}, \sigma^x_{[t_{i-1}, t_i)}\right). \quad (10)$$

Define

$$N_1 = \#\left\{i \leq N \colon d\left(p_{[(i-1)kT, ikT)}, \sigma^x_{[t_{i-1}, t_i)}\right) > \mu\right\},$$
$$N_2 = N - N_1.$$

It can be seen that

$$N_1 \leq \lceil 2N\varpi \rceil,$$
$$N_2 \geq \lfloor N(1 - 2\varpi) \rfloor,$$

where $\lceil a \rceil$ ($\lfloor a \rfloor$) denotes the smallest (largest) integer equal or greater (less) than a. Based on the above facts, routine calculation gives

$$\|\Phi(NkT, 0, \sigma^x)\| \leq \mu^{N_2}\eta^{N_1\mu} \leq \frac{\eta^\mu}{\mu}\left(\mu^{1-2\varpi}\eta^{2\varpi}\right)^N, \quad (11)$$

where $\eta = \max\{e^{\|A_1\|}, \cdots, e^{\|A_m\|}\}$.

Finally, let $\varpi = -\frac{\ln \mu}{4(\ln \eta - \ln \mu)}$, which is clearly independent of τ_0. Let $\alpha = -\frac{\ln \mu}{2kT}$, and $\beta = \frac{\eta^\mu}{\mu}$. Then, it can be seen from Inequality (11) that

$$\|\Phi(NkT, 0, \sigma^x)\| \leq \beta e^{-\alpha NkT},$$

which clearly exhibits that σ^x makes the switched system exponentially convergent from the initial state x.

5. CONCLUDING REMARKS

In this work, we first introduced the notion of distance between two switching paths, then extended the notion to relative distance between two switching signals. By means of the distance notions, we established a robust switching property for a class of switched linear systems.

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